Operation Research for Business Decisions (MMS-608), MCA-II Year

UNIT-I

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Definition

- *Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.*

  – P M Morse and G E Kimball, 1951

- *Operations research is a scientific approach to problem-solving for executive management.*

  – H M Wagner

- *Operations research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management in determining its policy and actions scientifically.*

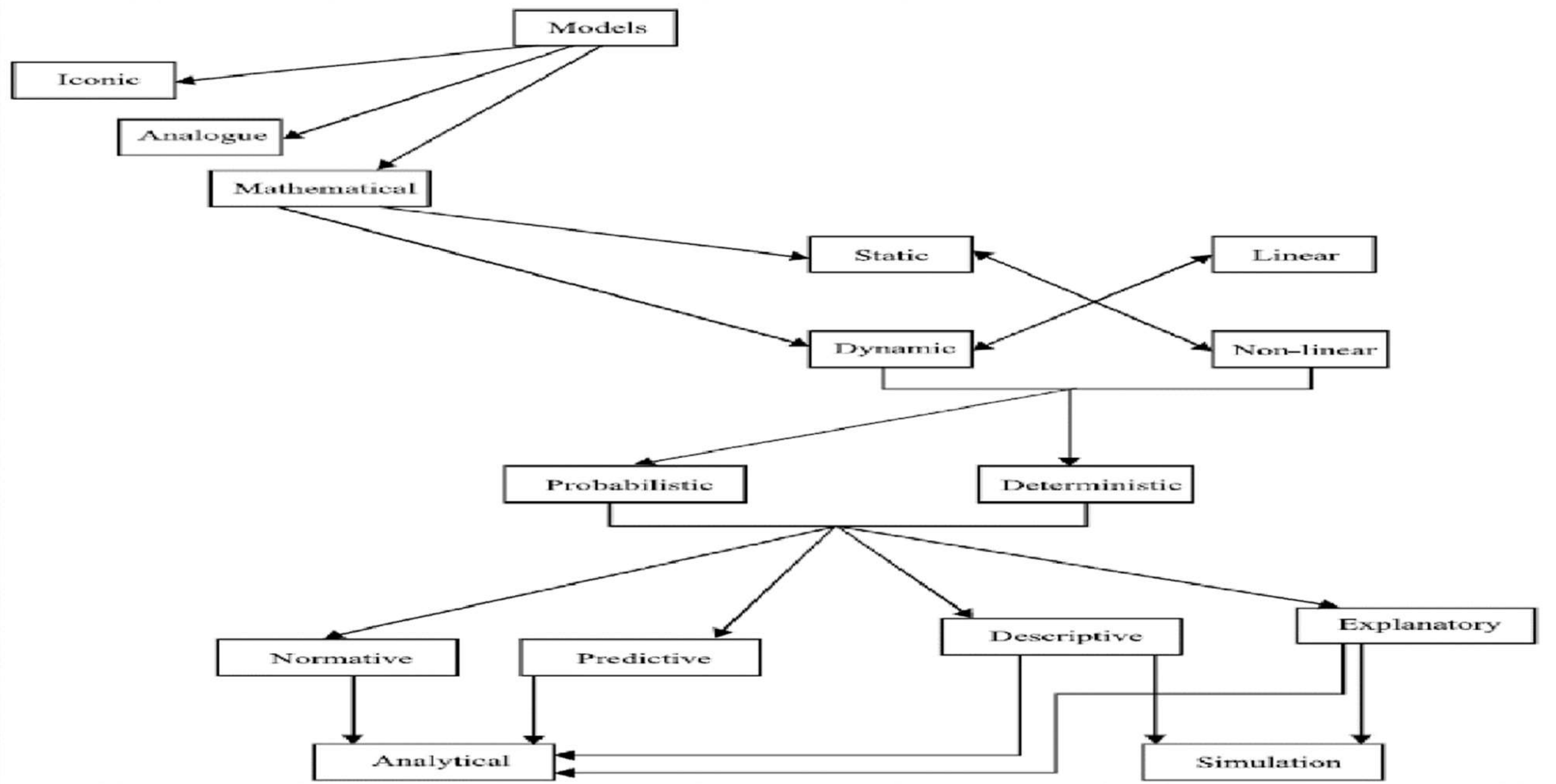
  – Operational Research Society, UK

SCOPE OF OPERATION RESEARCH

In recent years of organized development, operation research(OR) has entered successfully in many different areas of research. It is useful in the following various important fields:

- ***In agriculture,*** *with the sudden increase of population and resulting shortage of food, every country is facing the problem of, Optimum allocation of land to a variety of crops as per the climatic conditions and Optimum distribution of water from numerous resources like canal for irrigation purposes. Hence there is a requirement of determining best policies under the given restrictions. Therefore a good quantity of work can be done in this direction.*
- ***In finance****, recent times of economic crisis, it has become very essential for every government to do a careful planning for the economic progress of the country. OR techniques can be productively applied to determine the profit plan for the company, maximize the per capita income with least amount of resources also decide on the best replacement policies, etc.*
- ***In L.I.C,*** *OR approach is also applicable to facilitate the L.I.C offices to decide. What should be the premium rates for a range of policies? How well the profits could be allocated in the cases of with profit policies?*

OR Model



Models
Iconic
Analogue
Mathematical
Static
Linear
Dynamic
Non–linear
Probabilistic
Deterministic
Normative
Predictive
Descriptive
Explanatory
Analytical
Simulation

METHODS FOR SOLVING OPERATIONS RESEARCH MODELS

In general, the following three methods are used for solving OR models, where values of decision variables are obtained that optimize the given objective function (a measure of effectiveness).

- ***Analytical (or deductive) method -****In this method, classical optimization techniques such as calculus, finite difference and graphs are used for solving an OR model. The analytical methods are non-iterative methods to obtain an optimal solution of a problem.*
- ***Numerical (or iterative) method-*** *When analytical methods fail to obtain the solution of a particular problem due to its complexity in terms of constraints or number of variables, a numerical (or iterative) method is used to find the solution. In this method, instead of solving the problem directly, a general algorithm is applied for obtaining a specific numerical solution.*
- ***Monte Carlo method*** *This method is based upon the idea of experimenting on a mathematical model by inserting into the model specific values of decision variables for a selected period of time under different conditions and then observing the effect on the criterion chosen. In this method, random samples of specified random variables are drawn to know how the system is behaving for a selected period of time under different conditions.*

Linear Programming Problem

- ***Introduction***- *Linear Programming is a mathematical technique useful for allocation of 'scarce' or 'limited' resources, to several competing activities on the basis of a given criterion of optimality*.
- *Linear Programming is techniques to determine the choice among several courses of action, so as to get an optimal value of the measures of effectiveness (objective or goal), requires to formulate (or construct) a mathematical model. Such a model helps to represent the essence of a system that is required for decision-analysis.*
- *Linear Programming problem can be solved by following methods:*

1. *Graphical Method*
2. *Simplex Method*

GENERAL MATHEMATICAL MODEL OF LINEAR PROGRAMMING PROBLEM

The general linear programming problem (or model) with n decision variables and m constraints can be stated in the following form:

Optimize (Max. or Min.) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to the linear constraints,

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \;(\leq, =, \geq)\; b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \;(\leq, =, \geq)\; b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \;(\leq, =, \geq)\; b_m \end{array}$$

and $\quad x_1, x_2, \ldots, x_n \geq 0$

The above formulation can also be expressed in a compact form as follows.

Optimize (Max. or Min.) $Z = \sum_{j=1}^{n} c_j x_j$ (Objective function) (1)

subject to the linear constraints

$\sum_{j=1}^{n} a_{ij} x_j \;(\leq, =, \geq)\; b_i ;\quad i = 1, 2, \ldots, m$ (Constraints) (2)

and $\quad x_j \geq 0 ;\quad j = 1, 2, \ldots, n$ (Non-negativity conditions) (3)

where, the c_j s are coefficients representing the per unit profit (or cost) of decision variable x_j to the value of objective function. The a_{ij}'s are referred as *technological coefficients* (*or input-output coefficients*).

Example 1:-

A company manufactures two types of boxes, corrugated and ordinary cartons.

The boxes undergo two major processes: cutting and pinning operations.

The profits per unit are Rs. 6 and Rs. 4 respectively.

Each corrugated box requires 2 minutes for cutting and 3 minutes for pinning operation, whereas each carton box requires 2 minutes for cutting and 1 minute for pinning.

The available operating time is 120 minutes and 60 minutes for cutting and pinning machines.

The manager has to determine the optimum quantities to be manufacture the two boxes to maximize the profits.

Solution:-

Decision variables completely describe the decisions to be made (in this case, by Manager). Manager must decide how many corrugated and ordinary cartons should be manufactured each week. With this in mind, he has to define:

xl be the number of corrugated boxes to be manufactured.
x2 be the number of carton boxes to be manufactured

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Manager can concentrate on maximizing the total weekly profit (z).
Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs).
Hence Manager's objective function is:

Maximize $Z = 6X_2 + 4X_2$

Constraints show the restrictions on the values of the decision variables. Without constraints manager could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Available machine-hours for each machine
Time consumed by each product

Sign restrictions are added if the decision variables can only assume nonnegative values (Manager can not use negative negative number machine and time never negative number)

Continue..

All these characteristics explored above give the following **Linear Programming** (LP) problem

$$\max z = 6x_1 + 4x_2 \quad \text{(The Objective function)}$$

$$\text{s.t.} \quad 2x_1 + 3x_2 \le 120 \quad \text{(cutting timeconstraint)}$$

$$2x_1 + x_2 \le 60 \quad \text{(pinning constraint)}$$

$$x_1, x_2 \ge 0 \quad \text{(Sign restrictions)}$$

A value of (x_1, x_2) is in the **feasible region** if it satisfies all the constraints and sign restrictions.

Example 2. *A company has two grades of inspectors 1 and 2, the members of which are to be assigned for a quality control inspection. It is required that at least 2,000 pieces be inspected per 8-hour day. Grade1 inspectors can check pieces at the rate of 40 per hour, with an accuracy of 97 per cent. Grade 2 inspectors check at the rate of 30 pieces per hour with an accuracy of 95 per cent. The wage rate of a Grade 1 inspector is Rs 5 per hour while that of a Grade 2 inspector is Rs 4 per hour. An error made by an inspector costs Rs 3 to the company. There are only nine Grade 1 inspectors and eleven Grade 2 inspectors available to the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as an LP model so as to minimize the daily inspection cost.* ***(Do yourself…..)***

Graphical Method

Step 1: *Convert the inequality constraint as equations and find co-ordinates of the line.*

Step 2: *Plot the lines on the graph.*

(***Note:*** If the constraint is ≥ type, then the solution zone lies away from the centre.
If the constraint is ≤ type, then solution zone is towards the centre.)

Step 3: *Obtain the feasible zone.*

Step 4: *Find the co-ordinates of the objectives function (profit line) and plot it on the* graph representing it with a dotted line.

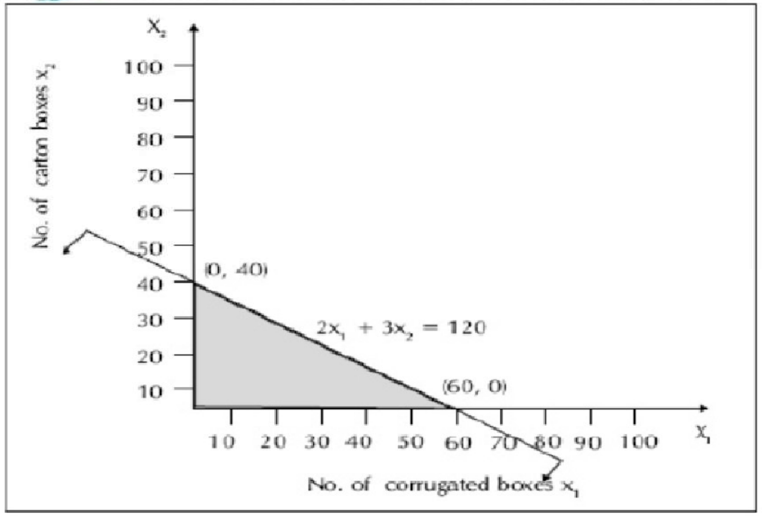
Step 5: *Locate the solution point.*

(*Note: If the given problem is maximization,* ***Zmax*** *then locate the solution point at* the far most point of the feasible zone from the origin and if minimization, Zmin then locate the solution at the shortest point of the solution zone from the origin).

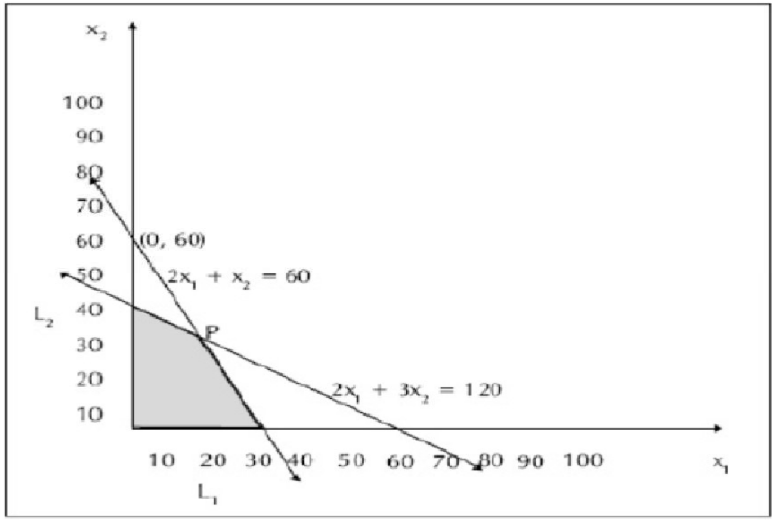
Step 6: Solution type

i. If the solution point is a single point on the line, take the corresponding values of x_1 and x_2.
ii. If the solution point lies at the intersection of two equations, then solve for x_1 and x_2 using the two equations.
iii. If the solution appears as a small line, then a multiple solution exists.
iv. If the solution has no confined boundary, the solution is said to be an unbound solution.

Continue..(Example 1)



Graph Considering First Constraint



Graph Showing Feasible Area

The inequality constraint of the first line is (less than or equal to) ≤ type which means the feasible solution zone lies towards the origin.

(Note: If the constraint type is ≥ then the solution zone area lies away from the origin in the opposite direction). Now the second constraints line is drawn.

When the second constraint is drawn, you may notice that a portion of feasible area is cut. This indicates that while considering both the constraints, the feasible region gets reduced further. Now any point in the shaded portion will satisfy the constraint equations.

the objective is to maximize the profit. The point that lies at the furthermost point of the feasible area will give the maximum profit. To locate the point, we need to plot the objective function (profit) line.

Continue..

Objective function line (Profit Line)

Equate the objective function for any specific profit value Z,

Consider a Z-value of 60, i.e.,

$6x_1 + 4x_2 = 60$

Substituting $x_1 = 0$, we get $x_2 = 15$ and

if $x_2 = 0$, then $x_1 = 10$

Therefore, the co-ordinates for the objective function line are (0,15), (10,0) as indicated objective function line. The objective function line contains all possible combinations of values of xl and x2.

Therefore, we conclude that to maximize profit, 15 numbers of corrugated boxes and 30 numbers of carton boxes should be produced to get a maximum profit. Substituting

$x_1 = 15$ and $x_2 = 30$ in objective function, we get

$Z_{max} = 6x_1 + 4x_2$

$= 6(15) + 4(30)$

Maximum profit : Rs. 210.00

Questions

Example 1 Solve the following LP problem graphically:

Maximize $Z = -x_1 + 2x_2$

subject to the constraints

(i) $x_1 - x_2 \leq -1$; (ii) $-0.5x_1 + x_2 \leq 2$

and $x_1, x_2 \geq 0$.

Example 2 Use the graphical method to solve the following LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

(i) $5x_1 + x_2 \geq 10$, (ii) $x_1 + x_2 \geq 6$, (iii) $x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$.

Example 3 Use the graphical method to solve the following LP problem.

Minimize $Z = -x_1 + 2x_2$

subject to the constraints

(i) $-x_1 + 3x_2 \leq 10$, (ii) $x_1 + x_2 \leq 6$, (iii) $x_1 - x_2 \leq 2$

and $x_1, x_2 \geq 0$.

Simplex Method

- When decision variables are ***more than 2***, we always use Simplex Method
- **Slack Variable**: Variable added to a ≤ constraint to convert it to an equation (=).
  - A slack variable represents unused resources
  - A slack variable contributes nothing to the objective function value.
- **Surplus Variable:** Variable subtracted from a ≥ constraint to convert it to an equation (=).
  - A surplus variable represents an excess above a constraint requirement level.
  - Surplus variables contribute nothing to the calculated value of the objective function.

Continue..

Basic solution Given a system of m simultaneous linear equations with n ($> m$) variables: $\mathbf{Ax} = \mathbf{B}$, where $\mathbf{A}$ is an $m \times n$ matrix and rank ($\mathbf{A}$) $= m$. Let $\mathbf{B}$ be any $m \times m$ non-singular submatrix of $\mathbf{A}$ obtained by reordering m linearly independent columns of $\mathbf{A}$. Then, a solution obtained by setting $n - m$ variables not associated with the columns of $\mathbf{B}$, equal to zero, and solving the resulting system is called a *basic solution* to the given system of equations.

The m variables (may be all non zero) are called *basic variables*. The $m \times m$ non-singular sub-matrix $\mathbf{B}$ is called a *basis matrix* and the columns of $\mathbf{B}$ as *basis vectors*.

If $\mathbf{B}$ is the basis matrix, then the basic solution to the system of equations will be $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$.

Basic feasible solution A basic solution to the system of simultaneous equations, $\mathbf{Ax} = \mathbf{b}$ is called *basic feasible* if $\mathbf{x_B} \geq 0$.

Degenerate solution A basic solution to the system of simultaneous equations, $\mathbf{Ax} = \mathbf{b}$ is called *degenerate* if one or more of the basic variables assume zero value.

Cost vector Let $\mathbf{x_B}$ be a basic feasible solution to the LP problem.

Maximize $Z = \mathbf{cx}$

subject to the constraints

$$\mathbf{Ax} = \mathbf{b}, \quad \text{and} \quad \mathbf{x} \geq 0.$$

Then the vector $\mathbf{c_B} = (c_{B1}, c_{B2}, \ldots, c_{Bm})$, is called *cost vector* that represents the coefficient of basic variable, $x_{\mathbf{B}}$ in the basic feasible solution.

Continue..(Initial simplex Table)

		$c_j \rightarrow$	c_1	c_2	. . .	c_n	0	0	. . .	0
Basic Variables Coefficient ($\boldsymbol{c_B}$)	*Basic Variables* $\boldsymbol{B}$	*Basic Variables Value* $\boldsymbol{b}$ (= $\boldsymbol{x_B}$)	*Variables* x_1	x_2	. . .	x_n	s_1	s_2	. . .	s_m
c_{B1}	s_1	$x_{B1} = b_1$	a_{11}	a_{12}	. . .	a_{1n}	1	0	. . .	0
c_{B2}	s_2	$x_{B2} = b_2$	a_{21}	a_{22}	. . .	a_{2n}	0	1	. . .	0
.	.	.	.	.		.	.	.		.
.	.	.	.	.		.	.	.		.
.	.	.	.	.		.	.	.		.
c_{Bm}	s_m	$x_{Bm} = b_m$	a_{m1}	a_{m2}	. . .	a_{mn}	0	0	. . .	1
$Z = \Sigma c_{Bi} x_{Bi}$		$z_j = \Sigma c_{Bi} x_j$	0	0	. . .	0	0	0	. . .	0
		$c_j - z_j$	$c_1 - z_1$	$c_2 - z_2$	. . .	$c_n - z_n$	0	0	. . .	0

Continue..

After having set up the initial simplex table, locate the identity matrix (contains all *zeros* except positive elements 1's on the diagonal). The identity matrix so obtained is also called a *basis matrix* [because basic feasible solution is represented by $\mathbf{B} = \mathbf{I}$].

The columns of identity matrix represent the coefficients of slack variables that have been added to the constraints. Each column of the identity matrix also represents a basic variable.

Assign values of the constants (b_i's) to the column variables in the identity matrix [because $\mathbf{x_B} = \mathbf{B}^{-1}\,\mathbf{b} = \mathbf{I}\,\mathbf{b} = \mathbf{b}$].

The variables corresponding to the columns of the identity matrix are called *basic variables* and the remaining ones are called *non-basic variables*. In general, if an LP model has n variables and m ($< n$) constraints, then m variables would be basic and $n - m$ variables non-basic. In certain cases one or more than one basic variables may also have *zero* values. It one or more basic variables have zero value, then this situation is called *degeneracy* and will be discussed later.

The first row in Table indicates the coefficients c_j of variables in the objective function. These values represent the cost (or profit) per unit associated with a variable in the objective function and these are used to determine the variable to be entered into the basis matrix **B**.

The column '$\mathbf{c_B}$' lists the coefficients of the current basic variables in the objective function. These values are used to calculate the value of Z when one unit of any variable is brought into the solution. Column '$\mathbf{x_B}$' represents the values of the basic variables in the current basic solution.

Numbers, a_{ij} in the columns under each variable are also called *substitution rates* (or *exchange coefficients*) because these represent the rate at which resource i ($i = 1, 2, \ldots, m$) is consumed by each unit of an activity j ($j = 1, 2, \ldots, n$).

Continue..

The values z_j represent the amount by which the value of objective function Z would be decreased (or increased) if one unit of the given variable is added to the new solution. Each of the values in the $c_j - z_j$ row represents the net amount of increase (or decrease) in the objective function that would occur when one unit of the variable represented by the column head is introduced into the solution. That is:

$$c_j - z_j \text{ (net effect)} = c_j \text{ (incoming unit profit/cost)} - z_j \text{ (outgoing total profit/cost)}$$

where z_j = Coefficient of basic variables column × Exchange coefficient column j

Step 3: Test for optimality Calculate the $c_j - z_j$ value for all non-basic variables. To obtain the value of z_j multiply each element under *'Variables'* column (columns, $\mathbf{a_j}$ of the coefficient matrix) with the corresponding elements in $\mathbf{c_B}$-column. Examine values of $c_j - z_j$. The following three cases may arise:

(i) If all $c_j - z_j \leq 0$, then the basic feasible solution is optimal.

(ii) If at least one column of the coefficients matrix (i.e. $\mathbf{a}_k$) for which $c_k - z_k > 0$ and all other elements are negative (i.e. $a_{ik} < 0$), then there exists an unbounded solution to the given problem.

(iii) If at least one $c_j - z_j > 0$, and each of these columns have at least one positive element (i.e. a_{ij}) for some row, then this indicates that an improvement in the value of objective function Z is possible.

Continue..

Step 4: Select the variable to enter the basis If Case (iii) of Step 3 holds, then select a variable that has the largest $c_j - z_j$ value to enter into the new solution. That is,

$$c_k - z_k = \text{Max } \{(c_j - z_j); \quad c_j - z_j > 0\}$$

The column to be entered is called the *key* or *pivot* column. Obviously, such a variable indicates the largest per unit improvement in the current solution.

Step 5: Test for feasibility (variable to leave the basis) After identifying the variable to become the basic variable, the variable to be removed from the existing set of basic variables is determined. For this, each number in $\mathbf{x}_\mathbf{B}$-column (i.e. b_i values) is divided by the corresponding (but positive) number in the key column and a row is selected for which this ratio is non-negative and minimum. This ratio is called the *replacement* (*exchange*) *ratio*. That is,

$$\frac{x_{Br}}{a_{rj}} = \text{Min}\left\{\frac{x_{Bi}}{a_{rj}};\ a_{rj} > 0\right\}$$

Continue..

This ratio restricts the number of units of the incoming variable that can be obtained from the exchange. *It may be noted that the division by a negative or by a zero element in key column is not permitted.*

The row selected in this manner is called the *key* or *pivot row* and it represents the variable which will leave the solution.

The element that lies at the intersection of the key row and key column of the simplex table is called *key* or *pivot* element.

Step 6: Finding the new solution

(i) If the key element is 1, then the row remains the same in the new simplex table.

(ii) If the key element is other than 1, then divide each element in the key row (including the elements in $\mathbf{x_B}$-column) by the key element, to find the new values for that row.

(iii) The new values of the elements in the remaining rows of the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero. In other words, for each row other than the key row, we use the formula:

$$\begin{array}{l}\text{Number in}\\ \text{new row}\end{array} = \begin{pmatrix}\text{Number in}\\ \text{old row}\end{pmatrix} \pm \left[\begin{pmatrix}\text{Number above or below}\\ \text{key element}\end{pmatrix} \times \begin{pmatrix}\text{Corresponding number in}\\ \text{the new row, that is row}\\ \text{replaced in Step 6 (ii)}\end{pmatrix}\right]$$

The new entries in $\mathbf{c_B}$ and $\mathbf{x_B}$ columns are updated in the new simplex table of the current solution.

Step 7: Repeat the procedure Go to Step 3 and repeat the procedure until all entries in the $c_j - z_j$ row are either negative or zero.

Continue..

Example 4.1 Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to the constraints

(i) $2x_1 + 3x_2 \leq 8$, (ii) $2x_2 + 5x_3 \leq 10$, (iii) $3x_1 + 2x_2 + 4x_3 \leq 15$

and $x_1, x_2, x_3 \geq 0$

		$c_j \rightarrow$	3	5	4	0	0	0	
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	*Min Ratio* x_B/x_2
0	s_1	8	2	3	0	1	0	0	8/3 $\rightarrow$
0	s_2	10	0	2	5	0	1	0	10/2
0	s_3	15	3	2	4	0	0	1	15/2
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	3	5 $\uparrow$	4	0	0	0	

Continue..

		$c_j \rightarrow$	3	5	4	0	0	0	
Basics Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	*Min Ratio* x_B/x_3
5	x_2	8/3	2/3	1	0	1/3	0	0	–
0	s_2	14/3	– 4/3	0	5	– 2/3	1	0	(14/3)/5 →
0	s_3	29/3	5/3	0	4	– 2/3	0	1	(29/3)/4
$Z = 40/3$		z_j	10/3	5	0	5/3	0	0	
		$c_j - z_j$	– 1/3	0	4	– 5/3	0	0	
					↑				

The improved basic feasible solution can be read from Table 4.4 as: $x_2 = 8/3$, $s_2 = 14/3$, $s_3 = 29/3$ and $x_1 = x_3 = s_1 = 0$. The improved value of the objective function becomes:

$$Z = (\text{Basic variable coefficients, } \mathbf{c_B}) \times (\text{Basic variable values, } \mathbf{x_B})$$

$$= 5\ (8/3) + 0\ (14/3) + 0\ (29/3) = 40/3$$

Once again, calculate values of $c_j - z_j$ to check whether the solution shown in Table 4.4 is optimal or not. Since $c_3 - z_3 > 0$, the current solution is not optimal.

Continue..

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value b (= x_B)	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio x_B/x_1
		$c_j \rightarrow$	3	5	4	0	0	0	
5	x_2	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3) = 4
4	x_3	14/15	– 4/15	0	1	– 2/15	1/5	0	–
0	s_3	89/15	41/15	0	0	2/15	– 4/5	1	(89/15)/(41/15) = 2.17 $\rightarrow$
Z = 256/15		z_j	34/15	5	4	17/15	4/5	0	
		$c_j - z_j$	11/15	0	0	– 17/15	– 4/5	0	
			$\uparrow$						

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value b (= x_B)	x_1	x_2	x_3	s_1	s_2	s_3
		$c_j \rightarrow$	3	5	4	0	0	0
5	x_2	50/41	0	1	0	15/41	8/41	– 10/41
4	x_3	62/41	0	0	1	– 6/41	5/41	4/41
3	x_1	89/41	1	0	0	– 2/41	– 12/41	15/41
Z = 765/41		z_j	3	5	4	45/41	24/41	11/41
		$c_j - z_j$	0	0	0	– 45/41	– 24/41	– 11/41

In Table 4.6, all $c_j - z_j < 0$ for non-basic variables. Therefore, the optimal solution is reached with, $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$ and the optimal value of $Z = 765/41$.

Simplex Method (Minimization case)

In certain cases, it is difficult to obtain an initial basic feasible solution of the given LP problem. Such cases arise

(i) when the constraints are of the $\leq$ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq b_i , \quad x_j \geq 0$$

and value of few right-hand side constants is negative [i.e. $b_i < 0$]. After adding the non-negative slack variable s_i ($i = 1, 2, \ldots, m$), the initial solution so obtained will be $s_i = -b_i$ for a particular resource, i. This solution is not feasible because it does not satisfy non-negativity conditions of slack variables (i.e. $s_i \geq 0$).

(ii) when the constraints are of the $\geq$ type,

$$\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i , \quad x_j \geq 0$$

Continue..

After adding surplus (negative slack) variable s_i, the initial solution so obtained will be $-s_i = b_i$ or $s_i = -b_i$

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i, \quad x_j \geq 0, s_i \geq 0$$

This solution is not feasible because it does not satisfy non-negativity conditions of surplus variables (i.e. $s_i \geq 0$). In such a case, artificial variables, A_i ($i = 1, 2, \ldots, m$) are added to get an initial basic feasible solution. The resulting system of equations then becomes:

$$\sum_{j=1}^{n} a_{ij} x_j - s_i + A_i = b_i$$

$$x_j, s_i, A_i \geq 0, \quad i = 1, 2, \ldots, m$$

These are m simultaneous equations with $(n + m + m)$ variables (n decision variables, m artificial variables and m surplus variables). An initial basic feasible solution of LP problem with such constraints can be obtained by equating $(n + 2m - m) = (n + m)$ variables equal to zero. Thus the new solution to the given LP problem is: $A_i = b_i$ ($i = 1, 2, \ldots, m$), which is not the solution to the original system of equations because the two systems of equations are not equivalent. Thus, to get back to the original problem, artificial variables must be removed from the optimal solution. There are two methods for removing artificial variables from the solution.

- *Two-Phase Method*
- *Big-M Method or Method of Penalties*

Question

Example Use two-phase simplex method to solve the following LP problem:

Minimize $Z = x_1 + x_2$

subject to the constraints

(i) $2x_1 + x_2 \geq 4$, (ii) $x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0$

Solution Converting the given LP problem objective function into the maximization form and then adding surplus variables s_1 and s_2 and artificial variables A_1 and A_2 in the constraints, the problem becomes:

Maximize $Z^* = -x_1 - x_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I: This phase starts by considering the following auxiliary LP problem:

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

(i) $2x_1 + x_2 - s_1 + A_1 = 4$, (ii) $x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Continue…

		$c_j \rightarrow$	0	0	0	0	−1	−1	
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	
−1	A_1	4	2	1	−1	0	1	0	
−1	A_2	7	1	7	0	−1	0	1	→
$Z^* = -11$		z_j	−3	−8	1	1	−1	−1	
		$c_j - z_j$	3	8	−1	−1	0	0	
				↑					

		$c_j \rightarrow$	0	0	0	0	−1	−1	
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b(= x_B)$	x_1	x_2	s_1	s_2	A_1	A_2*	
−1	A_1	3	13/7	0	−1	1/7	1	−1/7	→
0	x_2	1	1/7	1	0	−1/7	0	1/7	
$Z^* = -3$		z_j	−13/7	0	1	−1/7	−1	1/7	
		$c_j - z_j$	13/7	0	−1	1/7	0	−8/7	
						↑			

Continue…

		$c_j \rightarrow$	0	0	0	0	−1	−1
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	A_1^*	A_2^*
0	s_2	21	13	0	−7	1	7	−1
0	x_2	4	2	1	−1	0	1	0
$Z^* = 0$		z_j	0	0	0	0	0	0
		$c_j - z_j$	0	0	0	0	−1	−1

		$c_j \rightarrow$	−1	−1	0	0	
Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value $b(=x_B)$	x_1	x_2	s_1	s_2	Min Ratio x_B/x_1
0	s_2	21	13	0	−7	1	21/13 →
−1	x_2	4	2	1	−1	0	4/2
$Z^* = -4$		z_j	−2	−1	1	0	
		$c_j - z_j$	1	0	−1	0	
			↑				

Continue..

LP problem is: $x_1 = 21/13$, $x_2 = 10/13$ and Max $Z^* = -31/13$ or Min $Z = 31/13$.

		$c_j \rightarrow$	-1	-1	0	0
Basic Variables Coefficient c_B	*Basic Variables* B	*Basic Variables Value* $b (= x_B)$	x_1	x_2	s_1	s_2
-1	x_1	21/13	1	0	$-7/13$	1/13
-1	x_2	10/13	0	1	1/13	$-2/13$
$Z^* = -31/13$		z_j	-1	-1	6/13	1/13
		$c_j - z_j$	0	0	$-6/13$	$-1/13$

Dual linear Programming

In general, the primal-dual relationship between a pair of LP problems can be expressed as follows:

Primal	*Dual*
Max $Z_x = \sum_{j=1}^{n} c_j x_j$	Min $Z_y = \sum_{i=1}^{m} b_i y_i$
subject to the constraints	subject to the constraints
$\sum_{j=1}^{n} a_{ij} x_j \le b_i \, ; \; i = 1, 2, \ldots, m$	$\sum_{i=1}^{m} a_{ji} y_i \ge c_j \, ; \; j = 1, 2, \ldots, n$
$a_{ij} = a_{ji}$	
and $x_j \ge 0; \quad j = 1, 2, \ldots, n$	and $y_i \ge 0; \quad i = 1, 2, \ldots, m$

A summary of the general relationships between primal and dual LP problems is given in Table

If Primal	*Then Dual*
(i) Objective is to maximize	(i) Objective is to minimize
(ii) jth primal variable, x_j	(ii) jth dual constraint
(iii) ith primal constraint	(iii) ith dual variable, y_i
(iv) Primal variable x_j unrestricted in sign	(iv) Dual constraint j is = type
(v) Primal constraint i is = type	(v) Dual variable y_i is unrestricted in sign
(vi) Primal constraints $\le$ type	(vi) Dual constraints $\ge$ type

Ques1-

Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \geq 7$, (ii) $6x_1 + x_2 + 3x_3 \geq 4$, (iii) $7x_1 - 2x_2 - x_3 \leq 10$

(iv) $x_1 - 2x_2 + 5x_3 \geq 3$, (v) $4x_1 + 7x_2 - 2x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

Sol- Minimize $Z_x = 3x_1 - 2x_2 + 4x_3$

subject to the constraints

(i) $3x_1 + 5x_2 + 4x_3 \geq 7$, (ii) $6x_1 + x_2 + 3x_3 \geq 4$, (iii) $-7x_1 + 2x_2 + x_3 \geq -10$

(iv) $x_1 - 2x_2 + 5x_3 \geq 3$, (v) $4x_1 + 7x_2 - 2x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

If y_1, y_2, y_3, y_4 and y_5 are dual variables corresponding to the five primal constraints in the given order, the dual of this primal LP problem is stated as:

Maximize $Z_y = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$

subject to the constraints

(i) $3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$, (ii) $5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$

(iii) $4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$

and $y_1, y_2, y_3, y_4, y_5 \geq 0$

Ques2-

Minimize $Z = x_1 + 2x_2$

subject to the constraints

(i) $2x_1 + 4x_2 \leq 160$, (ii) $x_1 - x_2 = 30$, (iii) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Solution Since the objective function of the primal LP problem is of minimization, change all $\leq$ type constraints to $\geq$ type constraints by multiplying the constraint on both sides by -1. Also write $=$ type constraint equivalent to two constraints of the type $\geq$ and $\leq$. Then the given primal LP problem can be rewritten as:

Minimize $Z_x = x_1 + 2x_2$

subject to the constraint

(i) $-2x_1 - 4x_2 \geq -160$, (ii) $x_1 - x_2 \geq 30$

(iii) $x_1 - x_2 \leq 30$ or $-x_1 + x_2 \geq -30$, (iv) $x_1 \geq 10$

and $x_1, x_2 \geq 0$

Let y_1, y_2, y_3 and y_4 be the dual variables corresponding to the four constraints in the given order. The dual of the given primal LP problem can then be formulated as follows:

Continue…

Maximize $Z_y = -160y_1 + 30y_2 - 30y_3 + 10y_4$

subject to the constraints

(i) $-2y_1 + y_2 - y_3 + y_4 \leq 1$, (ii) $-4y_1 - y_2 + y_3 \leq 2$

and $y_1, y_2, y_3, y_4 \geq 0$

Let $y = y_2 - y_3$ ($y_2, y_3 \geq 0$). The above dual problem then reduces to the form

Maximize $Z_y = -160y_1 + 30y + 10y_4$

subject to the constraints

(i) $-2y_1 + y + y_4 \leq 1$, (ii) $-4y_1 - y \leq 2$

and $y_1, y_4 \geq 0$; y unrestricted in sign

Remark Since second constraint in the primal LP problem is equality, therefore as per rule 6 corresponding second dual variable $y\ (= y_2 - y_3)$ should be unrestricted in sign.

Dual Simplex Method

A firm manufactures two products A and B on machines I and II as shown below:

Machine	*Product*		*Available Hours*
	A	*B*	
I	30	20	300
II	5	10	110
Profit per unit (Rs)	6	8	

The total time available is 300 hours and 110 hours on machines I and II, respectively. Products A and B contribute a profit of Rs 6 and Rs 8 per unit, respectively. Determine the optimum product mix. Write the dual of this LP problem and give its economic interpretation.

Mathematical formulation The primal and the dual LP problems of the given problem are

Primal problem

x_1 and x_2 = number of units of A and B to be produced, respectively

$$\text{Max } Z_x = 6x_1 + 8x_2$$

subject to the constraints

$$30x_1 + 20x_2 \le 300$$
$$5x_1 + 10x_2 \le 110$$

and $x_1, x_2 \ge 0$

Dual problem

y_1 and y_2 = cost of one hour on machines I and II, respectively

$$\text{Min } Z_y = 300y_1 + 110y_2$$

subject to the constraints

$$30y_1 + 5y_2 \ge 6$$
$$20y_1 + 10y_2 \ge 8$$

and $y_1, y_2 \ge 0$

Continue..

Solution of the primal problem The optimal solution of the primal problem is given in Table

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value x_B (= b)	x_1	x_2	s_{1p}	s_{2p}
		$c_j \rightarrow$	6	8	0	0
6	x_1	4	1	0	1/20	– 1/10
8	x_2	9	0	1	– 1/10	3/20
Z = 96		$c_j - z_j$	0	0	– 1/10	– 6/10

Table 5.2 indicates that the optimal solution is to produce: x_1 = 4 units of product A; x_2 = 9 units of product B and Z_x = total maximum profit, Rs 96.

Solution of the dual problem The optimal solution of the dual problem can be obtained by applying the Simplex method. The optimal solution is shown in Table

Basic Variables Coefficient c_B	Basic Variables B	Basic Variables Value y_B	y_1	y_2	s_{1d}	s_{2d}
		$b_i \rightarrow$	300	110	0	0
300	y_1	1/10	1	0	– 1/20	1/40
110	y_2	6/10	0	1	1/10	– 3/20
Z = 96		$b_i - z_j$	0	0	4	9

The optimal solution as given in Table 5.3 is: y_1 = Rs 1/10 per hour on machine I; y_2 = Rs 6/10 per hour on machine II and Z_y = total minimum cost, Rs 96.

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Theorem : The dual of the dual is the primal.

Proof: Consider the primal problem in canonical form:

Minimize $Z_x = \mathbf{cx}$

subject to $\quad \mathbf{Ax} \geq \mathbf{b};\ \ \mathbf{x} \geq 0 \qquad (1)$

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}^T \in E^m$ and $\boldsymbol{c}$, $\boldsymbol{x}^T \in E^n$.

Applying the transformation rules, the dual of this problem is:

Maximize $Z_y = \mathbf{b}^T\mathbf{y}$

subject to $\quad \mathbf{A}^T\mathbf{y} \leq \boldsymbol{c}^T;\ \ \mathbf{y} \geq 0 \qquad (2)$

This dual problem can also be written as:

Minimize $Z_y^* = (-\mathbf{b})^T\mathbf{y} \qquad (3)$

subject to $\quad (-\mathbf{A}^T)\mathbf{y} \geq (-\mathbf{c}^T);\ \ \mathbf{y} \geq 0,$ where $Z_y^* = -Z_y$

If we consider LP problem (3) as primal, then its dual can be constructed by considering $\mathbf{x}$ as the dual variable. Thus, we have

Maximize $Z_x^* = (-\mathbf{c}^T)^T\mathbf{x} = (-\mathbf{c})\mathbf{x}$

subject to $\quad (-\mathbf{A}^T)^T\mathbf{x} \leq (-\mathbf{b}^T)^T$ or $(-\mathbf{A})\mathbf{x} \leq (-\mathbf{b})$

and $\quad \mathbf{x} \geq 0 \qquad (4)$

But LP problem (4) is identical to the given primal LP problem (1). This completes the proof of the theorem.

SPECIAL CASES IN LINEAR PROGRAMMING

- **Alternative (or Multiple) Optimal Solutions**

There are two conditions that should be satisfied for an alternative optimal solution to exist:

(i) The slope of the objective function should be the same as that of the constraint forming the boundary of the feasible solutions region, and

(ii) The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint should be an active constraint.

Remark The constraint is said to be *active* (*binding* or *tight*), if at the point of optimality, the left-hand side of a constraint equals the right-hand side. In other words, an equality constraint is always active, and inequality constraint may or may not be active.

Geometrically, an *active* constraint is one that passes through one of the extreme points of the feasible solution space.

Example 3.20 Use the graphical method to solve the following LP problem.

Maximize $Z = 10x_1 + 6x_2$

subject to the constraints

(i) $5x_1 + 3x_2 \leq 30$, (ii) $x_1 + 2x_2 \leq 18$

and $x_1, x_2 \geq 0$.

Solution The constraints are plotted on a graph by first treating these as equations and then their inequality signs are used to identify feasible region (shaded area) as shown in Fig. . The extreme points of the region are O, A, B and C.

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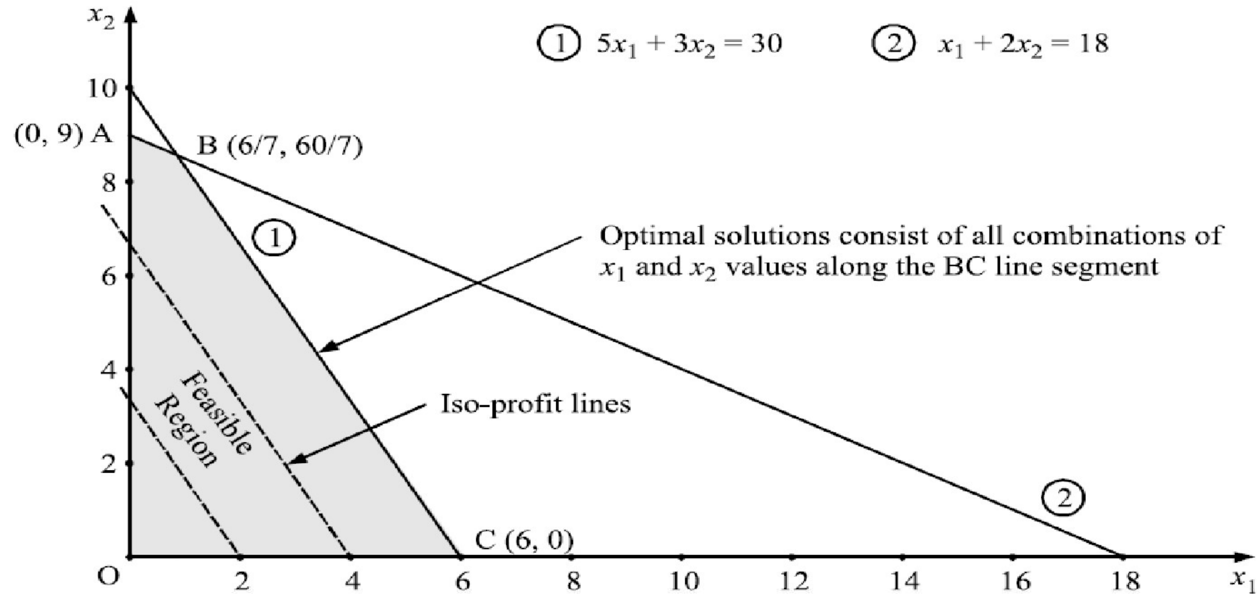

Since objective function (iso-profit line) is parallel to the line BC (first constraint : $5x_1 + 3x_2 = 30$), which also falls on the boundary of the feasible region. Thus, as the iso-profit line moves away from the origin, it coincides with the line BC of the constraint equation line that falls on the boundary of the feasible region. This implies that an optimal solution of LP problem can be obtained at any point between B and C including extreme points B and C on the same line. Therefore, several combinations of values of x_1 and x_2 give the same value of objective function.

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The value of variables x_1 and x_2 obtained at extreme points B and C should only be considered to establish that the solution to an LP problem will always lie at an extreme point of the feasible region.

The value of objective function at each of the extreme points is shown in Table 3.19.

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 10x_1 + 6x_2$
O	(0, 0)	10(0) + 6(0) = 0
A	(0, 9)	10(0) + 6(9) = 54
B	(6/7, 60/7)	10(6/7) + 6(60/7) = 60
C	(6, 0)	10(6) + 6(0) = 60

Since value (maximum) of objective function, Z = 60 at two different extreme points B and C is same, therefore two alternative solutions: $x_1 = 6/7$, $x_2 = 60/7$ and $x_1 = 6$, $x_2 = 0$ exist.

Remark If slope of a constraint is parallel to the slope of the objective function, but it does not fall on the boundary of the feasible region, the multiple solutions will not exist. This type of a constraint is called *redundant constraint* (a constraint whose removal does not change the feasible region.)

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- **Unbounded Solution-** Unbounded solution exists when the value of a decision variables can be made infinitely large without violating any of the LP problem

Example Use the graphical method to solve the following LP problem:

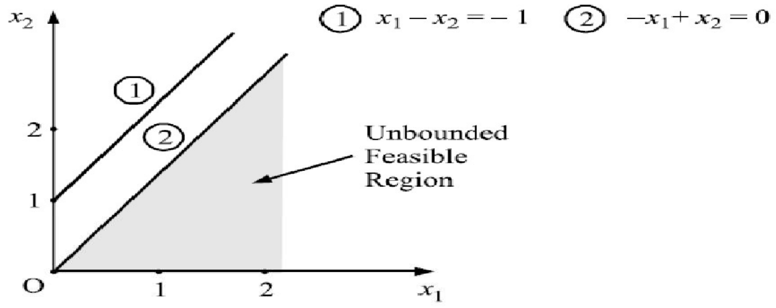
Maximize $Z = 3x_1 + 4x_2$

subject to the constraints

(i) $x_1 - x_2 = -1$ (ii) $-x_1 + x_2 \leq 0$

and $x_1, x_2 \geq 0.$

Solution Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region (shaded area) as shown in Fig.



It may be noted from Fig. that there exist an infinite number of points in the convex region for which the value of the objective function increases as we move from the extreme point (origin), to the right. That is, the value of variables x_1 and x_2 can be made arbitrarily large and accordingly the value of objective function Z will also increase. Thus, the LP problem has an unbounded solution.

Continue..

- **Infeasible Solution-**

Example Use the graphical method to solve the following LP problem:

$$\text{Maximize } Z = x_1 + \frac{x_2}{2}$$

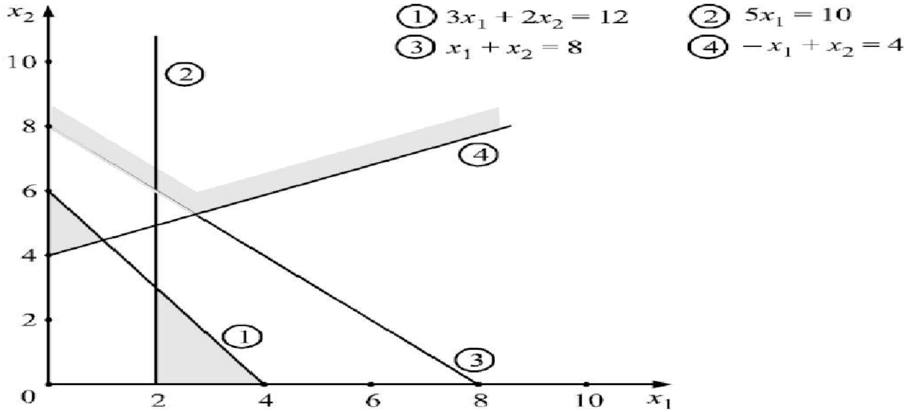
subject to the constraints

(i) $3x_1 + 2x_2 \leq 12$ (ii) $5x_1 = 10$

(iii) $x_1 + x_2 \geq 8$ (iv) $-x_1 + x_2 \geq 4$

and $x_1, x_2 \geq 0$

Solution The constraints are plotted on graph as usual and feasible regions are shaded as shown in Fig. . The three shaded areas indicate non-overlapping regions. All of these can be considered feasible solution space because they all satisfy some subsets of the constraints.



However, there is no unique point (x_1, x_2) in these shaded regions that can satisfy all the constraints simultaneously. Thus, the LP problem has an infeasible solution.

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- **Redundancy-** Redundancy is a situation in which one or more constraints do not affect the feasible solution region. In the given Fig. equation 4 is the redundant because it does not affect the feasible region.