A beam is a structural member whose longitudinal dimension is large compared to the transverse dimension. The beam is supported along its length and is acted upon by a system of loads at right angles to its axis. Due to external loads and couples, shear force and bending moment develop at any section of the beams. For the design of beams, information about the shear force and bending moment is desired. Accordingly, it is appropriate to learn about the variation of shear force and bending moment along the length of the beam.

11.1. SHEAR FORCE AND BENDING MOMENT

Consider a simply supported beam acted upon by different point loads as shown in Fig. 11.1.

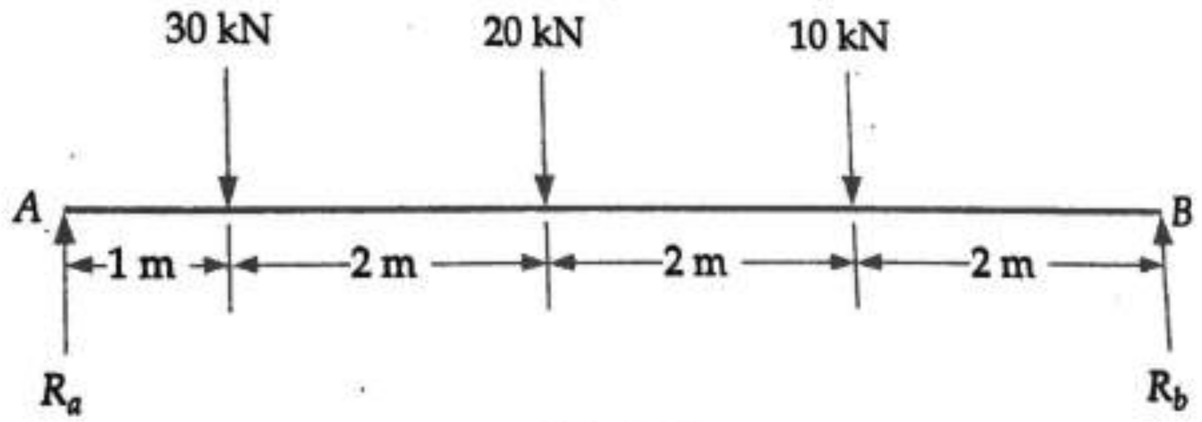


Fig. 11.1

The loads are transferred to supports and for equilibrium conditions

$$R_a + R_b = W_1 + W_2 + W_3 = 30 + 20 + 20 = 60 \text{ kN}$$

Also ΣM_a (moments about support A) = 0. That gives

$$R_b \times 7 = 10 \times 5 + 20 \times 3 + 30 \times 1 = 140$$

$$\therefore \quad R_b = \frac{140}{7} = 20 \text{ kN and } R_a = 60 - 20 = 40 \text{ kN}$$

Imagine the beam to be cut at an arbitrary section xx at distance $x = 4$ m from the end A, and draw separately the free body diagrams of the two portions (Fig. 11.2).

- Considering equilibrium of forces on each portion of the beam, the net resultant vertical forces are:

$$F_{left} = 30 + 20 - 40 = 10 \text{ kN} \quad \text{(downward)}$$
$$F_{right} = 10 - 20 = -10 \text{ kN} \quad \text{(upward)}$$

It is to be noted that forces on the left and right sides of the section xx are equal in magnitude but opposite in direction.

Obviously, section xx is subjected to a force of 10 kN which is trying to shear the beam. The force of 10 kN is called as shear force at section xx.

"Shear force at a section in a beam is the force that is trying to shear off the section. It is obtained as algebraic sum of all the forces acting normal to the axis of beam ; either to the left or to the right of the section."

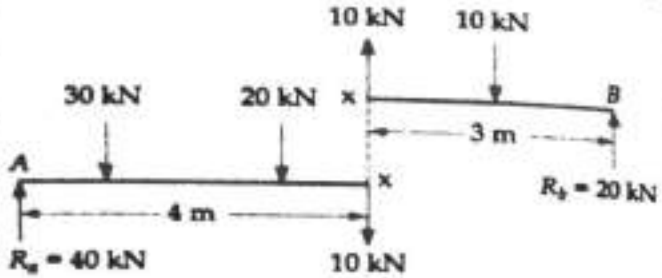


Fig. 11.2

- Considering equilibrium of moments on each portion of the beam,

$$M_{left} = 40 \times 4 - 30 \times 3 - 20 \times 1 = 50 \text{ kNm} \quad \text{(clockwise)}$$
$$M_{right} = 20 \times 3 - 10 \times 1 = 50 \text{ kNm} \quad \text{(anti-clockwise)}$$

It is to be noted that moments on the left and right sides of the section xx are equal in magnitude but opposite in direction. Obviously section xx is acted upon by a moment of 50 kNm. Since this moment is trying to bend the beam, it is called bending moment.

"Bending moment at a section in a beam is the moment that tends to bend the beam and is obtained as algebraic sum of moment of all the forces about the section, acting either to the left or to the right of the section."

Sign Conversion: The following sign conventions are normally adopted for the shear force and bending moment.

(i) Shear force is taken positive if it tends to move the left portion upward with respect to the right portion.

(ii) Bending moment is taken positive if it tends to sag (concave upward) the beam, and it is taken negative if it tends to hog (concave down) the beam.

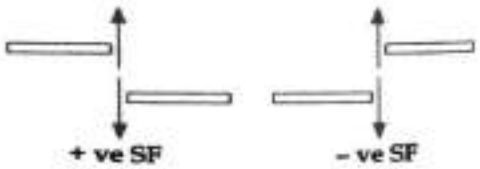


Fig. 11.3

The shear force and bending moment vary along the length of the beam and this variation is represented graphically. The plots are known as shear force and bending moment diagrams. In these diagrams, the abscissa indicates the position of section along the beam, and the ordinate represents the value of SF and BM respectively. These plots help to determine the maximum value of each of these quantities.

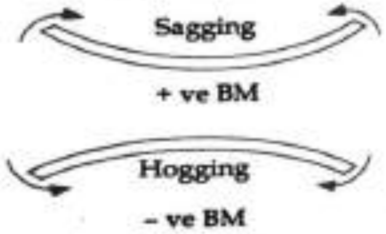


Fig. 11.4

11.2. TYPES OF BEAMS AND LOADS

The shear force and the bending moment that develop at any cross-section of the beam depend upon the types of beams and the types of loads acting on them.

Beam are generally classified as:

- Cantilever beam (Fig. 11.5a): A beam having its one end fixed or built-in and the other end free to deflect. There is no deflection or rotation at the fixed end.
- Fixed beam (Fig. 11.5b): A beam having both of its ends fixed or built-in.
- Simply supported beam (Fig. 11.5c): A beam made to freely rest on supports which may be knife edges or rollers. The term 'freely supported' implies that these supports exert only forces but no moments on the beam. The horizontal distance between the supports is called span.
- Overhanging beam (Fig. 11.5d): A beam having one or both ends extended over the supports. The end portion or portions extend in the form of cantilever beyond the support/supports.
- Continuous beam (Fig. 11.5e): A beam provided with more than two supports. Further such a beam may or may not have overhang.

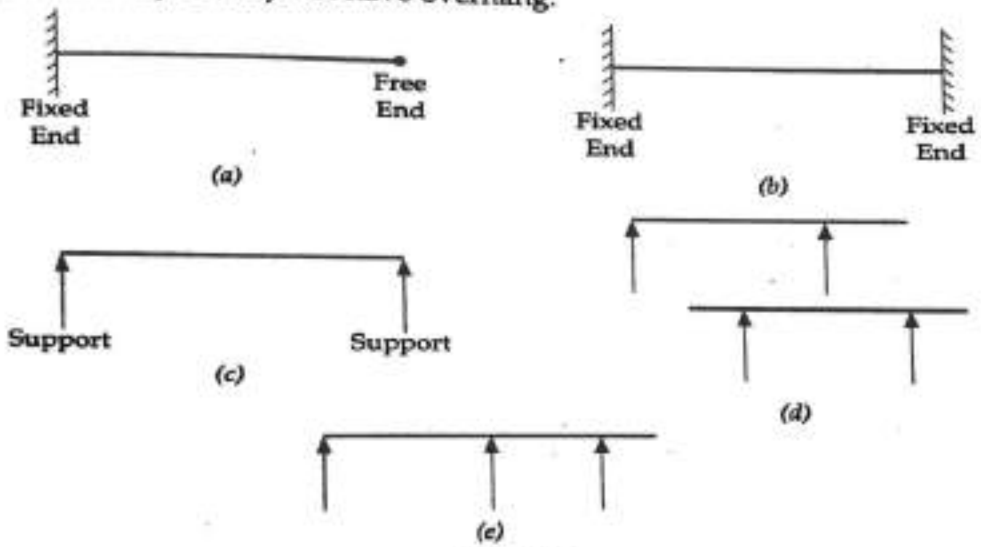


Fig. 11.5

The different types of loads acting on a beam are:

- Concentrated load: The load acts at a point on the beam. This point load is applied through a knife edge.
- Uniformly distributed load: The load is evenly distributed over a part or the entire length of the beam. The total udl is assumed to act at the centre of gravity of the load. The udl is expressed as N/m length of beam.
- Uniformly varying load: The load whose intensity varies linearly along the length of beam over which it is applied.
- A beam may be loaded by a couple whose magnitude is expressed as Nm.

A beam may carry any one of the above load systems or combination of two or more loads at a time.

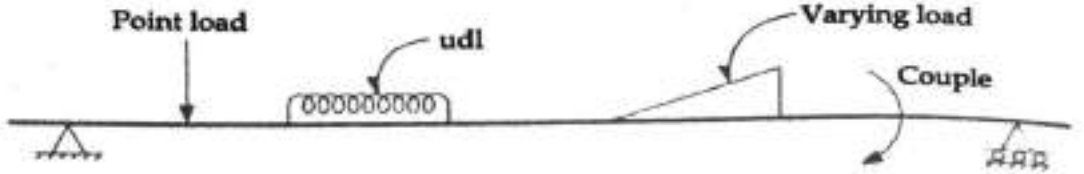


Fig. 11.6

11.3. RELATION BETWEEN LOAD INTENSITY, SF AND BM

Consider a beam subjected to any type of transverse load of the general form shown in Fig. 11.7. Isolate from the beam an element of length dx at a distance x from left end and draw its free body diagram as shown in Fig. 11.7. Since the element is of extremely small length, the loading over the beam can be considered to be uniform and equal to w kN/m. The element is subject to shear force F on its left hand side and shear force $(F + dF)$ on its right hand side. Further, the bending moment M acts on the left side of the element and it changes to $(M + dM)$ on the right side.

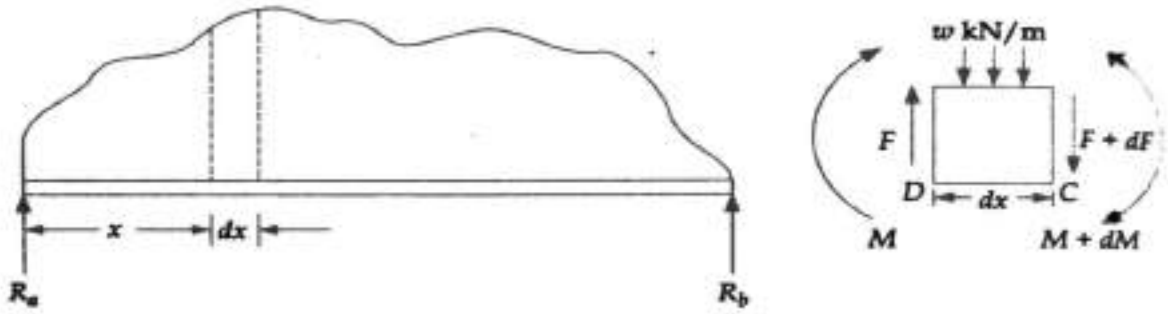


Fig. 11.7

Taking moments about point C on the right side,

$\Sigma M_c = 0$:
$$M - (M + dM) + F \times dx - (w \times dx) \times \frac{dx}{2} = 0$$

The udl is considered to be acting at its CG

$$dM = F\,dx - \frac{w\,(dx)^2}{2} = 0$$

The last term consists of the product of two differentials and can be neglected.

$$\therefore \qquad dM = F\,dx \text{ or } F = \frac{dM}{dx}$$

Thus the shear force is equal to the rate of change of bending moment with respect to x.

Applying the condition $\Sigma F_y = 0$ for equilibrium, we obtain

$$F - w\,dx - (F + dF) = 0$$

or
$$w = \frac{dF}{dx}$$

That is the intensity of loading is equal to rate of change of shear force with respect to x.

Draw the shear force and bending moment diagrams for the beam loaded and supported as shown in Fig. 11.12.

Solution: The line of action of the reaction will be at right angles to the roller base at end A. The reaction at a hinge can have two components acting in the horizontal and the vertical directions. Since there is no horizontal external force acting on the beam, the reaction at the hinged end B will be only in the vertical direction.

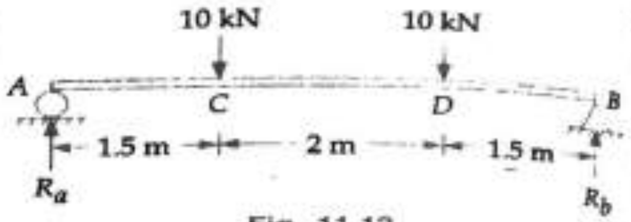


Fig. 11.12

Due to symmetry of loading,

$$R_a = R_b = \frac{10 + 10}{2} = 10 \text{ kN}$$

Shear force

SF at $A = 10$ kN
SF just on left of $C = 10$ kN
SF just on right of $C = 10 - 10 = 0$
SF just on left of $D = 0$
SF just on right of $D = 0 - 10 = -10$ kN
SF just on left of $B = -10$ kN
SF just on right of $B = -10 + 10 = 0$

Bending moment. Taking a section at distance x from end A and considering forces on left side.

Portion AC:

$$M = R_a \times x = 10x \qquad \text{(linear variation)}$$

At $x = 0$: $M_a = 0$
At $x = 1.5$ m : $M_c = 10 \times 1.5 = 15$ kNm

Portion CD:

$$M = R_a \times x - 10(x - 1.5)$$
$$= 10x - 10x + 15 = 15 \text{ kNm}$$

The bending moment remains constant at 15 kNm within the portion CD.

Portion DB:

$$M = R_a \times x - 10(x - 1.5) - 10(x - 3.5)$$
$$= 10x - 10x + 15 - 10x + 35$$
$$= -10x + 50 \qquad \text{(linear variation)}$$

At $x = 3.5$ m :

$$M_a = -10 \times 3.5 + 50 = 15 \text{ kNm}$$

At $x = 5$ m :

$$M_b = -10 \times 5 + 50 = 0$$

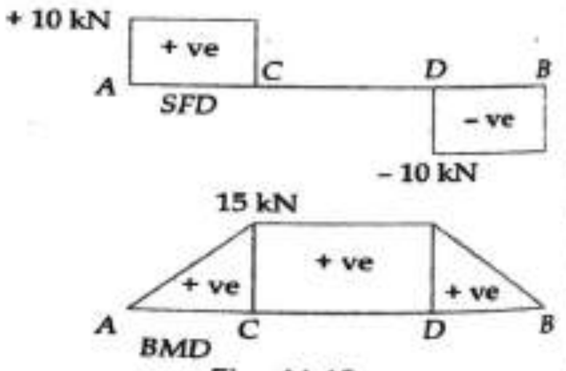


Fig. 11.13

The variation of shear force and bending moment for the entire length of the beam has been depicted in Fig. 11.13.

Construct the shear force and bending moment diagram for the cantilever beam loaded as shown in Fig. 11.14.

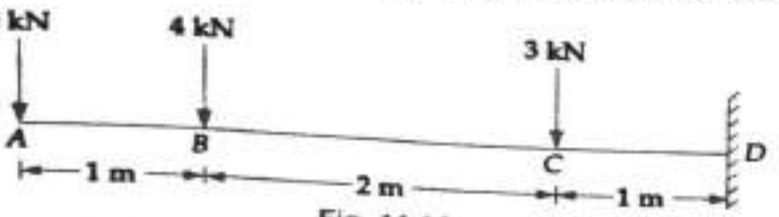


Fig. 11.14

Solution: For shear force calculations, consider any section at distance x from the free end A

At $x = 0$: $SF = -5$ kN

The shear force is being taken – ve because it tends to move the left portion downward with respect to the right portion.

At $x = 1$ m

just left of B : $SF = -5$ kN   just right of B : $SF = -5 - 4 = -9$ kN

At $x = 3$ m

just left of C : $SF = -9$ kN   just right of C : $SF = -9 - 3 = -12$ kN

Bending moment

Portion AB: Imagine a section between A and B, and at distance x from end A. Then

$M_x = -5x$ (linear variation)

At $x = 0$: $M_a = 0$

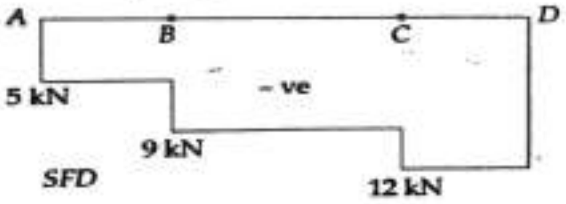
At $x = 1$ m : $M_b = -5 \times 1 = -5$ kNm

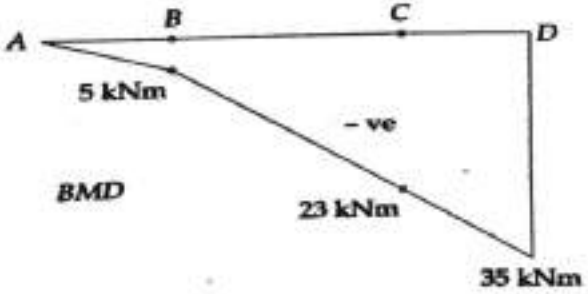
Portion BC: Consider the section to be between B and C, and at distance x from end A. Then

$M_x = -5x - 4(x - 1)$ (linear variation)

At $x = 1$ m : $M_b = -5 \times 1 - (1 - 1) = -5$ kNm as calculated above

At $x = 3$ m : $M_c = -5 \times 3 - 4(3 - 1) = -23$ kNm





Portion CD: Consider the section to be between C and D, and at distance x from end A. Then

$$M_x = -5x - 4(x-1) - 3(x-3) \qquad \text{(linear variation)}$$

At $x = 3$ m : $M_c = -5 \times 3 - 4(3-1) - 3(3-3) = -23$ kNm

At $x = 4$ m : $M_d = -5 \times 4 - 4(4-1) - 3(4-3) = -35$ kNm

The shear force and the bending moment for the entire beam are shown in Fig. 11.15.

EXAMPLE 11.3

Construct the shear force and bending moment diagrams for the cantilever beam loaded as shown in Fig. 11.16.

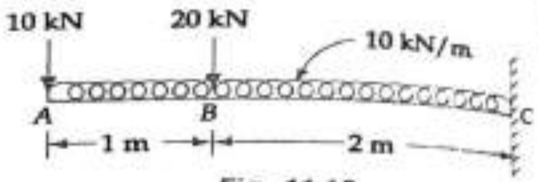


Fig. 11.16

Solution: For shear force calculations *for portion AB*, take section at distance x from end A.

$SF = -10 - 10x$ (linear variation)

At $x = 0$; $SF = -10$ kN

At $x = 1$ m (just to left of point B) ;

$SF = -10 - 10 = -20$ kN

For *portion BC*, again we consider a section at distance x from the end A,

$$SF = -10 - 20 - 10x \qquad \text{(linear variation)}$$

At $x = 1$ m (just to left of point B);

$SF = -10 - 20 - 10 = -40$ kN

At $x = 3$ m (fixed end) ;

$SF = -10 - 20 - 10 \times 3 = -60$ kN

The shear force diagram indicating the values of shear force at salient points is as shown in Fig. 11.17.

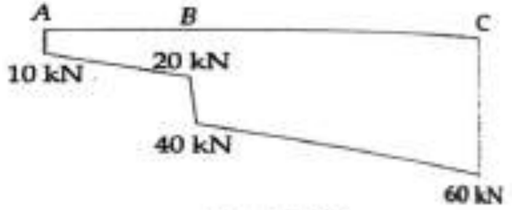


Fig. 11.17

(*b*) For bending moment for *portion AB*, take section at distance x from the free end A.

$$BM = -10x - 10x \times \frac{x}{2} \qquad \text{(parabolic variation)}$$

The *udl* is taken to be acting at its CG

At $x = 0$; $BM = 0$

At $x = 1$ m ; $BM = -10 \times 1 - 10 \times 1 \times \frac{1}{2} = -15$ kNm

For portion BC, again we consider a section at distance x from the end A

$$BM = -10x - 20(x-1) - 10x \times \frac{x}{2}$$

At $x = 1$ m ; $BM = -10 - 20(1-1) - 10 \times 1 \times \frac{1}{2} = -15$ kNm

At $x = 3$ m (fixed end) :

$$BM = -10 \times 3 - 20(3-1) - 10 \times 3 \times \frac{3}{2}$$
$$= -30 - 40 - 45 = -115 \text{ kNm}$$

The bending moment diagram indicating the value of bending moment at salient points is as shown in Fig. 11.18.

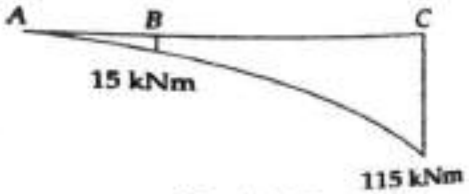


Fig. 11.18

EXAMPLE

Determine the reactions and construct the shear force and bending moment diagrams for the beam loaded as shown in Fig. 11.19. Also find the point of contraflexture, if any.

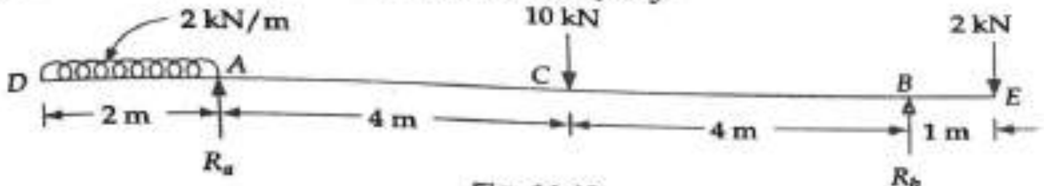


Fig. 11.19

Solution: A point of contraflexture is a point where bending moment is zero. From conditions of static equilibrium ($\Sigma V = 0$ and $\Sigma M = 0$), we have

$$R_a + R_b = 2 \times 2 + 10 + 2 = 16 \quad ...(i)$$

$$-2 \times 2 \times 10 + R_a \times 9 - 10 \times 5 + R_b \times 1 = 0 \,;\, 9R_a + R_b = 90 \quad ...(ii)$$

The *udl* is considered to be concentrated at its CG.

From expression (*i*) and (*ii*) : $R_a = 9.25$ kN and $R_b = 6.75$ kN

Shear Force:

At $D = 0$

Just left of $A = -2 \times 2 = -4$ kN ; Just right of $A = -4 + 9.25 = 5.25$ kN

Just left of $C = 5.25$ kN ; Just right of $C = 5.25 - 10 = -4.75$ kN

Just left of $B = -4.75$ kN ; Just right of $B = -4.75 + 6.75 = 2$ kN

Just left of $E = 2$ kN ; Just right of $E = 2 - 2 = 0$ kN

Bending moment

$$M_D = 0$$

At distance x from D (within portion DA)

$$M_x = -2x \times \frac{x}{2} = -x^2$$

$\therefore$ M (at $x = 1$m) $= 1$ and M (at $x = 2$m) $= -4$

$$M_A = -4 \text{ kNm}$$

$$M_C = -2 \times 2 \times 5 + 9.25 \times 4 = -20 + 37 = 17 \text{ kNm}$$

Apparently there is a point of contraflexture between A and C as bending moment changes sign between A and C.

Bending moment at x between A and C with x measured from D

$$M_x = -4(x-1) + 9.25(x-2) = 5.25x - 14.5$$

$\therefore$ $5.25x - 14.5 = 0$ for point of contraflexture

That gives $x = \dfrac{14.5}{5.25} = 2.76$ m

$M_B = -2 \times 1 = -2$ kNm (considering the segment EB from right hand side)

Since bending moment at C is + ve and at B is − ve, there is also a point of contraflexture between C and B.

Bending moment at distance x measured from end E towards left,

$$M_x = -2x + 6.75(x-1) = 4.75x - 6.75$$

$\therefore$ $4.75x - 6.75 = 0$ for point of contraflexture.

That gives $x = \dfrac{6.75}{4.75} = 1.42$ m

The shear force and bending moment diagrams for the entire beam are shown in Fig. 11.20 along with position of points of contraflexture.

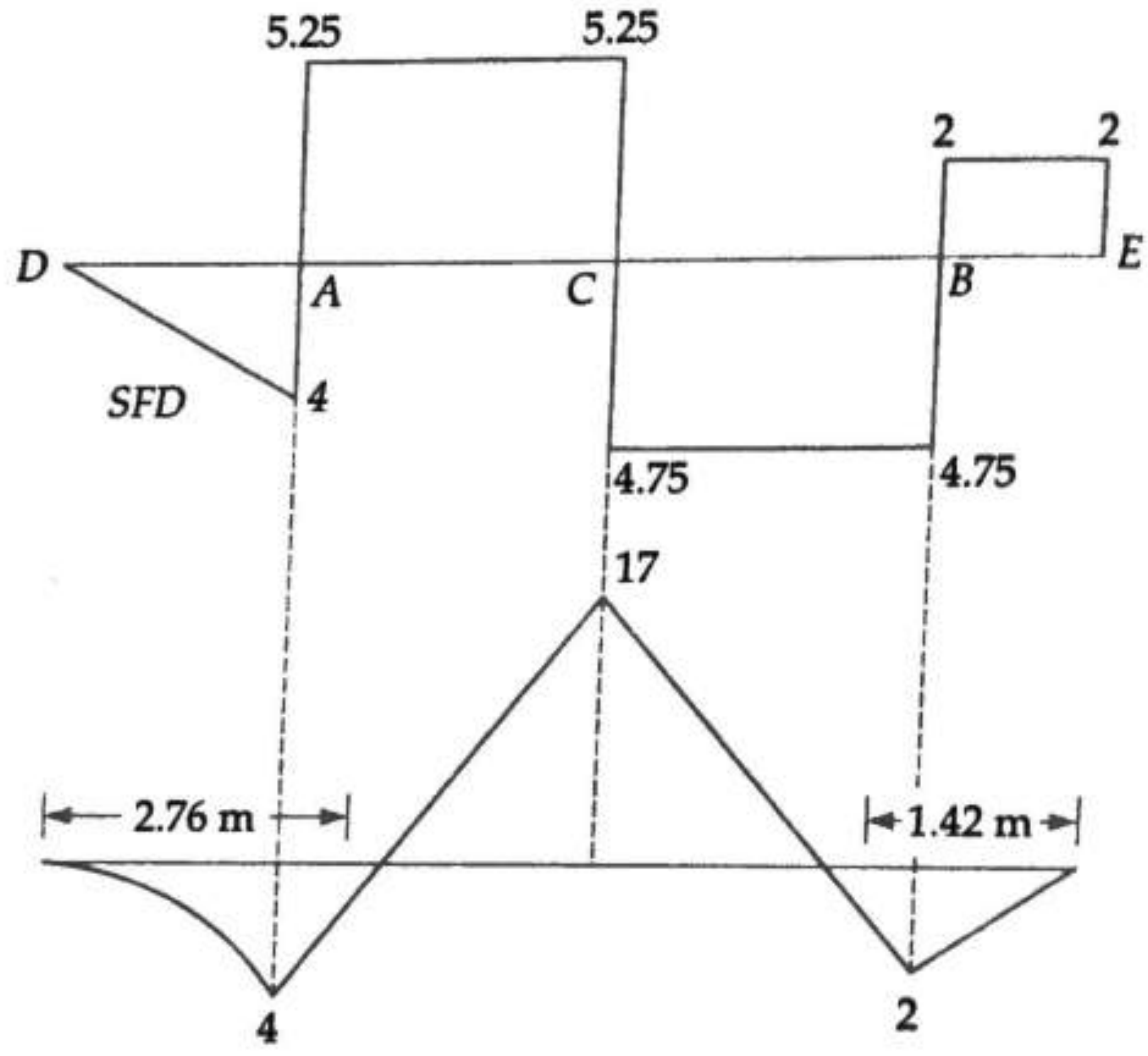


Fig. 11.20

A simply supported beam with 8 m span is loaded as shown in the figure given below:

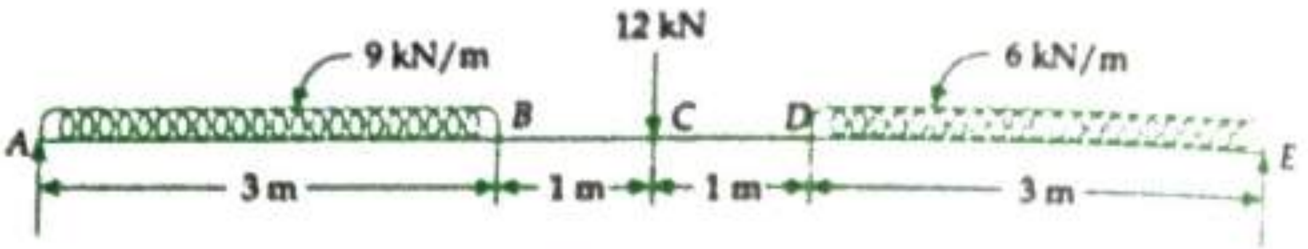


Fig. 11.30

Draw the shear force and bending moment diagrams. Also determine the magnitude and position of maximum bending moment on the beam.

Solution : Considering equilibrium of beam

$\Sigma F_y = 0$: $\quad R_a + R_e = (9 \times 3) + 12 + (6 \times 3) = 57$ kN

$\Sigma M = 0$: Taking moments about end point A (clockwise moments +ve)

$$27 \times 1.5 + 12 \times 4 + 18 \times 6.5 - R_e \times 8 = 0$$

The *udl* is considered to be concentrated at *CG*.

$$R_e = \frac{27\times1.5+12\times4+18\times6.5}{8}$$

$$= \frac{40.5+48+117}{8} = 25.69 \text{ kN}$$

and $\quad R_a = 57 - 25.69 = 31.31$ kN

Shear Force:

Portion AB : Consider any section at distance x from end support A

$SF = 31.31 - 9x$ (linear variation)

At point A, $\quad x = 0$ and $SF = 31.31$ kN

At point B, $\quad x = 3$ m and $SF = 31.31 - 9 \times 3 = 4.31$ kN

Portion BCD : The shear force remains constant at 4.32 kN between B and just left of C.

Just right of CD: $SF = 4.31 - 12 = 7.69$ kN

The shear force remains constant at 7.69 kN between and just left of D.

Portion DE : Consider any section between DE and at distance x from end support A.

$$SF = 31.31 - 9 \times 3 - 12 - (x - 5) \times 6$$
$$= 22.31 - 6x$$

At point D, $x = 5$ m

and $SF = 22.31 - 6 \times 5 = -7.69$ kN

At just left of point E, $x = 8$ m

and $SF = 22.31 - 6 \times 8 = -25.69$ kN

At point E, $SF = 25.69 - 25.69 = 0$

Bending Moment :

Portion AB : Consider any section between AB at distance x from the end support A.

$$BM = 31.31x - 9\,\frac{x^2}{2} \qquad \text{(parabolic variation)}$$

At point A, $x = 0$ and $BM = 0$

At point B, $x = 3$ m

and $BM = 31.31 \times 3 - \frac{9 \times 3^2}{2} = 53.43$ kN m

Portion BCD :

At point C, $BM = R_a \times 4 - (9 \times 2) \times (1.5 + 1)$
$= 31.31 \times 4 - 67.5$
$= 57.74$ kN m

At point D, $BM = R_a \times 5 - (9 \times 3) \times (1.5 + 2)$
$= 31.31 \times 5 - 94.5 - 12$
$= 50.05$ kN m

Portion DE : Consider any section within DE at distance x from the end support A.

$$BM = 31.31x - (9 \times 3) \times (x - 1.5) - 12 \times (x - 4) - 6x\,(x - 5) \times \frac{x-5}{2}$$
$$= 31.31x - 27 \times (x - 1.5) - 12\,(x - 4) - 3\,(x - 5)^2$$

BM at D (at $x = 5$)

$$= 31.31 \times 5 - 27\,(5 - 1.5) - 12\,(5 - 4) - 3\,(5 - 5)^2$$
$$= 50.05 \text{ kN m}$$

BM at E (at $x = 8$)

$$= 31.31 \times 8 - 27\,(8 - 1.5) - 12\,(8 - 4) - 3\,(8 - 5)^2 = 0$$

Since there is *udl* in the segment DE, the variation in bending moment is parabolic.

The variation in shear force and bending moment for the entire beam are as shown in Fig. 11.31.

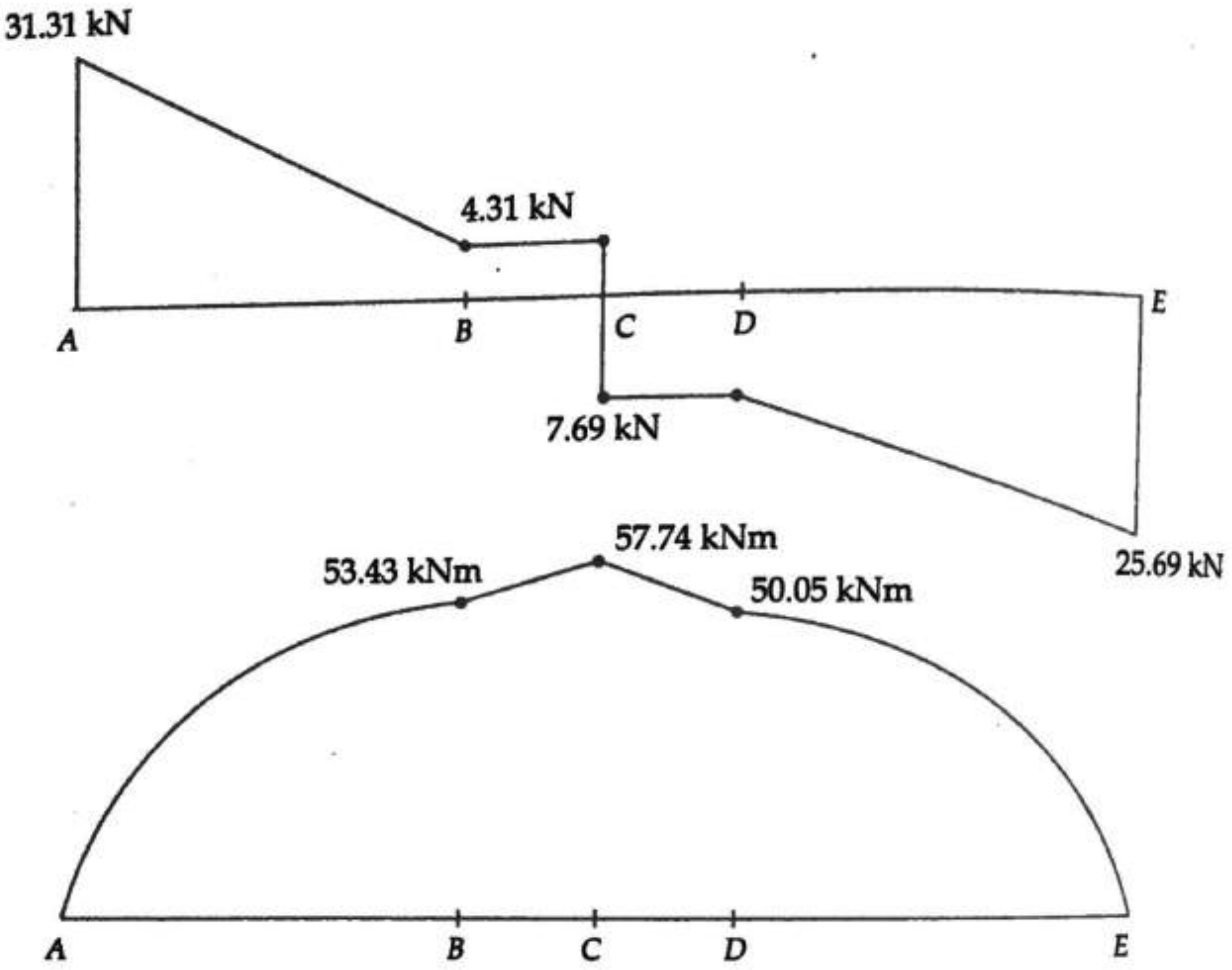

31.31 kN
4.31 kN
A
B
C
D
E
7.69 kN
25.69 kN
53.43 kNm
57.74 kNm
50.05 kNm
A
B
C
D
E

Fig. 11.31

A horizontal beam 10 m long carries a uniformly distributed load of 8 kN/m together with concentrated loads of 40 kN at the left end and 60 kN at the right end. The beam is supported at two points 6 m, so chosen that reaction is the same at the each support. Determine the position of props and show the variation of shear force and bending moment over the entire length of the beam.

Solution: Refer Fig. 11.34 for the beam loaded and supported as per the statement. Let the prop C be at distance *a* from end *A*.

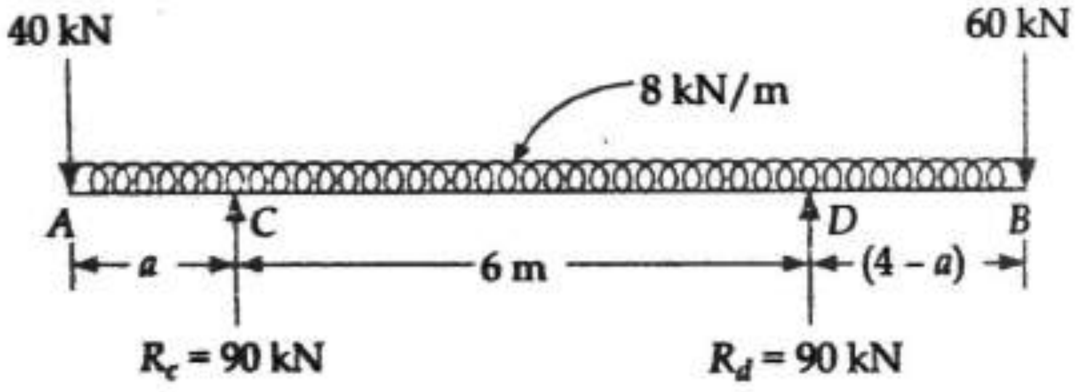


Fig. 11.34

Then the prop *D* is at distance (4 – *a*) from end *B*.

Total load on the beam = 40 + 60 + (10 × 8) = 180 kN. Since reaction is the same at each support

$$R_c = R_d = \frac{180}{2} = 90 \text{ kN}$$

Taking moments about end *A*,

$$60 \times 10 + (8 \times 10) \times \frac{10}{2} = 90 \times a + 90\,(6 + a)$$

or $$600 + 400 = 90a + 540 + 90a$$

$$\therefore \quad a = \frac{(600 + 400) - 540}{180} = 2.55 \text{ m}$$

Thus the left support is 2.55 m from A and the right support is $(4 - 2.55) = 1.45$ m from B.

Shear force:

SF at $A = -40$ kN

SF just on left side of $C = -40 - 8 \times 2.55 = -60.40$ kN

SF just on right side of $C = -60.40 + 90 = 29.60$ kN

SF just on left side of $D = 29.60 - 8 \times 6 = -18.40$ kN

SF just on right side of $D = -18.40 + 90 = 71.60$ kN

SF just on left side of $B = 71.60 - 8 \times 1.45 = 60$ kN

SF just on right side of $B = 60 - 60 = 0$

The point of zero shear stress as measured from end A and lying between CD can be worked out from the equation.

$$-40 + 90 - 8x = 0; \quad x = \frac{50}{8} = 6.25 \text{ m}$$

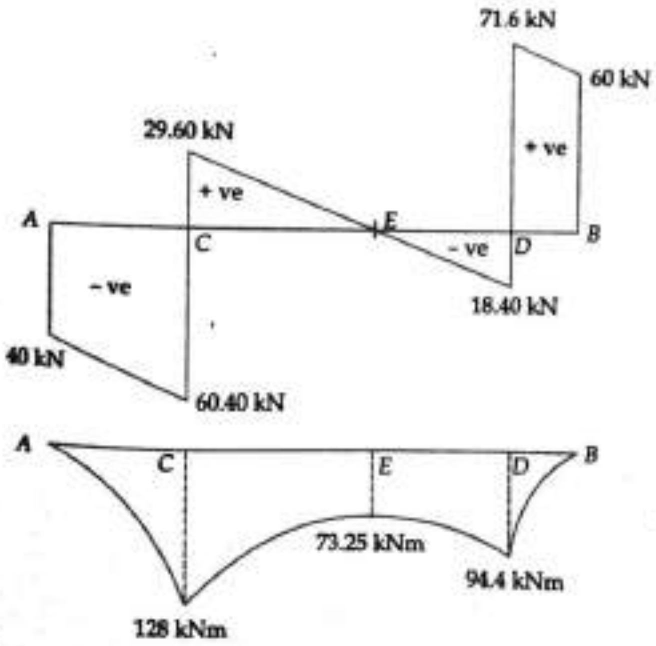
Bending moment :

BM at $A = 0$

$$BM \text{ at } C = -40 \times 2.55 - (8 \times 2.55) \times \frac{2.55}{2} = -128 \text{ kNm}$$

$$BM \text{ at } D = -40 \times 8.55 - (8 \times 8.55) \times \frac{8.55}{2} + 90 \times 6$$

$$= -342 - 292.4 + 540 = -94.4 \text{ kNm}$$



BM at a distance of 6.25 m from A,

$$= -40 \times 6.25 - (8 \times 6.25) \times \frac{6.25}{2} + 90 \times (6.25 - 2.55)$$

$$= -250 - 156.25 + 333 = -73.25 \text{ kNm}$$

The variation of shear force and bending moment length of the beam has been depicted in Fig 11.35.

EXAMPLE 11.12

A horizontal beam AB of span 10 m carries a uniformly distributed load of intensity 160 N/m and a point load of 400 N at the left end A. The beam is supported at a point C which is 1m from A and at D which is in the right half of the beam. If the point of contraflexture is at the mid point of the beam, determine the distance of support at D from the end B of the beam. Proceed to draw the shear force and bending moment diagrams for the arrangement.

Solution:

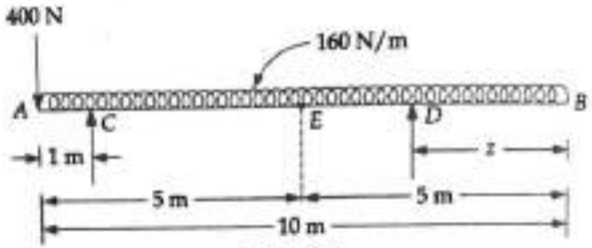


Fig. 11.36

The bending moment is zero at the point of contraflexture. Therefore

$$M_e = 0 = -400 \times 5 - 160 \times 5 \times \frac{5}{2} + R_c \times 4 \quad \text{(left half of beam)}$$

The *udl* is taken to be acting at its CG.

or $\quad 4R_c = 2000 + 2000 \,;\, R_c = 1000 \text{ N}$

Applying the condition $\Sigma F_y = 0$ for equilibrium of beam, we have

$$R_c + R_d = 400 + 160 \times 10 = 2000$$

$$\therefore \quad R_d = 2000 - R_c = 2000 - 1000 = 1000 \text{ N}$$

Again taking moments about the point of contraflexture E,

$$M_e = 0 = -R_d \times (5 - z) + 160 \times 5 \times \frac{5}{2} \quad \text{(right half of beam)}$$

$$1000 \times (5 - z) = 2000 \,;\, z = 3 \text{ m}$$

Thus the support D is at a distance of 3 m from end B.

Shear Force

Portion AC:

At A	:	$SF = -400$ N
Just left of C	:	$SF = -400 - 160 \times 1 = -560$ N
Just right of C	:	$SF = -560 + 1000 = +440$ N
Just left of D	:	$SF = 440 - 160 \times 6 = -520$ N
Just right of D	:	$SF = -520 + 1000 = 480$ N
At point B	:	$SF = 480 - 160 \times 3 = 0.$

The shear force changes sign between the section CD. The location of the point of zero shear stress can be obtained from the relations:

$-400 - 160x + 1000 = 0$; $x = 3.75$ m

Bending moment: Considering any section at distance x from end A,

Portion AC:

$$M_x = -400x - 160x \times \frac{x}{2} = -400 \times -80x^2$$

when $x = 0 : M_a = 0$

$x = 1$ m : $M_c = -400 - 80 = -480$ Nm

Between C and D

$$M_x = -400x - 160x \times \frac{x}{2} + R_b(x-1)$$

$$= -400x - 80x^2 + 1000(x-1).$$

when $x = 1$ m : $M_b = -400 \times 1 - 80 \times 1^2 + 1000(1-1) = -480$ Nm

$x = 3.75$ m : $M = -400 \times 3.75 - 80 \times 3.752 + 1000(3.75 - 1) = 125$ Nm

$x = 5$ m: $M_r = -400 \times 5 - 80 \times 52 + 1000(5-1) = 0$

$x = 7$ m : $M_d = -400 \times 7 - 80 \times 7^2 + 1000(7-1) = -720$ Nm

Between D and B:

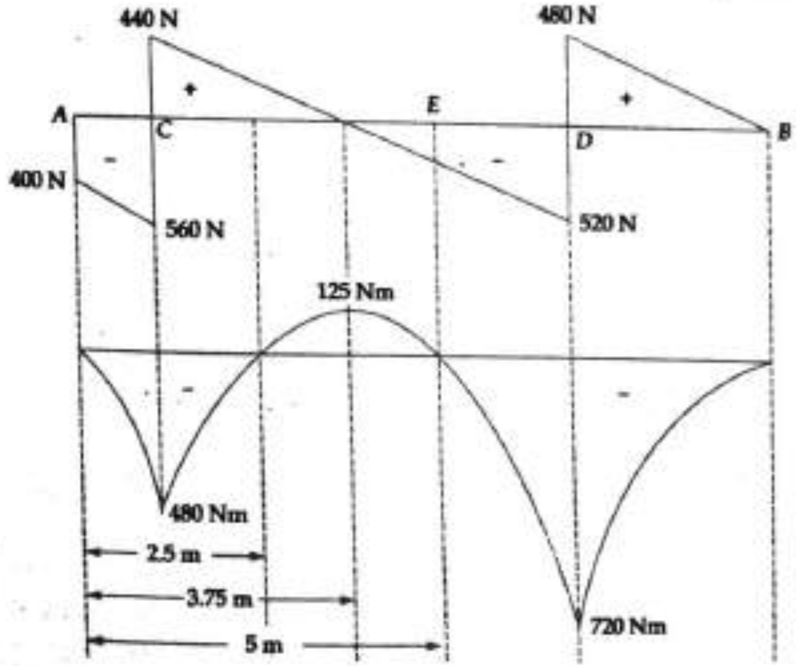
$$M_x = -400x - 80x^2 + 1000 \times (x-1) + 1000(x-7)$$

At $x = 7$ m : $M_d = -400 \times 7 - 80 \times 7^2 + 1000 \times (7-1) + 1000(7-7) = -720$ Nm

$x = 10$ m : $M_b = -400 \times 10 - 80 \times 10^2 + 1000(10-1) + 1000(10-7)$

$= -4000 - 8000 + 9000 + 3000 = 0.$

The shear force and bending moment for the entire beam are shown in Fig. 11.37.



A girder 10 m long rests on two supports with equal overhangs on either side and carries a uniformly distributed load of 20 kN/m over the entire length. Calculate the overhangs if the maximum bending moment, positive or negative, is to be as small as possible. Proceed to draw the shear force and bending moment diagrams for the arrangement.

Solution: Refer to Fig. 11.38 for the space diagram of the loaded girder. The overhang on each side has been indicated as a.

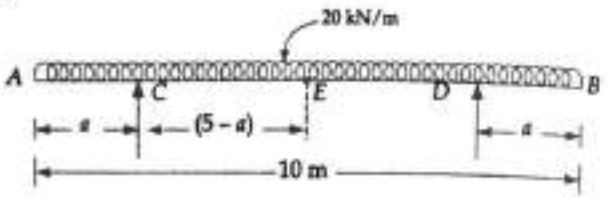


Fig. 11.38

Due to symmetrical arrangement, the total load on the beam will be shared equally between the two supports.

$$\therefore \qquad R_c = R_d = \frac{20 \times 10}{2} = 100 \text{ kN}$$

The maximum positive moment would occur at the mid span (point E) and the maximum negative would occur at the supports. Since these moments are stated to be equal in magnitude, we have

$$(20 \times a) \times \frac{a}{2} = 100\,(5 - a) - (20 \times 5) \times \frac{5}{2}$$

Simplification gives : $a^2 + 10a - 25 = 0$

$$\therefore \qquad a = \frac{-10 + \sqrt{10^2 - 4 \times 1 \times (-25)}}{2} = 2.07 \text{ m}$$

Shear force:

SF at $A = 0$

SF just on left of $C = -2.07 \times 20 = -41.40$ kN

SF just on right of $C = -41.40 + 100 = +58.60$ kN

SF at mid span (point E) $= 58.60 - 20\,(5 - 2.07) = 0$

Bending moment: Taking a section at distance x from end A and considering forces on left hand side.

Portion AC:

$$M = -(20 \times x) \times \frac{x}{2} = -10x^2 \qquad \text{(parabolic variation)}$$

At $x = 0$: $M_a = 0$

At $x = 2.07$ m : $M_c = -10 \times (2.07)^2 = -42.84$ kNm

Portion CD:

$$M = -(20 \times x) \times \frac{x}{2} + R_c\,(x - a)$$

$$= -10x^2 + 100\,(x - 2.07) \qquad \text{(parabolic variation)}$$

At $x = 2.07$ m : $M_c = -10 \times (2.07)^2 + 100\,(2.07 - 2.07) = -42.84$ kNm

At $x = 5$ m : $M_e = -10 \times 5^2 + 100\,(5 - 2.07) = 43$ kNm

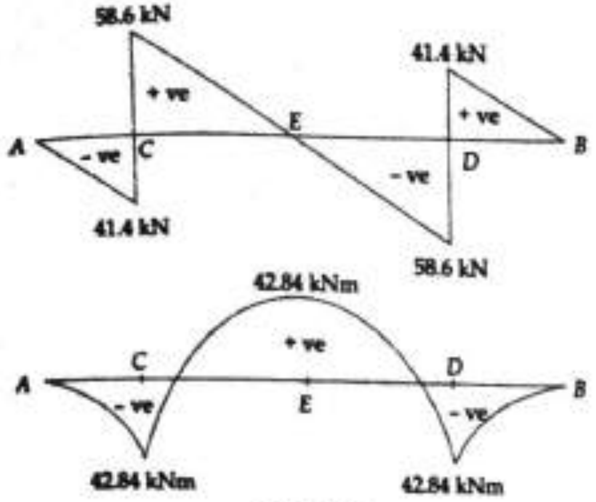


Fig. 11.39

The slight variation in the magnitude of bending moment at the support (point B) and at the centre (point E) is due to rounding off.

For locating the position of the point of contraflexture, we have

$$-10x^2 + 100\,(x - 2.07) = 0$$

or $$x^2 - 10x + 20.7 = 0$$

$\therefore$ $$x = \frac{10 \pm \sqrt{10^2 - 4 \times 20.7}}{2} = 2.97 \text{ m and } 7.07 \text{ m}$$

The shear force and the bending moment diagrams for the entire span of the girder are shown in Fig. 11.39.

Note: The *SF* and *BM* for the right half has been drawn making use of symmetry.

EXAMPLE 11.14

Draw the shear force and bending moment diagrams for the overhanging beam loaded as shown in the figure given below.

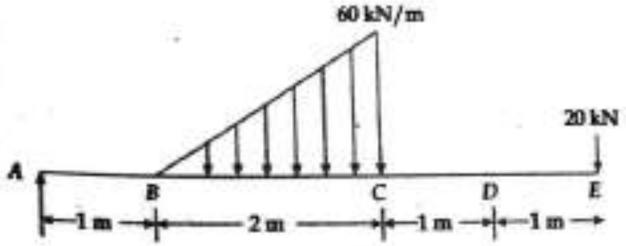


Fig. 11.40

Indicate values at salient points of the beam. Locate the position of the point of contraflexure if any.

Solution: From conditions of static equilibrium ($\Sigma Fy = 0$ and $\Sigma M = 0$), we have

$$R_a + R_d = \left(\frac{1}{2} \times 2 \times 60\right) + 20 = 80$$

Taking moment about A,

or $R_d \times 4 - 20 \times 5 - \frac{1}{2} \times 2 \times 60 \times \left(1 + \frac{2}{3} \times 2\right) = 0$

or $4R_d = 100 + 140 = 240$

$$R_d = \frac{240}{4} = 60 \text{ kN}$$

and $R_a = 80 - 60 = 20$ kN

Shear Force:

SF at $A = R_a = 20$ kN

Since there is no load in the segment AB, shear force remains constant at 20 kN within the portion of the beam.

Portion BC: Consider any section within portion BC and at distance x from end A.

Load intensity at this section $= \frac{x-1}{2} \times 60$

$$SF = 20 - \frac{1}{2}(x-1) \times \left\{\frac{x-1}{2} \times 60\right\}$$

$$= 20 - 15(x-1)^2 \qquad \text{(parabolic variation)}$$

At point B : $x = 1$ m and $SF = 20 - 15(1-1)^2 = 20$ kN

At point C : $x = 3$ m and $SF = 20 - 15(3-1)^2 = -40$ kN

The location of zero shear force can be worked out from the relation.

$$20 - 15(x-1)^2 = 0$$

or $x - 1 = \sqrt{\frac{20}{15}} = 1.54$ $\quad \therefore x = 2.154$ m

Since there is no load in portion CD of the beam, the shear force from point C to just left of point D will remain constant at -40 kN (the shear force at point C).

SF just an right side of $D = -40 + 60 = 20$ kN

This value of shear force remains constant within portion DE (because of no loading) and at point E, it takes the values

$$20 - 20 = 0 \text{ kN}$$

Bending Moment

Portion AB

BM at point $A = 0$

BM at point $B = R_a \times 1 = 20 \times 1 = 20$ kNm

Portion BC: Consider any section within portion BC and at distance x from end A.

Load intensity at this section $= \frac{x-1}{2} \times 60 = 30(x-1)$

$$BM = 20x - \left[\frac{1}{2}(x-1) \times 30(x-1)\right] \times \frac{x-1}{3}$$

Here $\frac{x-1}{3}$ is the distance of CG of triangular load from the section.

$$BM = 20x - 5(x-1)^3 \qquad \text{(cubic variation)}$$

At point $B: x = 1$ m and $BM = 20 \times 1 - 5(1-1)^3 = 20$ kNm

At point $C: x = 3$ m and $BM = 20 \times 3 - 5(3-1)^3 = 20$ kNm

The bending moment will be maximum at the point where shear force is zero, *i.e.*, at $x = 2.154$ m

Maximum bending moment

$$= 20 \times 2.154 - 5(2.154 - 1)^3 = 35.40 \text{ kNm}$$

Portion CD:

$$BM \text{ at point } C = 20 \text{ kNm (calculated above)}$$

$$BM \text{ at point } D = 20 \times 4 - \left(\frac{1}{2} \times 2 \times 60\right) \times \left(1 + \frac{1}{3} \times 2\right)$$

$$= 80 - 100 = -20 \text{ kNm}$$

Since shear forces remains constant due to no load in this section, the bending moment will have linear variation from 20 kNm (at point C) to -20 kNm (at point D).

Since the bending moment changes sign in the portion CD, there is a point of contraflexure in this portion and its location with respect to point A can be worked out from the relation

$$20 \times x - \left(\frac{1}{2} \times 2 \times 60\right)\left(x - 3 + \frac{2}{3}\right) = 0$$

or

$$20x - 60x + 180 - 40 = 0$$

That gives

$$x = \frac{-180 + 40}{-40} = 3.5 \text{ m}$$

Thus the point of contraflexure is located at 3.5 m from the end A.

Portion DE

BM at point $D = -20$ kNm (calculated above)

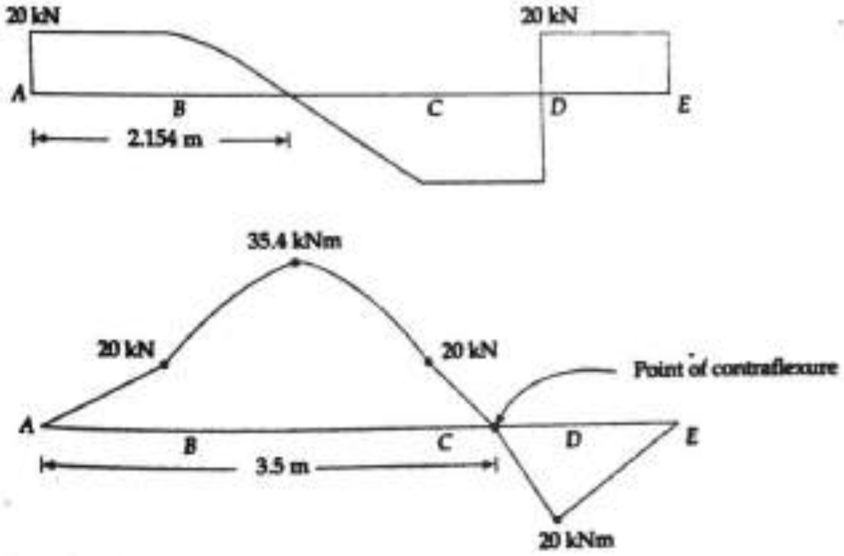


Fig. 11.41

9. Determine the reactions and construct the shear force and bending moment diagrams for the simply supported beam loaded as shown in Fig. 11.51. Also determine the position and magnitude of maximum bending moment.

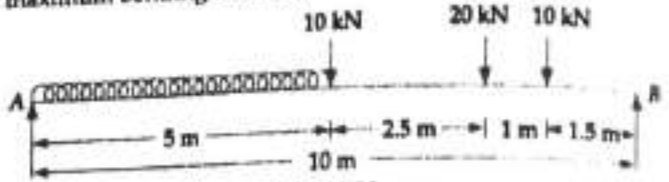


Fig. 11.51

$(M_{max} = 120.25$ kNm at 4.9 m from

10. For a symmetrically loaded overhang beam shown in Fig. 11.52, make calculations for the value of load W such that the bending moment becomes zero at the mid span of the beam.

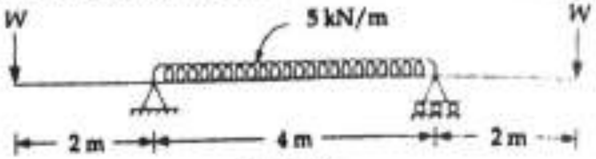


Fig. 11.52

$(W = 5$ kN

11. Draw the shear force and bending moment diagrams for the beam loaded and supported as shown in Fig. 11.53.

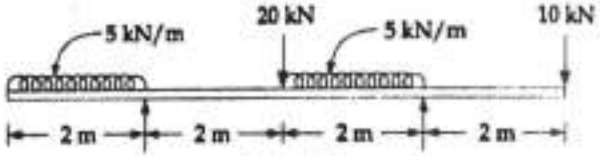


Fig. 11.53

12. Draw the shear force and bending moment diagram for the beam loaded as shown in Fig. 11.54.

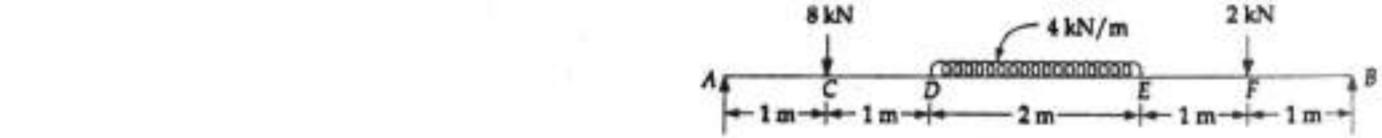


Fig. 11.54

13. Draw the shear force and bending moment diagrams for a cantilever beam loaded as shown in Fig. 11.55 given below

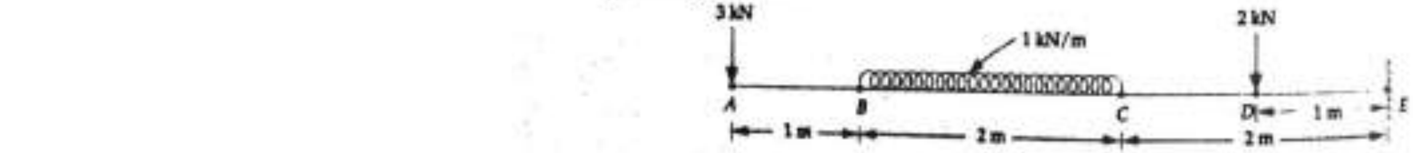


Fig. 11.55

Locate the position for maximum bending moment and determine its value.

The Centroid

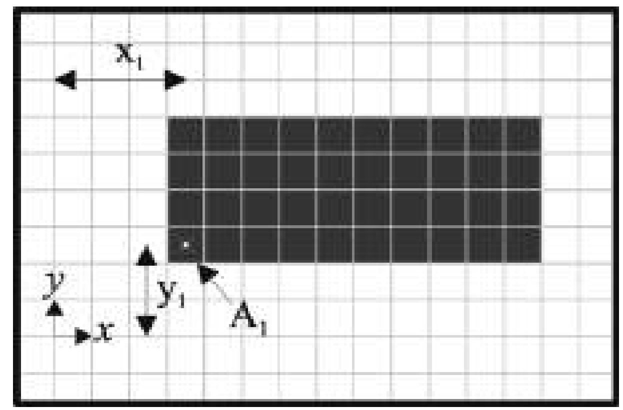
- The centroid is a point that locates the geometric center of an object.

- The position of the centroid depends only on the object's geometry (or its physical shape) and is independent of density, mass, weight, and other such properties.

- The average position along different coordinate axes locates the centroid of an arbitrary object.

The Centroid

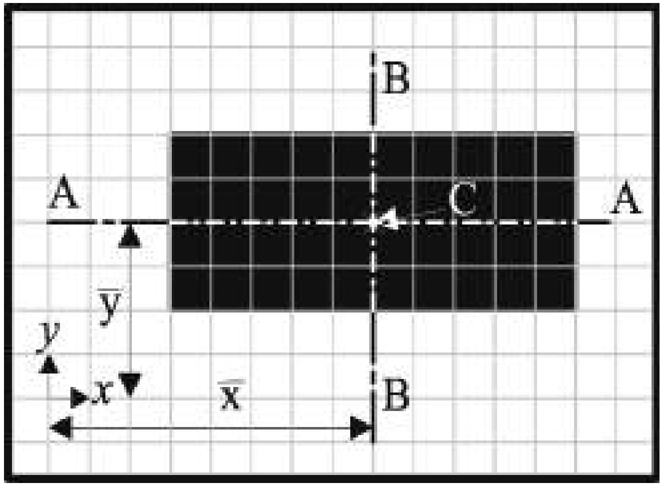
- We can divide the object into a number of very small finite elements A_1, A_2, … A_n.
- In this particular case, each small square grid represents one finite area.
- Let the coordinates of these areas be (x_1, y_1), (x_2, y_2), …, (x_n, y_n).
- The coordinates x_1 and y_1 extend to the center of the finite area.
- Now, the centroid is given by



$$\bar{x} = \frac{\sum_i x_i A_i}{\sum_i A_i} \qquad \bar{y} = \frac{\sum_i y_i A_i}{\sum_i A_i}$$

The Centroid

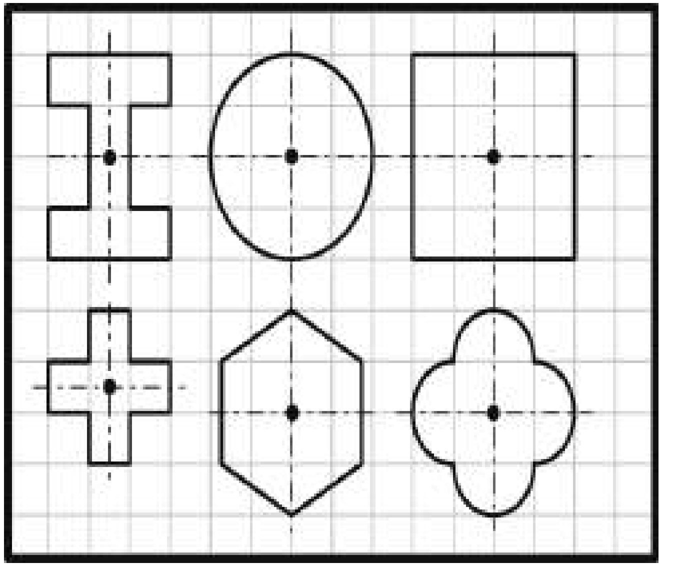
- The calculations will result in the location of centroid C.
- Because point C is at the center of the rectangle, the results intuitively make sense.
- Consider the moment due to the finite areas (instead of the forces) about two lines (AA and BB) parallel to the x- and y-axes passing through the centroid.
- Because the rectangle is symmetric about these two lines, the net moment will be zero.



The Centroid

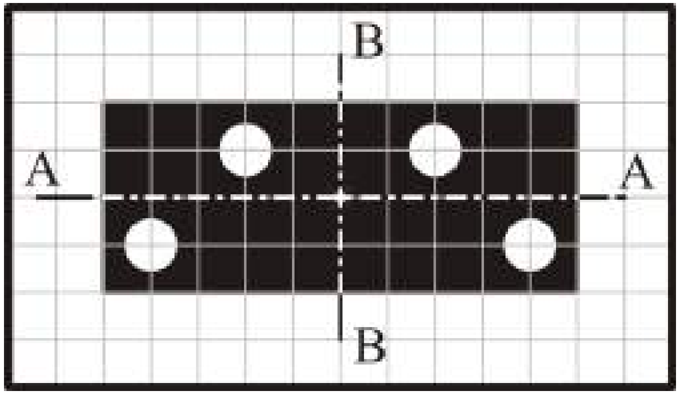
- Centroid always lies on the line of symmetry.

- For a doubly symmetric section (where there are two lines of symmetry), the centroid lies at the intersection of the lines of symmetry.



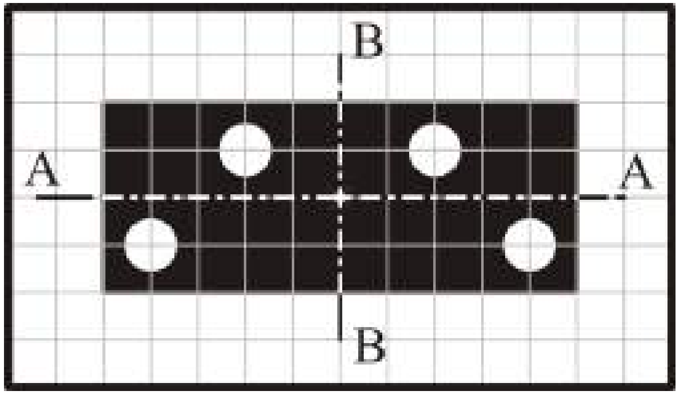
Functional Symmetry

- The area is symmetric about line BB, its centroid must lie on this line.
- The area is not symmetric about line AA.



Functional Symmetry

- The four holes are equidistant from line AA, and the moments from the two holes on the top of line AA counteract that of the two bottom holes.
- Even though the area is not physically symmetric about line AA, functionally line AA can be viewed as the line of symmetry.
- Therefore, the centroid lies on the intersection of the two lines.



The Centroid

- The calculation of the centroid for a composite section requires the following three steps:
  - Divide the composite geometry into simple geometries for which the positions of the centroid are known or can be determined easily.
  - Determine the centroid and area of individual components.
  - Apply the equation to determine the centroid location.

Example 8.1

Derive an expression for the centroid of a thin semicircular arc of mean radius, r.

Solution

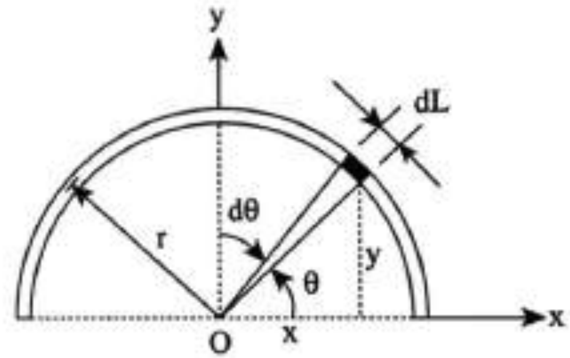


Figure 8.1 Centroid calculation of a semicircular arc

From Fig. 8.1,

$$dL = r d\theta \text{ and } L = \pi r$$

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

From Eq. 8.2,

$$\bar{y} = \frac{\int y\, dL}{L}$$

$$= \frac{\int_0^{\pi} r\sin\theta \cdot r d\theta}{\pi r}$$

$$= \frac{r}{\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{r}{\pi}\left[-\cos\theta\right]_0^{\pi} = \frac{r}{\pi}(1+1) = \frac{2r}{\pi}$$

$$\therefore \bar{y} = \frac{2r}{\pi}$$

$$\bar{x} = 0 \text{ (By symmetry)}$$

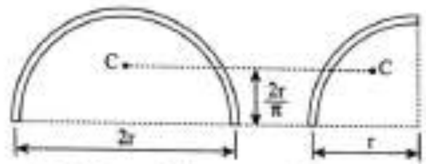


Figure 8.2 Centroid of semicircular and quarter arcs

This is a very important result which one must remember as a formula. Note that y-coordinate of the centroid of a quarter circle would also lie at the same level $\left(\bar{y} = \frac{2r}{\pi}\right)$ due to symmetry in left and right halves (Fig. 8.2). One can verify this result by substituting $\frac{\pi r}{2}$ for L and integrating between 0 and $\frac{\pi}{2}$. In fact, both $\bar{x}$ and $\bar{y}$ would come out to be the same due to symmetry.

Example 8.2

Derive an expression for the centroid of a thin arc of mean radius r and included angle 2α, selecting the symmetrical radial line as x-axis.

Solution

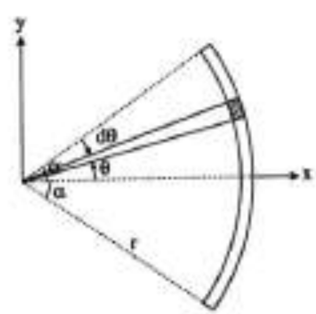


Figure 8.3 Centroid calculation of an arc of radius r and included angle 2α

From Fig. 8.3,

$$dL = r d\theta$$

$$L = 2r\alpha$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

From Eq. 8.2,

$$\bar{x} = \frac{\int x\, dL}{L}$$

$$= \frac{\int_{-\alpha}^{\alpha} r^2 \cos\theta\, d\theta}{2r\alpha}$$

$$= \frac{r}{2\alpha}[\sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{r\sin\alpha}{\alpha}$$

$$\bar{y} = 0 \text{ (By symmetry)}$$

One can verify that $\bar{x}$ reduces to $\frac{2r}{\pi}$ for $\alpha = \frac{\pi}{2}$, as expected for a semicircular arc.

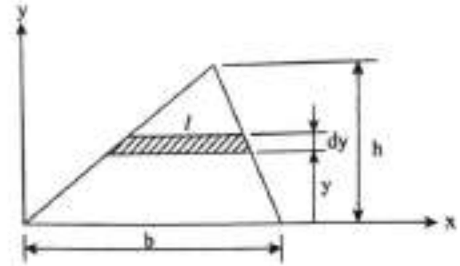


Figure 8.6 Centroid calculation of a triangle

By similar triangles,

$$\frac{l}{b} = \frac{h-y}{h}$$

$$\therefore l = \frac{b(h-y)}{h}$$

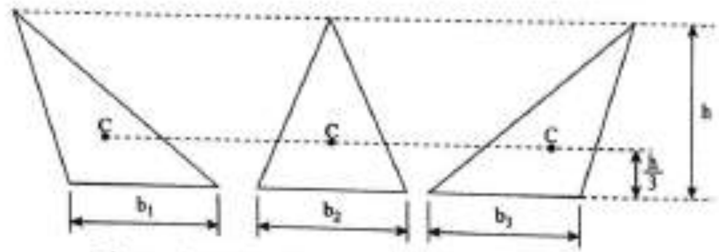


Figure 8.7 Centroid of triangles with the same altitude

From Eq. 8.3,

$$\bar{y}\,A = \int y\,dA$$

$$\bar{y}\left(\frac{bh}{2}\right) = \int_0^h y\,l\,dy$$

$$= \int_0^h \frac{b}{h}(hy - y^2)dy = \frac{b}{h}\left[h\,\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{bh^2}{6}$$

$$\therefore \bar{y} = \frac{h}{3}$$

This is a very important result. One must remember this as a formula (Fig. 8.7).

Calculation of $\bar{x}$ by direct integration in this example is possible but not convenient. A much better approach would be to use the concept of centroid of composite areas in conjunction with the result obtained in this example (*centroid divides the altitudes in the ratio of* 1:2). This is discussed later.

Example 8.4

Locate the centroid of a semicircular disk of radius r.

Solution

Method I (using horizontal strip)

A horizontal strip is more convenient than a vertical strip (Fig. 8.8).

$$A = \frac{\pi r^2}{2}$$

$$y = r\sin\theta \quad \therefore dy = r\cos\theta\, d\theta$$

$$dA = l\, dy = 2r\cos\theta\, dy = 2r^2\cos^2\theta\, d\theta$$

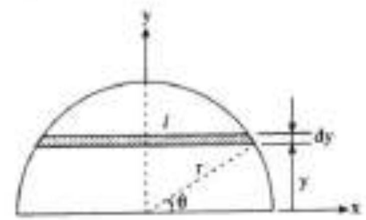


Figure 8.8 Centroid calculation of a semicircular disk using a horizontal strip

From centroid formula,

$$\bar{y}A = \int y\, dA = \int_0^r y\, l\, dy$$

$$= \int_0^{\frac{\pi}{2}} (r\sin\theta)(2r^2\cos^2\theta\, d\theta) = 2r^3\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\, d\theta$$

Note that the lower and the upper limits of θ correspond to $y = 0$ and $y = r$, respectively.

$$\text{Let } \cos\theta = u \quad \therefore -\sin\theta\, d\theta = du$$

$$\therefore \bar{y}A = -2r^3\int u^2\, du = -2r^3\frac{u^3}{3} = -\frac{2r^3}{3}\cos^3\theta\Big|_0^{\frac{\pi}{2}} = \frac{2r^3}{3}$$

$$\bar{y}\,\frac{\pi r^2}{2} = \frac{2r^3}{3}$$

$$\bar{y} = \frac{4r}{3\pi} \text{ and } \bar{x} = 0 \text{ (By symmetry)}$$

Example 8.5

Locate the centroid of a circular sector of radius r and included angle 2α, selecting the symmetrical radial line as the x-axis.

Solution

Though all the four methods described in Ex. 8.4 can be used, the method involving a triangular strip would be the most convenient. From Fig. 8.13,

$$A = \int dA = \int_{-\alpha}^{\alpha} \frac{1}{2} r^2 d\theta = r^2 \alpha$$

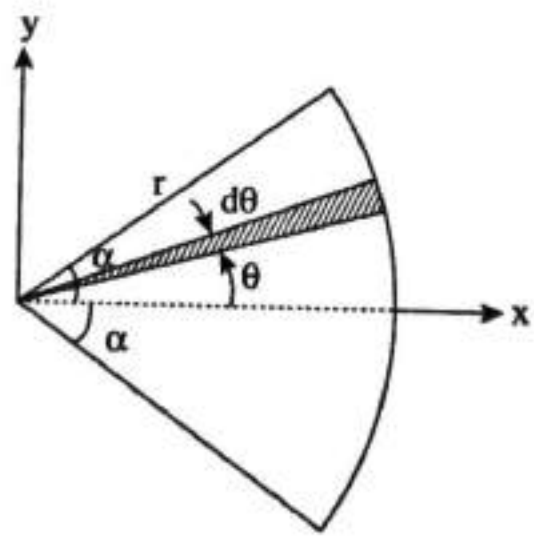


Figure 8.13 Centroid calculation of a circular sector

$$\bar{x}\, A = \int x \, dA$$

$$\bar{x}\, r^2\alpha = \int_{-\alpha}^{\alpha} \left(\frac{2r}{3}\cos\theta\right)\left(\frac{1}{2}r^2 d\theta\right) = \frac{r^3}{3}[\sin\theta]_{-\alpha}^{\alpha} = \frac{2r^3 \sin\alpha}{3}$$

$$\therefore\ \bar{x} = \frac{2r\sin\alpha}{3\alpha} \text{ and } \bar{y} = 0 \ \text{ (By symmetry)}$$

Note that $\bar{x}$ reduces to $\frac{4r}{3\pi}$ for $\alpha = \frac{\pi}{2}$, as expected for a semicircular disk.

Example 8.6

Locate the centroid of the area bounded by lines $x = a$, $y = 0$ and curve $x = \frac{ay^3}{b^3}$.

Solution

$x = a$ and $x = \frac{ay^3}{b^3}$, when solved together, give (a, b) as the point of intersection (Fig. 8.14).

$$A = \int dA = \int_0^a y\, dx = \int_0^a \left(\frac{b^3 x}{a}\right)^{\frac{1}{3}} dx$$

$$= \frac{b}{a^{\frac{1}{3}}}\left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^a = \frac{3b\,a^{\frac{4}{3}}}{4a^{\frac{1}{3}}} = \frac{3ab}{4}$$

$$\bar{x}\, A = \int x\, dA = \int_0^a x\, y\, dx = \int_0^a x\left(\frac{b^3 x}{a}\right)^{\frac{1}{3}} dx$$

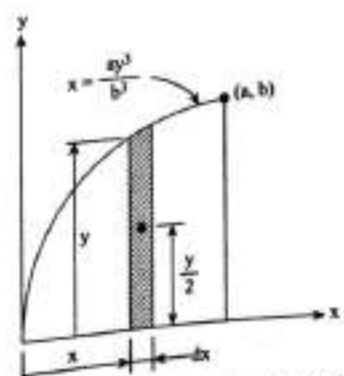


Figure 8.14 Centroid calculation using a vertical strip in Ex. 8.6

$$\bar{x}\,\frac{3ab}{4} = \frac{b}{a^{\frac{1}{3}}}\int_0^a x^{\frac{4}{3}}\,dx = \frac{3b\,a^{\frac{7}{3}}}{7a^{\frac{1}{3}}} = \frac{3a^2b}{7}$$

$$\therefore\ \bar{x} = \frac{4a}{7}$$

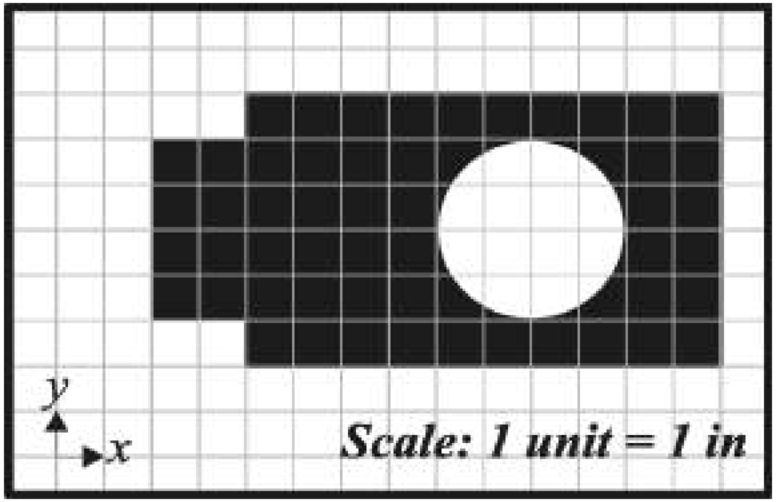
$$\bar{y}\,A = \int \frac{y}{2}\,dA \qquad \left(y\text{-coordinate of the area element is } \frac{y}{2}\right)$$

$$\bar{y}\,\frac{3ab}{4} = \int_0^a \frac{y^2}{2}\,dx = \int_0^a \frac{1}{2}\left(\frac{b^3x}{a}\right)^{\frac{2}{3}} dx = \frac{b^2}{2a^{\frac{2}{3}}}\left[\frac{3a^{\frac{5}{3}}}{5}\right] = \frac{3ab^2}{10}$$

$$\therefore\ \bar{y} = \frac{2b}{5}$$

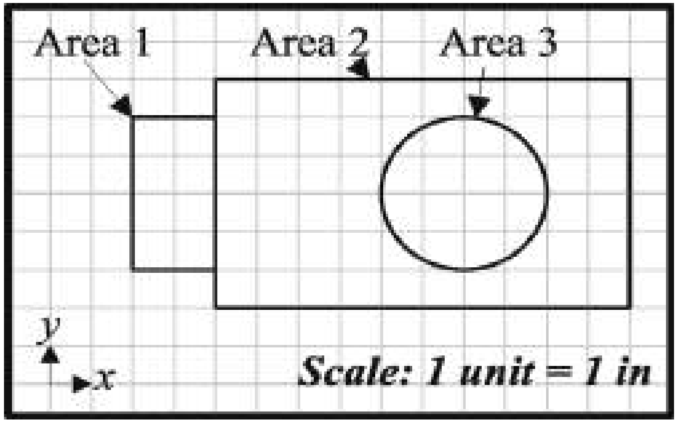
Example 1

- Determine the centroid of the composite section.



Example 1

- Step 1: *Divide the composite section into simple geometries*
  - The composite geometry can be divided into three parts:
    - two positive areas
    - one negative area (circular cutout).



Example 1

- Step II: *Determine the centroid and the area of individual component*

Part	**Dimensions**	**Area (sq. in)**	**x**	**y**
Area 1	$2'' \times 4''$	8	3	5
Area 2	$10'' \times 6''$	60	9	5
Area 3	$2''$ radius	-4π	10	5

Example 1

- Step III: *Determine the centroid location*

Part	Dimensions	Area (sq. in)	x	y	(in^3) $x_i A_i$	(in^3) $y_i A_i$
Area 1	2″×4″	8	3	5	24	40
Area 2	10″×6″	60	9	5	540	300
Area 3	2″ radius	-4π	10	5	-40π	-20π
		$\sum A_i =$ 55.434			$\sum x_i A_i =$ 438.34	$\sum y_i A_i =$ 277.17

Example 1

$$\bar{x} = \frac{\sum_i x_i A_i}{\sum_i A_i} \qquad \bar{y} = \frac{\sum_i y_i A_i}{\sum_i A_i}$$

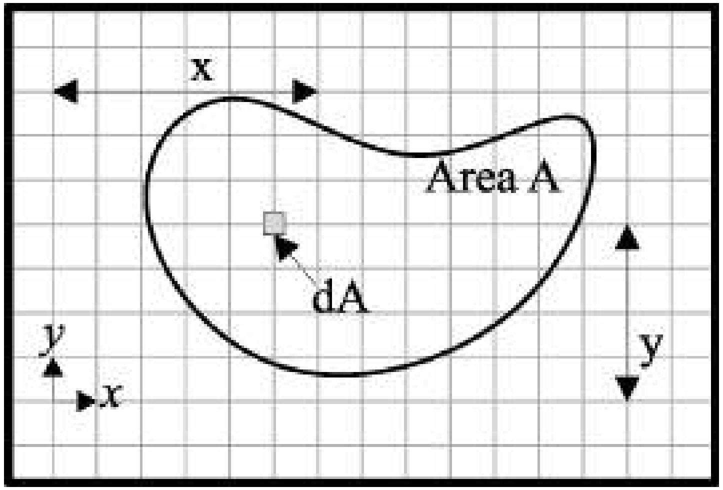
$$\bar{x} = \frac{438.34}{55.434} = 7.91\text{in} \qquad \bar{y} = \frac{277.17}{55.43} = 5.00\,\text{in}$$

Determining the location of the centroid using a Differential Element

- If x and y are the coordinates of a differential element *d*A, the centroid of a two-dimensional surface is given by

$$\bar{x} = \frac{\int_A x \, dA}{\int_A dA}$$

$$\bar{y} = \frac{\int_A y \, dA}{\int_A dA}$$



Determining the location of the centroid using a Differential Element

- The equation can be generalized to a three-dimensional surface as

$$\bar{x} = \frac{\int_A x\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A y dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A z dA}{\int_A dA}$$

- The same concepts can be used for determining the centroid of a line.

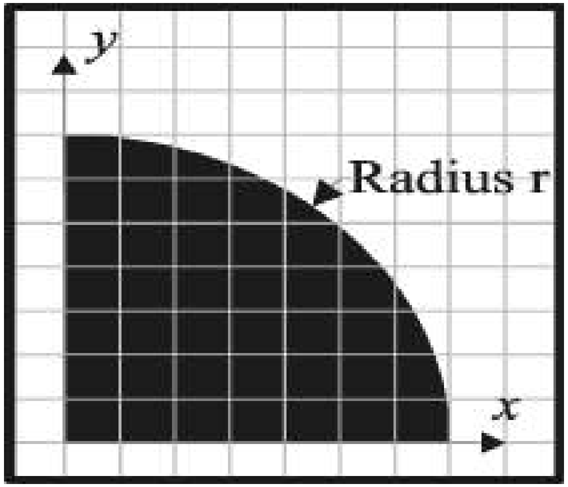
$$\bar{x} = \frac{\int_L x\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L y dL}{\int_L dL} \qquad \bar{z} = \frac{\int_L z dL}{\int_L dL}$$

- To determine the centroid of a volume, the equation takes the form of

$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V y dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V z dV}{\int_V dV}$$

Example 2

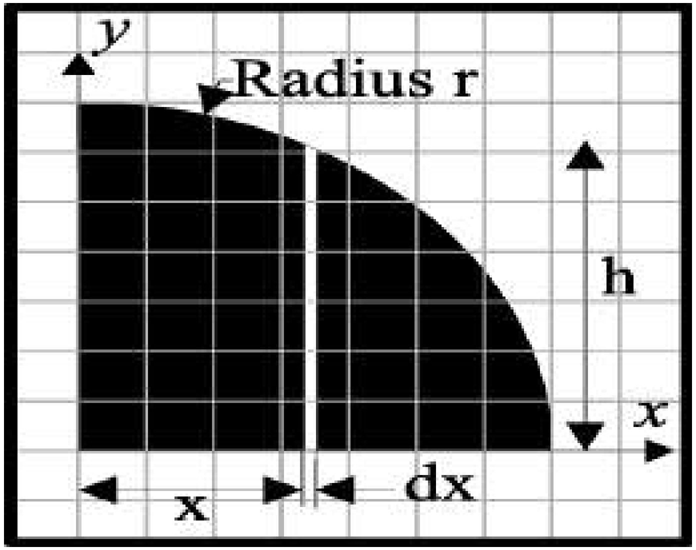
- Determine the centroid of the quarter circle.



Example 2

- The key step in solving this type of problems is to establish and define an appropriate differential element.

- Let us consider a vertical differential element with thickness dx and height h.

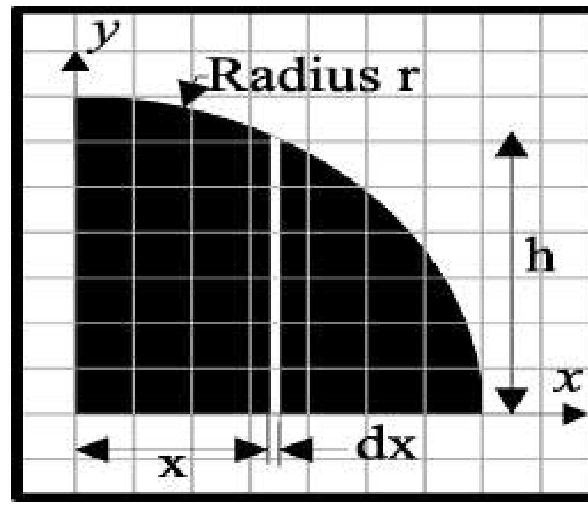


$$dA = h \cdot dx$$

$$h = \sqrt{r^2 - x^2}$$

Example 2

- Because the section is symmetric about a line that is at 45^0 to the x- and y-axes, the centroid lies on this line.



$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^R x\sqrt{r^2 - x^2}\,dx}{\int_A dA}$$

$$\bar{x} = \frac{\int_0^r x\sqrt{r^2 - x^2}\,dx}{\left(\frac{\pi}{4}r^2\right)} = \frac{\left[-\frac{\left(r^2-x^2\right)^{\frac{3}{2}}}{3}\right]_0^r}{\left(\frac{\pi}{4}r^2\right)}$$

$$\bar{x} = \frac{4}{3}\frac{r}{\pi}$$

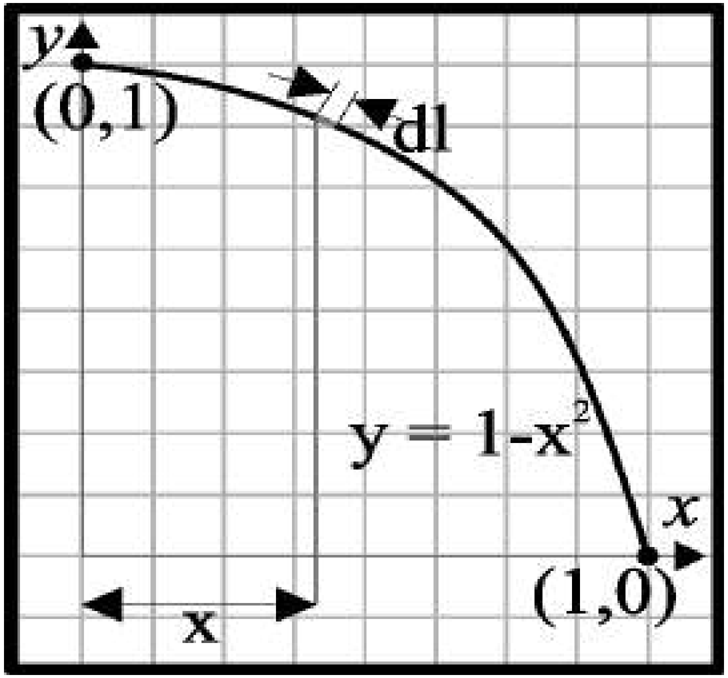
$$\bar{y} = \frac{4}{3}\frac{r}{\pi}$$

Example 3

- Locate the centroid of the line whose equation is

$$y=1-x^2$$

with x ranging from 0 to 1

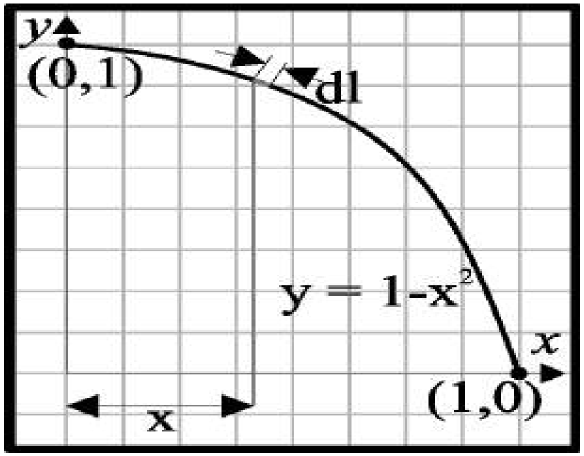


Example 3

$$dL = \sqrt{(dx)^2 + (dy)^2} == \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$\frac{dy}{dx} = -2x$$

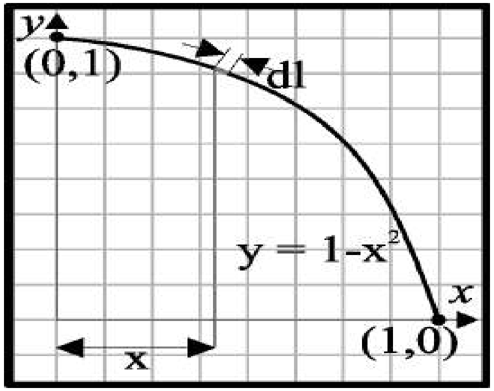
$$dL = \sqrt{1 + 4x^2}\ dx$$



Example 3

$$\bar{x} = \frac{\int_L x\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L y\, dL}{\int_L dL}$$

$$\bar{x} = \frac{\int_0^1 x\sqrt{1+4x^2}\; dx}{\int_0^1 \sqrt{1+4x^2}\; dx} \qquad \bar{y} = \frac{\int_0^1 \left(1-x^2\right)\sqrt{1+4x^2}\; dx}{\int_0^1 \sqrt{1+4x^2}\; dx}$$



$$\bar{x} = 0.8667 \qquad \bar{y} = 0.2861$$

Example 8.17

A uniform rod is bent into the shape as shown in Fig. 8.29. Determine the coordinates of its centroid.

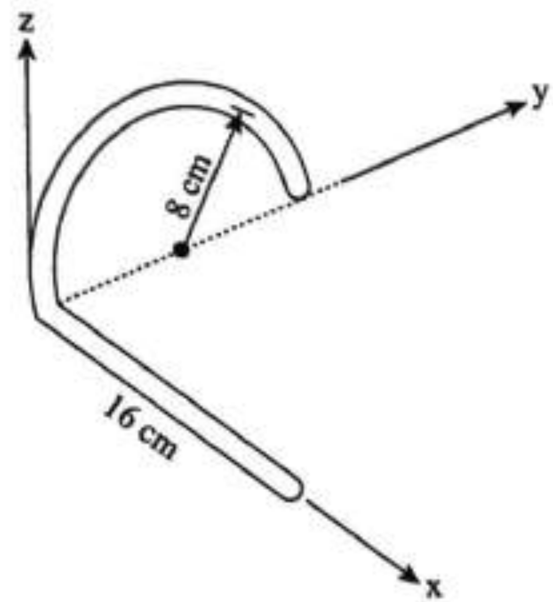


Figure 8.29 Figure for Ex. 8.17

Solution

The length of the straight part and the coordinates of its centroid are 16 cm and (8, 0, 0) cm, respectively. These are 8π cm and $\left(0, 8, \frac{16}{\pi}\right)$ cm for the circular part. For convenience, this problem would be solved in the tabular form given below.

Part	L_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{z}_i$	$L_i\bar{x}_i$	$L_i\bar{y}_i$	$L_i\bar{z}_i$
Straight	16	8	0	0	128	0	0
Circular	8π	0	8	$\frac{16}{\pi}$	0	64π	128
Total	41.13				128	201.06	128

Equation 8.5 can now be used for finding out the coordinates of the centroid:

$$\bar{x} = \frac{\sum L_i\bar{x}_i}{L} = \frac{128}{41.13} = 3.11 \text{ cm}$$

$$\bar{y} = \frac{\sum L_i\bar{y}_i}{L} = \frac{201.06}{41.13} = 4.89 \text{ cm}$$

$$\bar{z} = \frac{\sum L_i\bar{z}_i}{L} = \frac{128}{41.13} = 3.11 \text{ cm}$$

Example 8.18

The homogeneous wire ABCD is bent as shown in Fig. 8.30. It is attached to a hinge at C. Determine the length l for which portion BCD of the wire remains horizontal. All dimensions are in mm

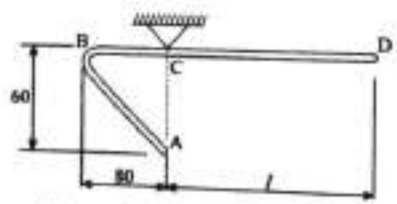


Figure 8.30 Figure for Ex. 8.18

Solution

$$AB = \sqrt{AC^2 + BC^2} = \sqrt{60^2 + 80^2} = 100 \text{ mm}$$

For equilibrium to be possible in the position shown, the centroid of the bent wire must lie on line AC. Centroids of both AB and BC lie $\frac{80}{2}$ (= 40 mm) towards left of AC, and that of CD is at $\frac{l}{2}$ towards right. We choose C as the origin and CD as the x-axis.

Part	L_i	$\bar{x}_i$	$L_i\bar{x}_i$
AB	100	-40	-4000
BC	80	-40	-3200
CD	l	$\frac{l}{2}$	$\frac{l^2}{2}$
Total	$180 + l$		$\frac{l^2}{2} - 7200$

$$\bar{x} = \frac{\Sigma L_i\bar{x}_i}{L} = \frac{\frac{l^2}{2} - 7200}{180 + l}$$

For $\bar{x}$ to be zero, $\frac{l^2}{2} = 7200$

$\therefore l = 120$ mm

Example 8.19

A wire is bent into a closed loop A–B–C–D–E–A as shown in Fig. 8.31. Portion AB is a circular arc of radius 5 m. Determine the centroid of the wire.

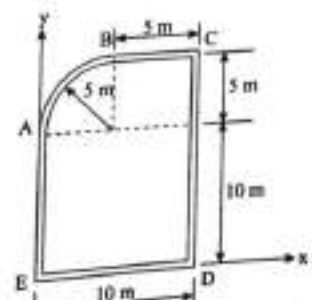


Figure 8.31 Figure for Ex. 8.19

Solution

We will use the result that the centroid of a quarter or a semicircular arc lies at a distance of $\frac{2r}{\pi}$ from its base (see Fig. 8.2).

Part	L_i	$\bar{x}_i$	$\bar{y}_i$	$L_i\ \bar{x}_i$	$L_i\ \bar{y}_i$
AB	$\frac{5\pi}{2}$	$5-\frac{10}{\pi}$	$10+\frac{10}{\pi}$	14.270	103.540
BC	5	7.5	15	37.5	75
CD	15	10	7.5	150	112.5
DE	10	5	0	50	0
EA	10	0	5	0	50
Total	47.854			251.77	341.04

$$\bar{x} = \frac{\sum L_i\ \bar{x}_i}{L} = \frac{251.77}{47.854} = 5.26 \text{ m}$$

$$\bar{y} = \frac{\sum L_i\ \bar{y}_i}{L} = \frac{341.04}{47.854} = 7.13 \text{ m}$$

Example 8.21

Locate the centroid of the composite area shown in Fig. 8.33.

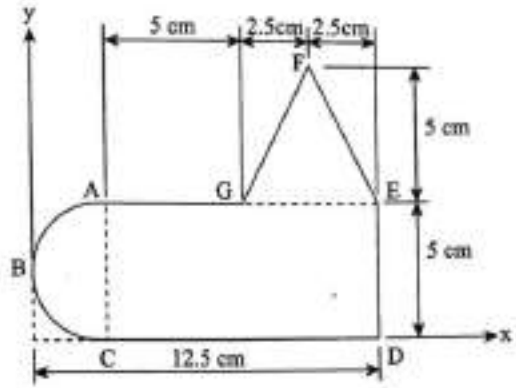


Figure 8.33 Figure for Ex. 8.21

Solution

We will use the results that the centroid of a semicircular disc of radius r lies at a distance of $\frac{4r}{3\pi}$ from its base (see Fig. 8.12), and that of a triangle of altitude h lies $\frac{h}{3}$ above its base (see Fig. 8.7).

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\,\bar{x}_i$	$A_i\,\bar{y}_i$
Semicircular sector ABC	$\frac{\pi\times 2.5^2}{2}$	$2.5-\frac{4\times 2.5}{3\pi}$	2.5	14.127	24.544
Rectangle ACDE	50	7.5	2.5	375	125
Triangle EFG	12.5	10	$5+\frac{5}{3}$	125	83.333
Total	72.317			514.127	232.877

$$\bar{x} = \frac{\sum A_i\,\bar{x}_i}{A} = \frac{514.127}{72.317} = 7.11 \text{ cm}$$

$$\bar{y} = \frac{\sum A_i\,\bar{y}_i}{A} = \frac{232.877}{72.317} = 3.22 \text{ cm}$$

Example 8.22

A triangle is removed from a semicircular disc as shown in Fig. 8.34. Locate the centroid of the remaining part (shaded).

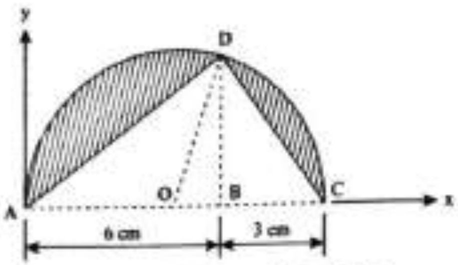


Figure 8.34 Figure for Ex. 8.22

Solution

Radius of the circle $= \frac{9}{2} = 4.5$ cm

Altitude (BD) of the triangle $= \sqrt{OD^2 - OB^2} = \sqrt{4.5^2 - (6-4.5)^2} = 4.243$ cm

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\,\bar{x}_i$	$A_i\,\bar{y}_i$
Semicircular disc	$\frac{\pi \times 4.5^2}{2}$	4.5	$\frac{4 \times 4.5}{3\pi}$	143.139	60.75
Triangle ABD	$-\frac{6 \times 4.243}{2}$	$6 - \frac{6}{3}$	$\frac{4.243}{3}$	-50.916	-18.003
Triangle BCD	$-\frac{3 \times 4.243}{2}$	$6 + \frac{3}{3}$	$\frac{4.243}{3}$	-44.552	-9.002
Total	12.715			47.671	33.745

$$\bar{x} = \frac{\sum A_i\,\bar{x}_i}{A} = \frac{47.671}{12.715} = 3.75 \text{ cm}$$

$$\bar{y} = \frac{\sum A_i\,\bar{y}_i}{A} = \frac{33.745}{12.715} = 2.65 \text{ cm}$$

Note that the concept of negative area has been used in this example because the shaded area is obtained by subtracting the areas of triangles ABD and BCD from the area of the semicircular disc. Subtracting an area is equivalent to adding a negative area of the same magnitude.

Example 8.23

Locate the centroid of the channel section shown in Fig. 8.35.

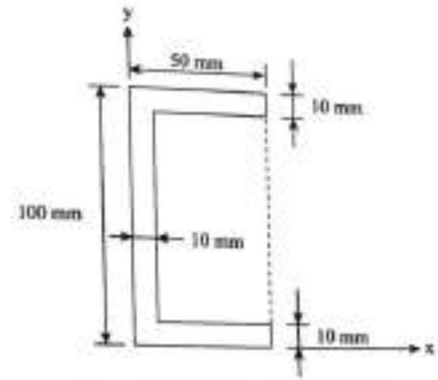


Figure 8.35 Figure for Ex. 8.23

Solution

This problem can be solved by considering three rectangles of areas 100 × 10, 40 × 10 and again 40 × 10 (other combinations are also possible). The other way is to consider the outer rectangle of area 100 × 50, and the inner rectangle of negative area 80 × 40. We will adopt the second approach since it would involve fewer calculations.

Part	A_i	$\bar{x}_i$	$\bar{y}_i$	$A_i\,\bar{x}_i$	$A_i\,\bar{y}_i$
Outer rectangle	5000	25	50	125000	250000
Inner rectangle	–3200	50–20	50	–96000	–160000
Total	1800			29000	90000

$$\bar{x} = \frac{\sum A_i\,\bar{x}_i}{A} = \frac{29000}{1800} = 16.11 \text{ mm}$$

$$\bar{y} = \frac{\sum A_i\,\bar{y}_i}{A} = \frac{90000}{1800} = 50 \text{ mm}$$

One may use the first approach also which would give the same answer. Take it as an exercise.

Note that we can use the symmetry argument to conclude that $\bar{y}$ is 50 mm.

EXAMPLE 7.6

Locate the centroid of the area shown in Fig. 7.10 with respect to the axes indicate in the figure.

Solution : The composite area has been divided into three segments namely a triangle, a rectangle and a semi-circle. The areas and the co-ordinates of centroid of these segments with respect to the given axes are:

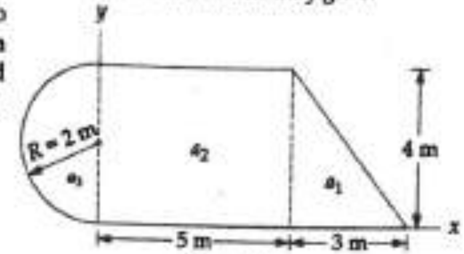


Fig. 7.10

Triangular segment:

$$a_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$$

$$x_1 = 5 + \frac{1}{3} \times 3 = 6 \text{ m} \; ; \quad y_1 = \frac{4}{3} = 1.33 \text{ m}$$

Rectangular segment: $a_2 = 5 \times 4 = 20 \text{ m}^2$

$$x_2 = \frac{5}{2} = 2.5 \text{ m} \; ; \quad y_2 = \frac{4}{2} = 2 \text{ m}$$

Semi-circular segment: $a_3 = \frac{1}{2} \times \pi \times 2^2 = 6.28 \text{ m}^2$

$$x_3 = -\frac{4R}{3\pi} = -\frac{4 \times 2}{3 \times \pi} = -0.849 \text{ m}$$

$$y_3 = \frac{4}{2} = 2 \text{ m}$$

The negative with co-ordinate x_3 stems from the fact that this co-ordinate lies on the left of y-axis

Then: $$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3} = \frac{6 \times 6 + 20 \times 2.5 + 6.28(-0.849)}{6 + 20 + 6.28} = 2.5 \text{ m}$$

and $$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3} = \frac{6 \times 1.33 + 20 \times 2 + 6.28 \times 2}{6 + 20 + 6.28} = 1.875 \text{ m}$$

EXAMPLE 7.7

A triangular plate in the form of an isosceles triangle ABC has base BC = 10 cm and altitude = 12 cm. From this plate, a portion in the shape of an isosceles triangle OBC is removed. If O is the mid-point of the altitude of triangle ABC, then determine the distance of CG of the remainder section from the base.

Solution : Refer Fig 7.11

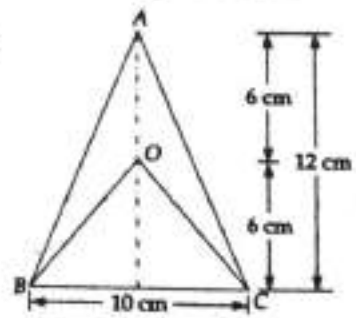


Fig. 7.11

For a triangle of height h, the CG lies on the axis at a distance $h/3$ from the base.

For triangle ABC,

$$\text{Area } A_1 = \frac{1}{2} \times 10 \times 12 = 60 \text{ cm}^2$$

$$y_1 = 12/3 = 4 \text{ cm from base } BC$$

For triangle OBC,

$$A_2 = \frac{1}{2} \times 10 \times 6 = 30 \text{ cm}^2$$

$$y_2 = 6/3 = 2 \text{ cm from base } BC.$$

Let y be the distance of CG of the section $ABOCA$ from the base line BC.

$$y = \frac{A_1y_1 + A_2y_2}{A_1 + A_2} = \frac{60 \times 4 + (-30) \times 2}{60 + (-30)}$$

The Center of Mass

- The center of mass is a point that locates the average position of the mass of an object.
- For an object with uniform density, it coincides with the centroid.
- It is often called the center of gravity because the gravitational pull on an object can be represented as a concentrated force acting at this point.

The Center of Mass

- The equation for finding the center of mass of a volume takes the form of

$$\bar{x} = \frac{\int_m x\,dm}{\int_m dm} \qquad \bar{y} = \frac{\int_m y\,dm}{\int_m dm} \qquad \bar{z} = \frac{\int_m z\,dm}{\int_m dm}$$

- For a three-dimensional surface of uniform thickness and density, the center of mass coincides with the centroid of the surface.

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A y\,dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A z\,dA}{\int_A dA}$$

- The same concepts can be used to determine the center of mass of a line. The equation takes the form of

$$\bar{x} = \frac{\int_L x\,dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L y\,dL}{\int_L dL} \qquad \bar{z} = \frac{\int_L z\,dL}{\int_L dL}$$

It may be recalled that the moment of force about a point is the product of force (F) and the perpendicular distance (x) between the point and the line of action of force.

$$\text{Moment of force} = F\,x$$

If this moment $F\,x$ is further multiplied by the distance x, then a quantity $F\,x^2$ is obtained which is referred to as the moment of moment or the second moment of force

$$\text{Moment of moment} = F\,x \times x = F\,x^2$$

If the term force F in the above identity is replaced by area or mass of the body, the resulting parameter is called the moment of inertia (*MOI*). Thus

$$\text{Moment of inertia of a plane area} = A\,x^2$$

$$\text{Mass moment of inertia of a body} = m\,x^2$$

where A and m respectively denote the area and mass of the body.

Inertia refers to the property of a body by virtue of which the body resists any change in its state of rest or of uniform motion. Area moment of inertia is considered only for plane figures for which the mass is assumed to be negligible. It is essentially a measure of resistance to bending, and is applied while dealing with the deflection or deformation of members in bending.

The mass moment of inertia pertains only to solid bodies having mass. It gives a measure of the resistance that body offers to change in angular velocity and accordingly is used in conjunction with rotation of rigid bodies.

8.1. MOMENT OF INERTIA AND RADIUS OF GYRATION

Moment of inertia (*MOI*) of any lamina is the second moment of all elemental areas dA comprising the lamina. With refence to Fig. 8.1.

$$I_{xx} = \text{moment of inertia about } x\text{-axis} = \Sigma\,(y\,dA)\,y$$

where $y\,dA$ is the first moment of area dA about x-axis and $(y\,dA)\,y$ is the moment of first moment (called second moment) of area dA about x-axis.

$$I_{xx} = \Sigma\, y^2\, dA$$

Likewise:

$$I_{yy} = \text{moment of inertia about } y\text{-axis}$$
$$= \Sigma\, x^2\, dA$$

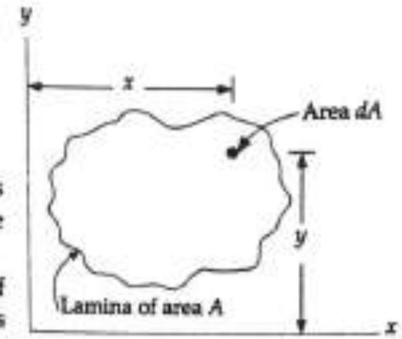


Fig. 8.1

Obviously moment of inertia of a section about an axis is prescribed by the cummulative product of area and square of the distance from that axis.

The units of moment of MOI are the fourth power of length. When the measurements are in mm, MOI has units of mm^4.

8.1.1. Parallel axis theorem

The moment of inertia of a plane lamina about any axis is equal to the sum of its MOI about a parallel axis through its centre of gravity G and the product of its area (mass) and the square of the distance between the two axes. With reference to Fig. 8.2.

$$I_{AA} = I_{gxx} + A\,h^2 \qquad \text{...(8.3)}$$

where I_{gxx} is MOI of the lamina about an axis x-x passing through its CG and I_{AA} is the MOI about any axis AA which is parallel to x-x and at a distance h from it.

Proof : The lamina consists of an infinite number of small elemental components parallel to the x-axis. Let one such elemental component of area dA be located at distance y from the x-axis. Obviously then its distance from the axis AA will be $(h + y)$.

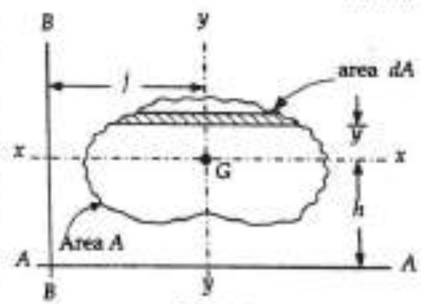


Fig. 8.2

Moment of inertia of the elemental component about axis AA will be

$$= dA\,(h + y)^2$$

Then moment of inertia of the entire lamina about axis AA

$$= \Sigma\, dA(h+y)^2$$
$$= \Sigma\, dA\,h^2 + \Sigma\, dA\,y^2 + \Sigma\, dA(2hy)$$
$$= h^2\,\Sigma\, dA + \Sigma\, dA\,y^2 + 2h\,\Sigma\, dA\,y$$

Now, $h^2\,\Sigma\, dA = Ah^2 \qquad (\because\ \Sigma\, dA = A)$

$\Sigma\, dA\,y^2$ = moment of inertia of the lamina about the axis x-x.

$\Sigma\, dA\,y = 0$ because $x - x$ is centroidal axis.

That gives :

$$I_{AA} = I_{xx} + Ah^2 \qquad \text{...(8.4)}$$

Also

$$I_{BB} = I_{yy} + Aj^2 \qquad \text{...(8.5)}$$

8.1.2. Perpendicular axis theorem

The moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about the two axes at right angles to each other and intersecting each other at the point where the perpendicular axis passes through it.

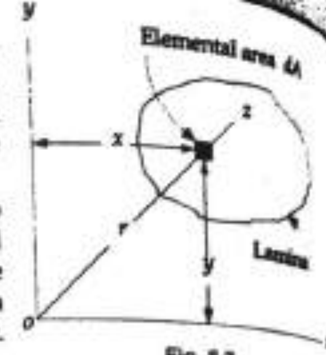


Fig. 8.3

Proof: With reference to Fig. 8.3, *ox* and *oy* are the two mutually perpendicular axes lying in the plane of lamina, and *oz* is the axis normal to the lamina and passing through *o* (the point of intersection of the axes *ox* and *oy*). The distance of an elemental component of area *dA* from *oz*, *i.e.*, from point *o* is *r*.

Moment of inertia of the elemental component about axis *oz*

$$= dA\, r^2 = dA\,(x^2 + y^2)$$

Moment of inertia of the elemental component about axis *oz*

$$I_{zz} = \Sigma\, dA\,(x^2 + y^2) = \Sigma\, dA\, x^2 + \Sigma\, dA\, y^2$$

But, $\Sigma\, dA\, x^2$ = moment of inertia of the lamina about the axis $oy = I_{yy}$

$\Sigma\, dA\, y^2$ = moment of inertia of the lamina about the axis $ox = I_{xx}$

$$\therefore \quad I_{zz} = I_{xx} + I_{yy} \quad \ldots(8.2)$$

8.1.3. Radius of gyration

If the entire area (or mass) of a lamina is considered to be concentrated at a point such that there is no change in the moment of inertia about a given axis, then distance of that point from the given axis is called the ***radius of gyration***.

The relation between radius of gyration *k* and moment of inertia *I* can be put in the form

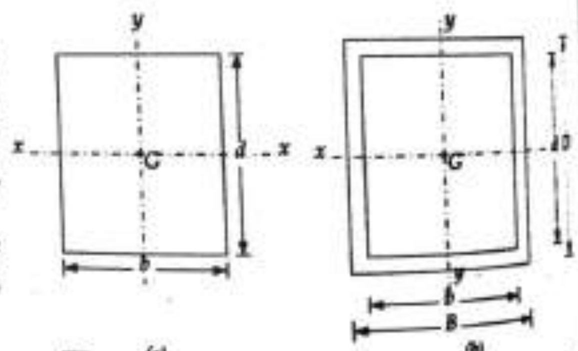


Fig. 8.4

$$I = Ak^2\,;\quad k = \sqrt{\frac{I}{A}} \quad \ldots(8.3)$$

Apparently the radius of gyration of a lamina is the square root of the ratio of its moment of inertia to its area.

The moment of inertia of common standard sections are presented below:

(*i*) For a rectangular section with breadth *b* and depth *d* (Fig. 8.4 *a*).

$$I_{xx} = \frac{bd^3}{12}\,;\quad I_{yy} = \frac{db^3}{12}$$

For a hollow rectangular section (Fig. 8.4 *b*)

$$I_{xx} = \frac{BD^3 - bd^3}{12}\,;\quad I_{yy} = \frac{DB^3 - db^3}{12}$$

(vi) For a triangular section (Fig. 8.5)

$$I_{xx} = \frac{bh^3}{36}$$

$$I_{AB} = \frac{bh^3}{12} \qquad ...(8.7)$$

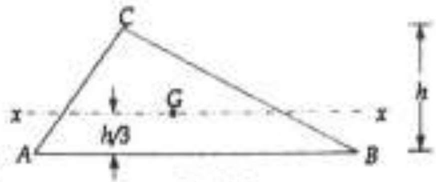


Fig. 8.5

(vii) For a circular section of diameter d (Fig. 8.6a)

$$I_{xx} = I_{yy} = \frac{\pi}{64} d^4$$

If zz is the axis perpendicular to lamina and passing through CG, then

$$I_{zz} = I_{xx} + I_{yy} = \frac{\pi}{32} d^4$$

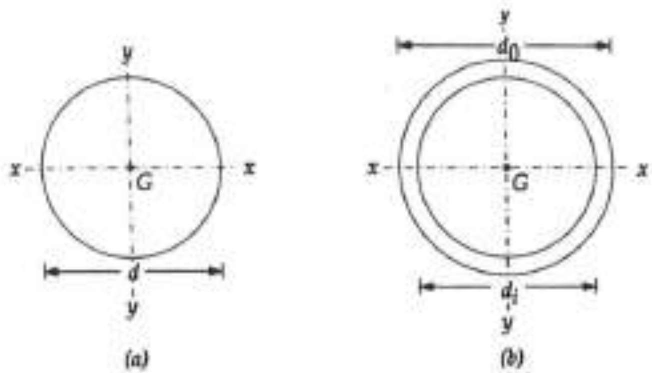


Fig. 8.6

For a hollow circular section with outer diameter d_0 and inner diameter d_i

$$I_{xx} = I_{yy} = \frac{\pi}{64}\left(d_0^4 - d_i^4\right)$$

If zz is the axis perpendicular to the plane of lamina and passing through CG then

$$I_{zz} = I_{xx} + I_{yy} = \frac{\pi}{32}\left(d_0^4 - d_i^4\right)$$

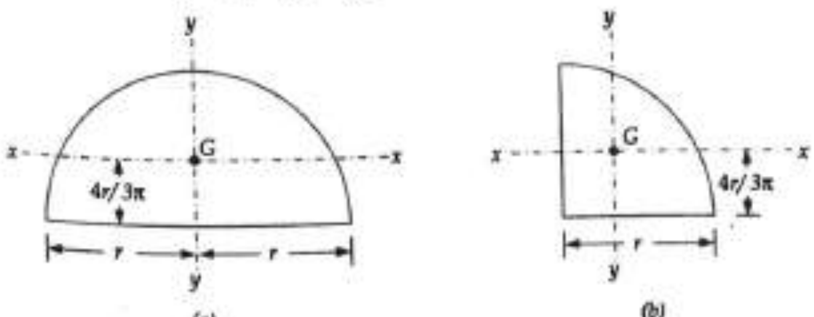


Fig. 8.7

(iv) For a semi-circle (Fig. 8.7 *a*)

$$I_{xx} = 0.11\, r^4 \text{ and}$$

$$I_{yy} = \frac{\pi r^4}{8}$$

For a quarter circle (Fig. 8.7*b*)

$$I_{xx} = I_{yy} = 0.055\, r^4$$

(v) For an ellipse (Fig 8.8),

$$I_{xx} = \frac{\pi a b^3}{4}, I_{yy} = \frac{\pi b a^3}{4}$$

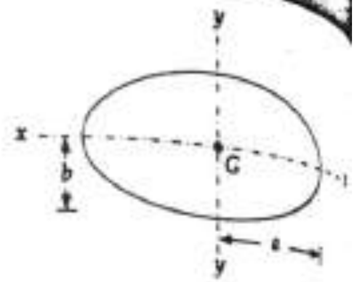


Fig. 8.8

8.2. MOMENT OF INERTIA OF LAMINAE OF DIFFERENT SHAPES

8.2.1. Rectangular lamina

Consider a rectangular lamina *ABCD* of width *b* and depth *d*. Let *xx* and *yy* be the axis which pass through the centroid of the area and are parallel to the sides of the lamina. The centroid lies at the mid point of the width as well as the depth.

Consider a small strip of thickness *dy* located at a distance *y* from the axis *xx*.

Area of the elemental strip = $b\,dy$

Moment of inertia of the elemental component about the axis *xx*,

$$= dA \times y^2 = b\,dy \times y^2 = by^2\,dy$$

Moment of inertia of the entire lamina about the axis *xx*,

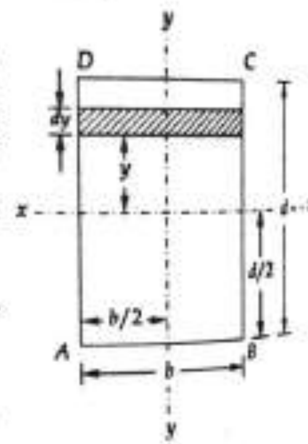


Fig. 8.9

$$I_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} by^2\,dy = b\left|\frac{y^3}{3}\right|_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right] = \frac{bd^3}{12}$$

Similarly the moment of inertia of the lamina about the axis *yy* is

$$I_{yy} = \frac{db^3}{12}$$

Let I_{AB} be the moment of inertia of the lamina about its bottom face *AB*. Then from the theorem of parallel axis

$$I_{AB} = I_{xx} + Ah^2 = \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2 = \frac{bd^3}{12} + \frac{bd^3}{4} = \frac{bd^3}{3}$$

Similarly the moment of inertia of the lamina about the face *AD* would be

$$I_{AD} = \frac{db^3}{3}$$

From the theorem of perpendicular axis, the polar moment of inertia I_p of the lamina is

$$I_p = I_{xx} + I_{yy} = \frac{bd^3}{12} + \frac{db^3}{12} \quad ...(8.10)$$

The polar moment of inertia is the inertia about the polar axis: an axis which passes through the centroid of the lamina and is normal to it.

For a rectangular lamina ($B \times D$) with a rectangular hole ($b \times d$) made centrally (Fig. 8.10) the moment of inertia about any centroidal axis is

= MOI of bigger rectangle
   – MOI of smaller rectangle

Thus: $$I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12} \quad ...(8.11)$$

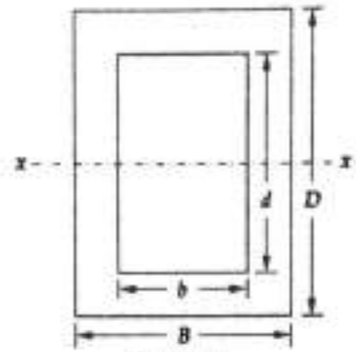


Fig. 8.10

8.2.2. Triangular lamina

Let ABC be the triangle of base width b and height h. Consider an elementary strip of width l, thickness dy and located at distance y from the base of the triangle. For this elemental strip,

area = $l\,dy$

moment of inertia of this strip about base BC

$= y^2\,dA = y^2 l\,dy$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express l in terms of y. For that, we have the following correlation from the similarity of triangles ADE and ABC,

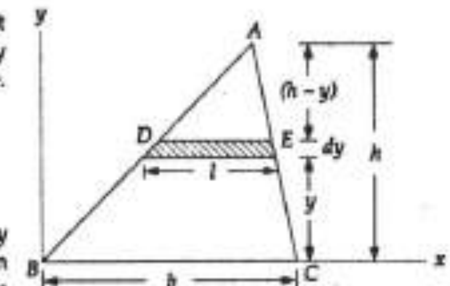


Fig. 8.11

$$\frac{l}{b} = \frac{h-y}{h}; \quad l = b\left(1 - \frac{y}{h}\right)$$

∴ Moment of inertia of the triangle about the base

$$I_{base} = \int_0^h y^2 b\left(1 - \frac{y}{h}\right)dy$$

$$= b\int_0^h\left(y^2 - \frac{y^3}{h}\right)dy = b\left|\frac{y^3}{3} - \frac{y^4}{4h}\right|_0^h = b\left(\frac{h^3}{3} - \frac{h^3}{4}\right) = \frac{bh^3}{12} \quad ...(8.12)$$

For a triangle, the centroidal axis I_{xx} is at a distance of $y_c = h/3$ from the base. Then from the theorem of parallel axis : $I_{base} = I_{xx} + Ay_c^2$, we have

$$\therefore \quad I_{xx} = I_{base} - Ay_c^2 = \frac{bh^3}{12} - \left(\frac{1}{2}bh\right) \times \left(\frac{h}{3}\right)^2 = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36} \quad ...(8.13)$$

8.2.3. Circular lamina

Consider an element of sides $r\,d\theta$ and dr within a circular lamina of radius R. Moment of this elemental area about the diametrical axis x-x,

$$= y^2\,dA = (r \sin\theta)^2 \times r d\theta\,dr = r^3 \sin^2\theta\,d\theta\,dr$$

∴ Moment of inertia of the entire circular lamina

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2\theta \, d\theta \, dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \frac{1-\cos 2\theta}{2} \, d\theta \, dr$$

$$= \int_0^R \frac{r^3}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} dr$$

$$= \int_0^R \frac{r^3}{2} (2\pi) \, dr = 2\pi \left|\frac{r^4}{8}\right|_0^R = \frac{\pi}{4} R^4$$

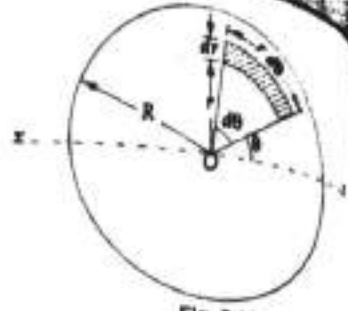


Fig. 8.12

If d is the diameter of the circular lamina, then

$$I_{xx} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{64} d^4 \qquad ...(8.14)$$

Likewise $\qquad I_{yy} = \frac{\pi}{64} d^4$

If zz is the axis through the centroid and normal to the plane of the lamina, then

$$I_{zz} = I_{xx} + I_{yy} = \frac{\pi}{64} d^4 + \frac{\pi}{64} d^4$$

$$= \frac{\pi}{32} d^4 \qquad ...(8.15)$$

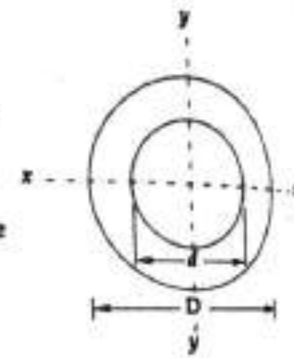


Fig. 8.13

The axis zz is called the polar axis and I_{zz} is referred to as the *polar moment of inertia*.

Polar moment of inertia has application in problems relating to torsion of cylindrical shafts and rotation of slabs.

For a circular lamina of diameter D with a central circular hole of diameter d (Fig. 8.13) the moment of inertia about any centroidal axis is

$$I_{xx} = I_{yy} = \frac{\pi}{64} (D^4 - d^4) \qquad ...(8.16)$$

The corresponding polar moment of inertia is

$$I_{zz} = I_p = \frac{\pi}{32} (D^4 - d^4) \qquad ...(8.17)$$

8.2.4. Semi-circular lamina

The moment of inertia of a circular lamina having diameter d about its diametrical axis AB

$$= \frac{\pi}{64} d^4$$

For the semi-circular lamina with AB as its base, the moment of inertia about AB would be

$$I_{AB} = \frac{1}{2} \times \left(\frac{\pi}{64} d^4\right) = \frac{\pi}{128} d^4$$

It can be obtained from first principles if the limit of integration is taken as 0 to π instead of 0 to 2π in the derivation of moment of inertia of a circular lamina about its diametrical axis. The

$$I_{AB} = \int_0^R \int_0^\pi r^3 \sin^2\theta \, d\theta \, dr = \frac{\pi}{8} R^4 = \frac{\pi}{128} d^4 \quad \text{...(8.18)}$$

The distance of centroidal axis xx of the semicircle from its base AB is

$$h = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

$$\text{Area of semicircle} = \frac{1}{2}\left(\frac{\pi}{4} d^2\right) = \frac{\pi d^2}{8}$$

From parallel axis theorem,

$$\frac{\pi d^4}{128} = I_{xx} + \frac{\pi d^2}{8} \times \left(\frac{2d}{3\pi}\right)^2 = I_{xx} + \frac{d^4}{18\pi}$$

$$\therefore \; I_{xx} = \frac{\pi d^4}{128} - \frac{d^4}{18\pi} = 0.00686\, d^4 = 0.11 R^4 \quad \text{...(8.19)}$$

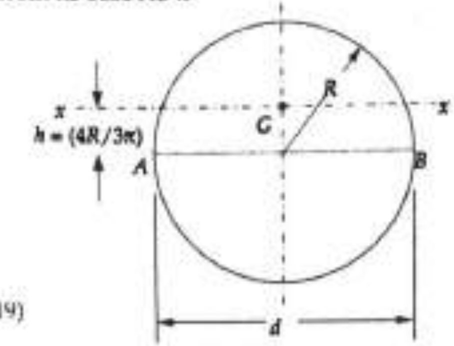


Fig. 8.14

8.2.5. Quarter of a circle

Reference Fig. 8.15, LAB is the quadrant of a circular lamina of diameter d. The moment of inertia of a quadrant equals 1/4th of the moment of inertia of the circular lamina.

$$\therefore \; I_{AB} = \frac{1}{4} \times \left(\frac{\pi}{64} d^4\right) = \frac{\pi}{256} d^4$$

It can be obtained from first principles if the limit of integration is taken as 0 to $\pi/2$ instead of 0 to 2π in the derivation of moment of inertia of a circular lamina about its diametral axis. That is

$$I_{AB} = \int_0^R \int_0^{\pi/2} r^3 \sin^2\theta \, d\theta \, dr$$

$$= \frac{\pi}{16} R^4 = \frac{\pi}{256} d^4 \quad \text{...(8.20)}$$

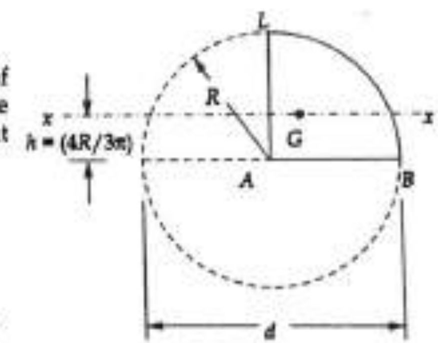


Fig. 8.15

The distance of the centroid of the quadrant LAB from AB is

$$h = \frac{4R}{3\pi} = \frac{2d}{3\pi}$$

$$\text{Area of quadrant} = \frac{1}{4} \times \left(\frac{\pi}{4} d^2\right) = \frac{\pi}{16} d^2$$

From parallel axis theorem,

$$I_{AB} = I_{xx} + Ah^2$$

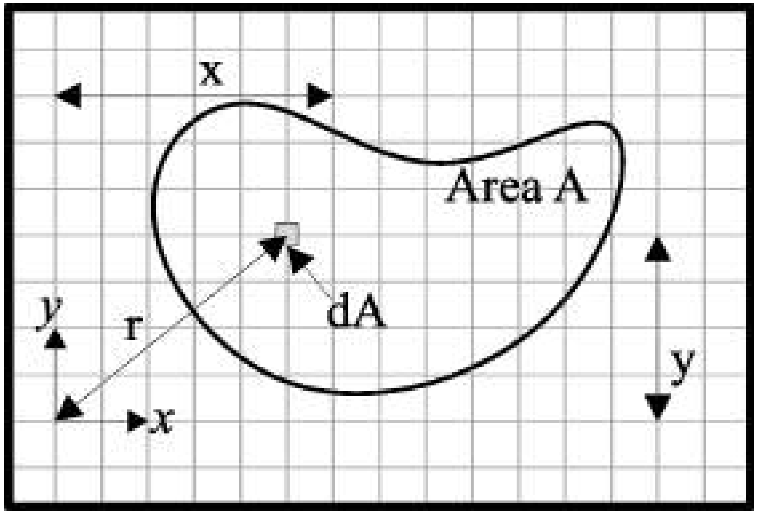
or

$$\frac{\pi d^4}{256} = I_{xx} + \frac{\pi d^2}{16} \times \left(\frac{2d}{3\pi}\right)^2 = I_{xx} + \frac{d^4}{36\pi}$$

$$\therefore \; I_{xx} = \frac{\pi d^4}{256} - \frac{d^4}{36\pi} = 0.00343\, d^4 = 0.055 R^4 \quad \text{...(8.21)}$$

The Moment of Inertia

Moment of inertia about x-axis:	$I_x = \int_A y^2 \, dA$
Moment of inertia about y-axis:	$I_y = \int_A x^2 dA$
Polar moment of inertia:	$J_O = \int_A r^2 dA$
Product of inertia:	$I_{xy} = \int_A x \, y \, dA$

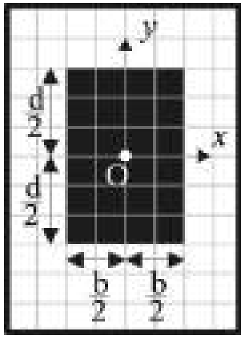


The moment of inertia is sometimes expressed in terms of the radius of gyration. The radius of gyration determines how the area is distributed around the centroid.

$$R_g = \sqrt{\frac{I}{A}}$$

Example 4

- Determine the moments of inertia about the x- and y-axes. Also, determine the polar moment of inertia.



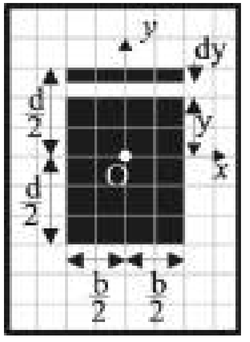
Example 4

$$I_x = \int_A y^2\, dA = \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 \left(b.dy\right) = b \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2\, dy$$

$$I_x = \frac{bd^3}{12}$$

$$I_y = \frac{db^3}{12}$$

$$J_O = I_x + I_y \qquad J_O = \frac{bd}{12}\left(b^2 + d^2\right)$$



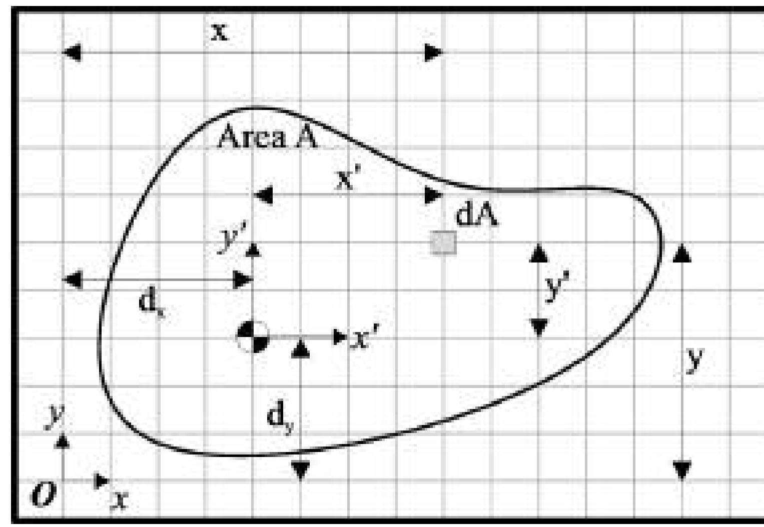
Parallel Axis Theorem

$$I_x = \int_A y^2 \, dA$$

$$I_x = \int_A \left(y' + d_y\right)^2 dA$$

$$= \int_A y'^2 dA + 2\int_A y' d_y dA + \int_A d_y{}^2 dA$$

$$= I_{x'} + 2d_y \int_A y' dA + d_y{}^2 \int_A dA$$



In the second term, is equal to zero as the x-axis passes through the centroid.

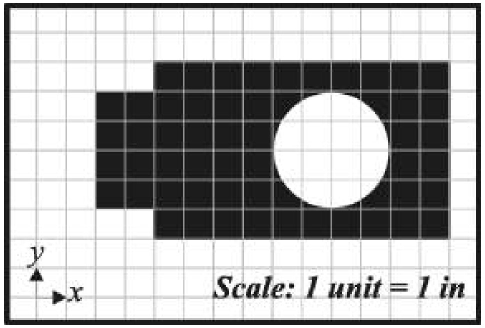
$$I_x = I_{x'} + Ad_y{}^2 \qquad I_y = I_{y'} + Ad_x{}^2$$

$$J_O = J_C + Ad^2$$

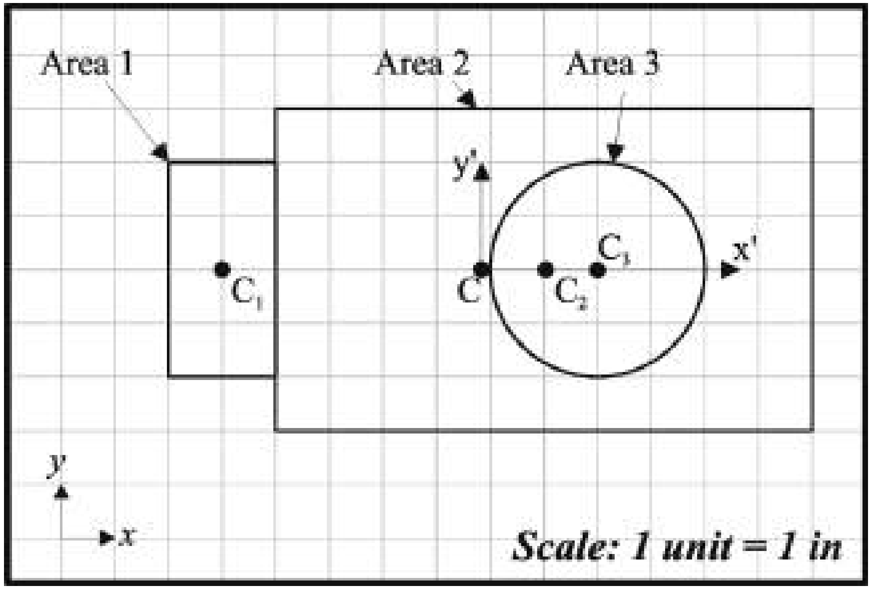
Example 5

- Determine the moments of inertia about the x′- and y′-axes about the centroid.

- Also, determine the polar moment of inertia.



Example 5



Example 5

Part	Dimen-sions	Area (sq. in)	x	y	(in³) x_iA_i	(in³) y_iA_i	$I_{x'}$	$I_{y'}$	d_x	d_y	Ad_x^2	Ad_y^2
Area 1	2″×4″	8	3	5	24	40	10.67	2.67	4.91	0	192.86	0
Area 2	10″×6″	60	9	5	540	300	180	500	1.09	0	71.286	0
Area 3	2″ radius	– 4π	10	5	– 40π	– 20π	-0.785	-0.785	2.09	0	-54.89	0
Summation		55.43			438.34	277.17	189.89	501.89			209.26	

Example 5

$$I_x = I_{x'} + \mathrm{Ad_y}^2 \qquad I_y = I_{y'} + \mathrm{Ad_x}^2$$

$$I_x = 189.89 \text{ in}^4 \qquad I_x = 711.15 \text{ in}^4$$

$$J_O = I_x + I_y$$

$$J_O = 901.04 \text{ in}^4$$

EXAMPLE 8.1

The moment of inertia of rectangular section beam about x-x and y-y axes passing though the centroid are 250×10^6 mm^4 and 40×10^6 mm^4 respectively. Calculate the size of the section.

Solution : Let b and d denote the breadth and depth respectively of the rectangular section beam. Then

$$I_{xx} = \frac{bd^3}{12}\ ; \quad 250 \times 10^6 = \frac{bd^3}{12} \qquad ...(i)$$

and

$$I_{yy} = \frac{db^3}{12}\ ; \quad 40 \times 10^6 = \frac{db^3}{12} \qquad ...(ii)$$

Dividing expression (*i*) by expression (*ii*)

$$5.25 = \left(\frac{d}{b}\right)^2 \quad \text{or} \quad \frac{d}{b} = 2.5$$

Substituting $d = 2.5\,b$ in expression (*i*), we get

$$\frac{b}{12}(2.5\,b)^3 = 250 \times 10^6$$

$$\text{or} \quad b^4 = \frac{250 \times 10^6 \times 12}{(2.5)^3} = 1.92 \times 10^8$$

That gives: $b = 117.7$ mm and $d = 2.5 \times 117.7 = 294.25$ mm

Therefore required size of the section is:

$$= 117.3 \text{ mm (breadth)} \times 294.25 \text{ mm (depth)}$$

EXAMPLE 8.2

Find the moment of inertia of a rolled steel joist girder of symmetrical I section shown in Fig. 8.16.

Solution : The areas of the three rectangles comprising the I-section are:

upper flange $A_1 = 6a \times a = 6a^2$

web $A_2 = 8a \times a = 8a^2$

lower flange $A_3 = 6a \times a = 6a^2$

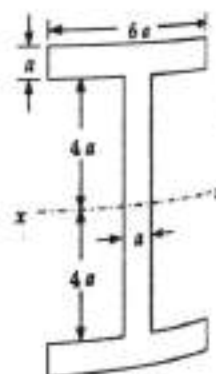


Fig. 8.16

MOI of upper flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 6a^2 \times \left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122\,a^4$$

MOI of web about x-axis $= \dfrac{a \times (8a)^3}{12} = \dfrac{128a^4}{3}$

MOI of lower flange about x-axis (using parallel axis theorem)

$$= \frac{6a \times a^3}{12} + 4a^2\left(4a + \frac{a}{2}\right)^2$$

$$= \frac{a^4}{2} + \frac{243a^4}{2} = 122\,a^4$$

∴ Total *MOI* of the given I-section about x-axis

$$= 122\,a^4 + \frac{128\,a^4}{3} + 122\,a^4$$

$$= \frac{860}{3}a^4$$

The *MOI* of the given I-section could also be worked out with reference to Fig. 8.17.

$$I_{xx} = I_{x1} - I_{x2}$$

$$= \frac{6a \times (10a)^3}{12} - \frac{5a \times (8a)^3}{12}$$

$$= 500\,a^4 - \frac{640}{3}a^4 = \frac{860}{3}a^4$$

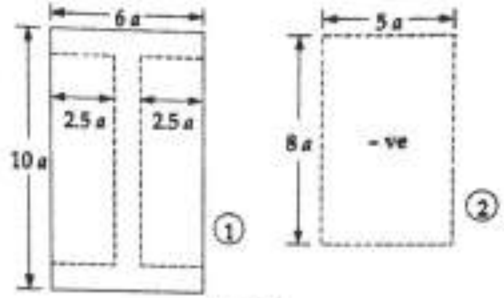


Fig. 8.17

EXAMPLE 8.3

Determine the moment of inertia of the T-section shown in Fig. 7.13 about an axis passing through the centroid and parallel to top most fibre of the section. Proceed to determine the moment of inertia about axis of symmetry and hence find out the radii of gyration.

Solution : From the calculations made in Example 7.10 the CG of the given T-section lies on the y-axis and at distance 43.71 mm from the top face of its flange

$$\bar{x} = 0 \quad \text{and} \quad \bar{y} = 43.71 \text{ mm}$$

Refering to this centroidal axis, the centroid of a_1 is (0.0, 38.71 mm) and that of a_2 is (0.0, 41.29 mm).

Moment of inertia of the section about centroid axis is

I_{xx} = *MOI* of area a_1 about centroidal axis + *MOI* of area a_2 about centroidal axis

$$= \left[\frac{160 \times 10^3}{12} + 1600 \times (38.71)^2\right] + \left[\frac{10 \times 150^3}{12} + 1500 \times (41.29)^2\right]$$

$$= 7780672 \text{ mm}^4$$

Similarly

$$I_{yy} = \frac{10 \times 160^3}{12} + \frac{150 \times 10^3}{12} = 3425833 \text{ mm}^4$$

The radius of gyration is given by $k = \sqrt{\frac{I}{A}}$

$$\therefore \quad k_{xx} = \sqrt{\frac{7780672}{3100}} = 50.1 \text{ mm}$$

$$k_{yy} = \sqrt{\frac{3425833}{3100}} = 34.24 \text{ mm}$$

EXAMPLE 8.4

Determine the moment of inertia of the area shown shaded in Fig. 8.18 about axis xx which coincides with the base edge *AB*.

Solution : The given section comprises the full rectangle *ABCD* minus the semi-circle *DEC*.

Moment of inertia of rectangle *ABCD* about *AB*

$$I_1 = I_{G1} + A_1 h_1^2$$

$$= \frac{2 \times 2.5^3}{12} + (2 \times 2.5) \times 1.25^2$$

$$= 2.604 + 7.812 = 10.416 \text{ cm}^4$$

Moment of inertia of semi-circle about AB

$$I_2 = I_{G2} + A_2 h_2^2$$

$$= 0.11 r^2 + \frac{1}{2} \pi r^2 \times \left(2.5 - \frac{4r}{3\pi}\right)^2$$

The parameter $\frac{4r}{3\pi}$ is the distance of centroid of semi-circle from DC.

$$\therefore \quad I_2 = 0.11 \times 1^2 + \frac{1}{2} \pi \times (1)^2 \times \left(2.5 - \frac{4 \times 1}{3\pi}\right)^2$$

$$= 0.11 + 6.76 = 6.87 \text{ cm}^4$$

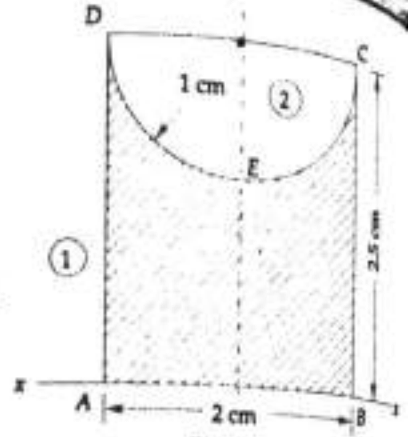


Fig. 8.18

$\therefore$ Moment of inertia of shaded area bout $AB = 10.416 - 6.87 = 3.546 \text{ cm}^4$

EXAMPLE 8.5

Determine the polar moment of inertia of the I-section shown in Fig. 8.19. Also make calculations for the radius of gyration with respect to x-axis and y-axis.

Solution: The I-section is symmetrical about y-axis and accordingly its CG lies at point G on the y-axis, i.e., $x = 0$. Further, the bottom fibre of lower flange has been chosen as reference axis to locate the centroid $\bar{y}$.

The areas and co-ordinates of centroids of the three rectangles comprising the given section are:

Lower flange: $a_1 = 10 \times 1 = 10 \text{ cm}^2$

$$y_1 = \frac{1}{2} = 0.5 \text{ cm}$$

Web: $a_2 = 12 \times 1 = 12 \text{ cm}^2$

$$y_2 = 1 + \frac{12}{2} = 7 \text{ cm}$$

Upper flange: $a_3 = 8 \times 18 \text{ cm}^2$

$$y_3 = 1 + 12 + \frac{1}{2} = 13.5 \text{ cm}$$

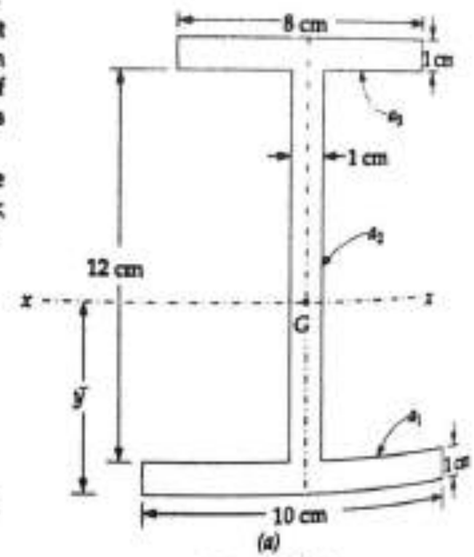


Fig. 8.19 (a)

Then: $\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$

$$= \frac{10 \times 0.5 + 12 \times 7 + 8 \times 13.5}{10 + 12 + 8} = \frac{5 + 84 + 108}{30} = 5.57 \text{ cm}$$

With reference to the centroidal axes, the centroid of the lower flange, web and upper flange are (0, 5.07), (0, 1.43) and (0, 7.93) respectively.

Moment of inertia of the I-section about centroidal axis is

= *MOI* of area a_1 about centroidal axis + *MOI* of area a_2 about centroidal axis + *MOI* of area a_3 about centroidal axis.

$$= (I_{G1} + A_1h_1^2) + (I_{G2} + A_2h_2^2) + (I_{G3} + A_3h_3^2)$$

$$= \left[\frac{10 \times 1^3}{12} + 10 \times (5.07)^2\right] + \left[\frac{1 \times 12^3}{12} + 12 \times (1.43)^2\right] + \left[\frac{8 \times 1^3}{12} + 8 \times (7.93)^2\right]$$

$$= (0.833 + 257.05) + (144 + 24.54) + (0.67 + 503.08)$$

$$= 930.17 \text{ cm}^4$$

and
$$I_{yy} = \frac{1 \times 10^3}{12} + \frac{12 \times 1^3}{12} + \frac{1 \times 8^3}{12}$$

$$= 83.33 + 1 + 42.64 = 127 \text{ cm}^4$$

Polar moment of inertia $= I_{xx} + I_{yy} = 930.17 + 127$

$$= 1057.17 \text{ cm}^4$$

(ii) The radius of gyration is given by: $k = \sqrt{\frac{I}{A}}$

$$\therefore \quad k_{xx} = \sqrt{\frac{930.17}{30}} = 5.567 \text{ cm}$$

$$k_{yy} = \sqrt{\frac{127}{30}} = 2.057 \text{ cm}$$

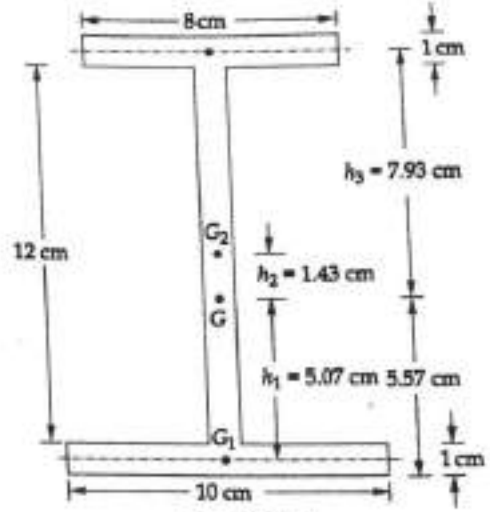


Fig. 8.19 (b)

EXAMPLE 8.6

Determine the moment of inertia about centroidal axes x-x and y-y of the channel section shown in Fig. 8.20.

Solution : The section is divided into three rectangles with areas

$$A_1 = 10 \times 1.5 = 15 \text{ cm}^2$$
$$A_2 = (40 - 1.5 - 1.5) \times 1 = 37 \text{ cm}^2$$
$$A_3 = 10 \times 1.5 = 15 \text{ cm}^2$$
$$\Sigma A = A_1 + A_2 + A_3$$
$$= 15 + 37 + 15 = 67 \text{ cm}^2$$

The given section is symmetrical about the horizontal axis passing through the centroid of rectangle A_2.

The distance of the centroid of the section with reference to section 1 – 1 is

$$\frac{\Sigma Ax}{\Sigma A} = \frac{(15 \times 5) + \left(37 \times \frac{1}{2}\right) + (15 \times 5)}{67} = 2.51 \text{ cm}$$

With reference to the centroidal axes x-x and y-y, the centroids of the rectangles are:

$$\left[(5 - 2.51), \left(\frac{40}{2} - \frac{1.5}{2}\right)\right] \quad \text{or} \quad (2.49, 19.25) \text{ for rectangle } A_1$$

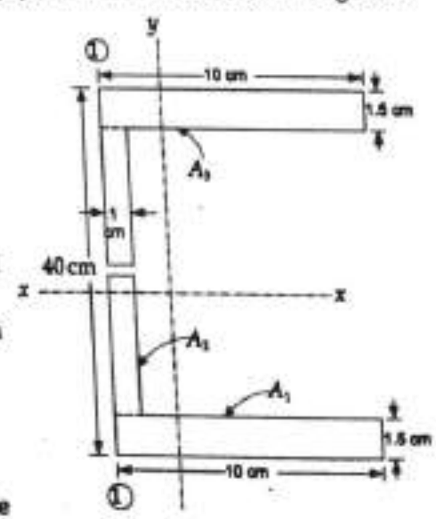


Fig. 8.20

$\left[\left(2.51-\frac{1}{2}\right), 0.0\right]$ or (2.01, 0.0) for rectangle A_2

$\left[(5-2.51), \left(\frac{40}{2}-\frac{1.5}{2}\right)\right]$ or (2.49, 19.25) for rectangle A_3

Then invoking parallel axis theorem, the moment of inertia of areas A_1, A_2, and A_3 about x-x,

$$I_{xx} = \left[\frac{10 \times 1.5^3}{12} + 15 \times 19.25^2\right] + \left[\frac{1 \times 37^3}{12}\right] + \left[\frac{10 \times 1.5^3}{12} + 15 \times 19.25^2\right]$$

$$= (2.812 + 5558.437) + (4221.083) + (2.812 + 558.437) = 155343.58 \text{ cm}^4$$

Similarly,

$$I_{yy} = \left[\frac{1.5 \times 10^3}{12} + 15 \times 2.49^2\right] + \left[\frac{37 \times 1^3}{12}\right] + \left[\frac{1.5 \times 10^3}{12} + 15 \times 2.49^2\right]$$

$$= (125 + 93.00) + 3.08 + (125 + 93.00) = 439.08 \text{ cm}^4$$

EXAMPLE 8.7

Determine I_{xx} and I_{yy} of the cross-section of a cast iron beam shown in Fig. 8.21.

Solution : The *MOI* of the given sections can be worked out by looking it as a rectangle minus two semi-circles.

$$\therefore \quad I_{xx} = I_{xx} \text{ of rectangle} - I_{xx} \text{ of circular part}$$

$$= \frac{bd^3}{12} - \frac{\pi r^4}{4}$$

$$= \frac{12 \times 15^3}{12} - \frac{\pi \times 5^4}{4}$$

$$= 33.75 - 490.87$$

$$= 2884.13 \text{ cm}^4$$

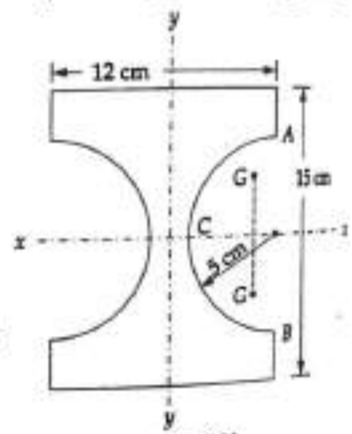


Fig. 8.21

Likewise : $I_{yy} = I_{yy}$ of rectangle $- I_{yy}$ of semi-circular parts

$$I_{yy} \text{ of rectangle} = \frac{15 \times 12^3}{12} = 2160 \text{ cm}^4$$

For the semi-circular part ACB;

MOI about its diameter, $I_{AB} = \frac{1}{2} \times \frac{\pi \times 5^4}{4} = 245.43 \text{ cm}^4$

Distance of its *CG* from the diameter,

$$h = \frac{4r}{3\pi} = \frac{4 \times 5}{3\pi} = 2.12 \text{ cm}$$

$$\text{Area } A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 5^2 = 39.27 \text{ cm}^2$$

From the correlation, $I_{AB} = I_{GG} + Ah^2$, the moment of inertia of semi-circular part about its centroidal axis

$$I_{GG} = 245.43 - 39.27 \times (2.12)^2 = 68.94 \text{ cm}^4$$

Again from the parallel axis theorem,

$$I_{yy} = I_{GG} + Ah_1^2$$

where h_1 = distance between axis and G-axis, = 6 − 2.12 = 3.88 cm

$$\therefore \quad I_{yy} = 68.94 + 39.27 \times 3.88^2 = 660.13 \text{ cm}^4$$

Since there are two semi-circular parts,

I_{yy} for two semi-circular parts = 2 × 660.13 = 1320.26 cm⁴

$\therefore I_{yy}$ for the section = 2160 − 1320.26 = 839.74 cm⁴

EXAMPLE 8.8

Determine the moments of inertia about the x and y centroidal axis of a beam whose cross-sectional area is as shown in Fig. 8.22. All dimensions are in cm.

Solution: The given section has been divided into three segments marked 1, 2 and 3

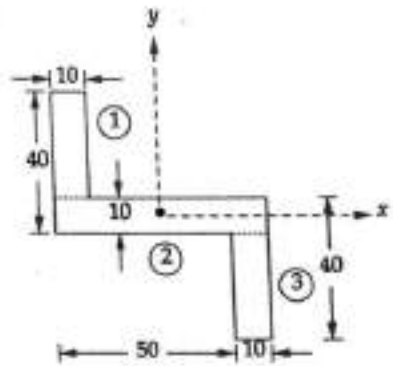


Fig. 8.22

$$(I_{xx})_1 = I_{G_1} + A_1 h_1^2 = I_{G_1} + A_1(\bar{y} - y_1)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)(35 - 15)^2$$

$$= 1.425 \times 10^5 \text{ cm}^4$$

$$(I_{xx})_2 = I_{G_2} + A_2 h_2^2 = I_{G_2} + A_2(\bar{y} - y_2)^2$$

$$= \frac{1}{12} \times 10^3 \times 60 + (60 \times 10) \times 0$$

$$= 0.05 \times 10^5 \text{ cm}^4$$

$$(I_{xx})_3 = I_{G_3} + A_3 h_3^2 = I_{G_3} + A_3(\bar{y} - y_3)^2$$

$$= \frac{1}{12} \times 10 \times 30^3 + (30 \times 10)(35 - 15)^2 = 1.425 \times 10^5 \text{ cm}^4$$

$$\therefore \quad I_{xx} = 1.425 \times 10^5 + 0.05 \times 10^5 + 1.425 \times 10^5 = 2.90 \times 10^5 \text{ cm}^4$$

$$(I_{yy})_1 = I_{G_1} + A_1 h_1^2 = I_{G_1} + A_1(\bar{x} - x_1)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2 = 1.9 \times 10^5 \text{ cm}^4$$

$$(I_{yy})_2 = I_{G_2} + A_2 h_2^2 = I_{G_2} + A_2(\bar{x} - x_2)^2$$

$$= \frac{1}{12} \times 60^3 \times 10 + (60 \times 10) \times 0 = 1.8 \times 10^5 \text{ cm}^4$$

$$(I_{yy})_3 = I_{G_3} + A_3 h_3^2 = I_{G_3} + A_3(\bar{x} - x_3)^2$$

$$= \frac{1}{12} \times 30 \times 10^3 + (30 \times 10) \times (30 - 5)^2$$

$$= 1.9 \times 10^5 \text{ cm}^4$$

$$\therefore \quad I_{yy} = 1.9 \times 10^5 + 1.8 \times 10^5 + 1.9 \times 10^5 = 5.6 \times 10^5 \text{ cm}^4$$

EXAMPLE 8.9

Find the moment of inertia about the centroid horizontal axis of the area shown shaded in Fig. 8.23. The section consists of triangle ABC, semi-circle on BC as diameter, and a circular hole of diameter 4 cm with its centre on BC.

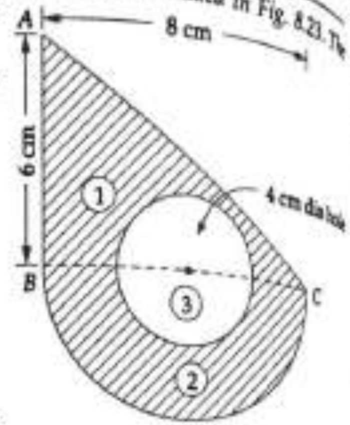


Fig. 8.23

Solution : The shaded area can be considered as a triangle (1), semicircle (2) and a circular hole (3)

Location of Centroid : For the triangular element,

$$a_1 = \frac{1}{2} \times 6 \times 8 = 24 \text{ cm}^2$$

y_1 (distance of centroid from BC)

$$= \frac{6}{3} = 2 \text{ cm}$$

For the semi-circular element,

$$a_2 = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 4^2 = 25.12 \text{ cm}^2$$

$$y_2 \text{(distance of centroid from } BC) = \frac{-4r}{3\pi} = \frac{-4 \times 4}{3\pi} = -1.7 \text{ cm}$$

The negative sign stems from the fact that it lies below BC.

For the circular hole

$$a_3 = \pi r^2 = \pi \times 2^2 = 12.56 \text{ cm}^2 \text{ (this area is removed)}$$

$$y_3 = 0 \text{ (centroid lies on } BC)$$

∴ Distance of the centroid of the shaded area from BC

$$= \frac{\Sigma ay}{\Sigma a} = \frac{a_1 y_1 + a_2 y_2 - a_3 y_3}{a_1 + a_2 - a_3} = \frac{24 \times 2 + 25.12 \times (-1.7) - 12.56 \times 0}{24 + 25.12 - 12.56} = 0.145 \text{ cm}$$

Moment of Inertia

I_1 = moment of inertia of triangle ABC about base BC

$$= \frac{1}{12} bh^3 = \frac{1}{12} \times 8 \times 6^3 = 144 \text{ cm}^4$$

I_2 = moment of inertia of semi-circle about BC

$$= \frac{1}{128}\pi d^4 = \frac{1}{128} \times \pi \times 8^4 = 100.48 \text{ cm}^4$$

I_3 = moment of inertia of circular hole about BC

$$= \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 4^4 = 12.56 \text{ cm}^2$$

∴ Moment of inertia of the shaded area about BC

$$= 144 + 100.48 - 12.56 = 231.92 \text{ cm}^4$$

Area of the shaded portion = 24 + 25.12 − 12.56 = 36.56 cm

Invoking parallel axis theorem,

Moment of inertia of shaded area about centroidal axis

$$I_G = I_{BC} - A h^2 = 231.92 - 36.56 \times 0.145^2 = 231.15 \text{ cm}^4$$

EXAMPLE 8.10 *Find the horizontal centroidal moment of inertia of the lamina ABCDEFG shown in Fig. 8.24.*

Solution: The composite figure is divided into the following simple figures:

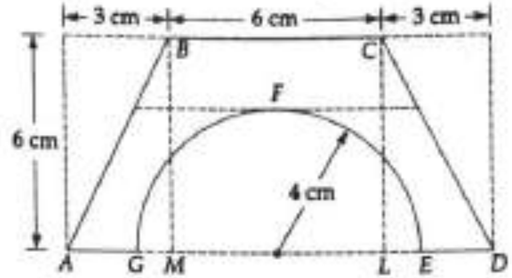


Fig. 8.24

1. A triangle ABM: $A_1 = \frac{1}{2} bh$

$$= \frac{1}{2} \times 3 \times 6 = 9 \text{ cm}^2$$

$$I_{G_1} = \frac{bh^3}{36} = \frac{3 \times 6^3}{36} = 18 \text{ cm}^4$$

y_1 (centroidal distance from line AD) $= \frac{6}{3} = 2$ cm

2. A square BCLM: $A_2 = 6 \times 6 = 36 \text{ cm}^2$

$$I_{G_2} = \frac{6 \times 6^3}{12} = 108 \text{ cm}^4$$

y_2 (centroidal distance from line AD) $= \frac{6}{2} = 3$ cm

3. A triangle CDL: $A_3 = \frac{1}{2} bh = \frac{1}{2} \times 3 \times 6 = 9 \text{ cm}^2$

$$I_{G_3} = \frac{bh^3}{36} = \frac{3 \times 6^3}{36} = 18 \text{ cm}^4$$

y_3 (centroidal distance from line AD) $= \frac{6}{3} = 2$ cm

4. A semi-circle GFE to be subtracted: $A_4 = \frac{\pi r^2}{2} = \frac{\pi \times 4^2}{2} = 25.12 \text{ cm}^2$ (– ve)

$$I_{G_4} = 0.11\, r^4 = 0.11 \times 4^4 = 28.16 \text{ cm}^4$$

y_4 (centroidal distance from line AD) $= \frac{4r}{3\pi} = \frac{4 \times 4}{3\pi} = 1.698$ cm

For the composite section

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \frac{(9 \times 2) + (36 \times 3) + (9 \times 2) - (25.12 \times 1.698)}{9 + 36 + 9 - 25.12}$$

$$= \frac{18 + 108 + 18 - 42.65}{28.88} = 3.51 \text{ cm}$$

Then

$$I_{xx} = I_{xx1} + I_{xx2} + I_{xx3} - I_{xx4}$$

$$= [18 + 9 \times (3.51 - 2)^2] + [108 + 36 \times (3.51 - 3)^2]$$

$$+ [18 + 9 \times (3.51 - 2)^2] - [28.16 + 25.12\,(3.51 - 1.698)^2]$$

The above relation has been written by applying the parallel axis theorem:

$$I_{xx} = I_{GG} + Ah^2$$

$\therefore$ $I_{xx} = 38.52 + 117.36 + 38.52 - 110.64 = 83.76 \text{ cm}^4$

8.3. MASS MOMENT OF INERTIA

The mass moment of inertia of a body about a particular axis is defined as *"the product of the mass and the square of the distance between the mass centre of the body and the axis"*.

The mass moment of inertia is an important term for the study of the rotational motion of a rigid body. It gives a measure of the resistance that the body offers to change in angular velocity.

The body can be considered to be split up into small masses. Let

$m_1, m_2 \ldots m_n$ be the masses of the various elements of the body

and $r_1, r_2 \ldots r_n$ be the distances of the above mentioned elements from the axis about which mass moment of inertia is to be determined.

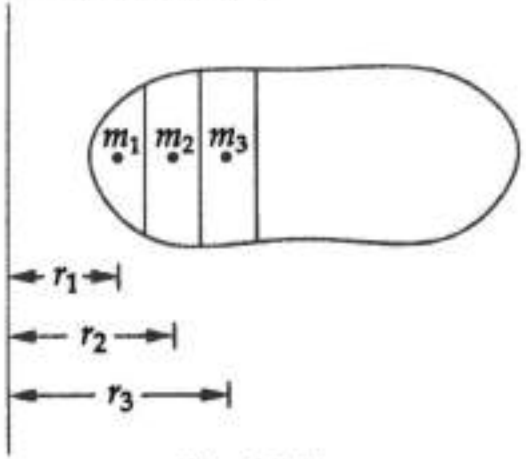


Fig. 8.30

The mass moment of inertia of the body can be written as

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots m_n r_n^2$$
$$= \Sigma mr^2$$

The summation of a large number of terms in the above expression can be replaced by integration. Consider a small mass dm rotating about and at distance r from the axis of rotation, then

$$I = \int r^2 dm$$

The radius of gyration k of the body with respect to the prescribed axis is defined by the relation

$$I = k^2 M \, ; \quad k = \sqrt{\frac{I}{M}}$$

where M is the mass of the body.

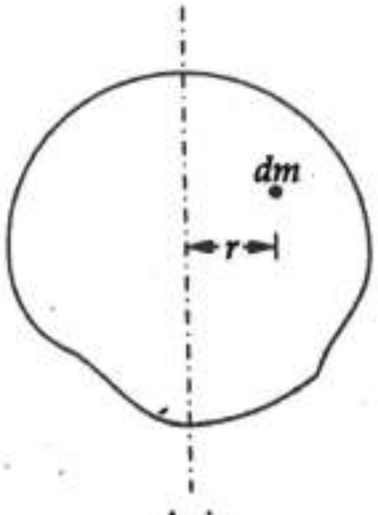


Fig. 8.31

8.4. MASS MOMENT OF INERTIA FOR SPECIFIED CASES

8.4.1. Thin uniform rod

Fig. 8.33 shows a thin uniform rod of length l. Consider a small elemental length dx located at distance x from the centroidal axis y-y normal to the rod.

Mass of the segmental element

$$dm = m\,dx$$

where m is the mass per unit of length.

Mass moment of inertia of the element about axis y-y

$$= dm\,x^2 = mx^2\,dx$$

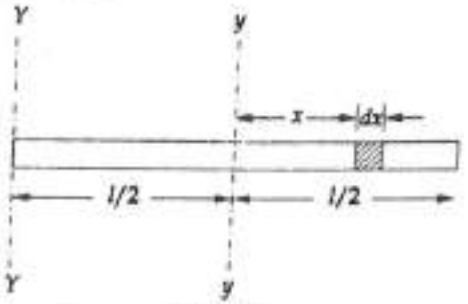


Fig. 8.33

The mass moment of inertia of the entire rod about axis y-y can be worked out by integrating the above expression between the limits $-\frac{l}{2}$ to $\frac{l}{2}$. That is

$$I_{yy} = m\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2\,dx = m\left|\frac{x^3}{3}\right|_{-\frac{l}{2}}^{\frac{l}{2}} = m\left(\frac{l^3}{24}+\frac{l^3}{24}\right)$$

$$= \frac{ml^3}{12} = \frac{Ml^2}{12}$$

where $M = ml$ is the mass of the whole rod.

If it is required to determine the mass moment of inertia of the rod about axis YY at the left end of the rod, we can use the parallel axis theorem

$$I_{YY} = I_{yy} + Mh^2$$

where h is the distance between the axis YY and the centroidal axis yy.

$$I_{YY} = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

8.4.2. Rectangular plate

Figure 8.34, shows a rectangular plate of width b, depth d and uniform thickness t. If ρ is the density of the plate material, then mass of the plate

$$M = \text{density} \times \text{volume}$$
$$= \rho\,b\,t\,d$$

Consider an elemental strip of depth dy located at distance y from the centroidal axis xx

mass of the elementa strip $dm = \rho\,b\,t\,dy$

mass moment of inertial of the strip about axis xx

$$= dm\,y^2 = \rho\,b\,t\,y^2 dy$$

The moment of inertia for the entire mass of plate about axis xx can be worked out by integrating the above expression between the limits $-\frac{d}{2}$ to $\frac{d}{2}$. That is

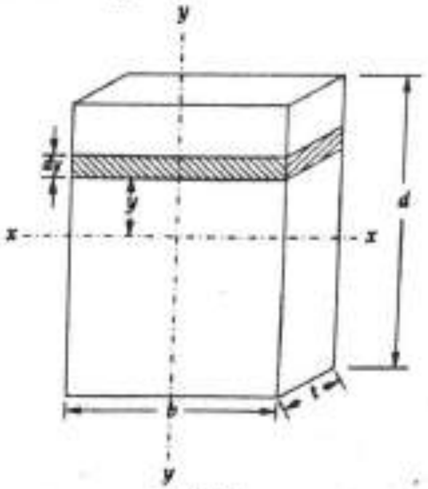


Fig. 8.34

$$I_{xx} = \rho bt \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 dy = \rho bt \left|\frac{y^3}{3}\right|_{-\frac{d}{2}}^{\frac{d}{2}}$$

$$= \rho bt\left(\frac{d^3}{24} + \frac{d^3}{24}\right) = (\rho bt\,d) \times \frac{d^2}{12}$$

$$= \frac{1}{12} Md^2 \qquad \ldots(8.23\ a)$$

Likewise the mass moment of inertia of the rectangular plate about the centroidal axis yy is

$$I_{yy} = \frac{1}{12} Mb^2 \qquad \ldots(8.23\ b)$$

From perpendicular axis theorem, the moment of inertia about axis zz is

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{1}{12} Md^2 + \frac{1}{12} Mb^2 = \frac{1}{12} M(d^2 + b^2) \qquad \ldots(8.24)$$

8.4.3. Triangular plate

Figure 8.35 shows a triangular plate of base width b, height h and uniform thickness t. If ρ is the density of the plate material, then

mass of the plate,

$$M = \text{density} \times \text{volume}$$

$$= \text{density} \times (\text{area} \times \text{thickness})$$

$$= \rho \times \frac{1}{2} b\,h\,t$$

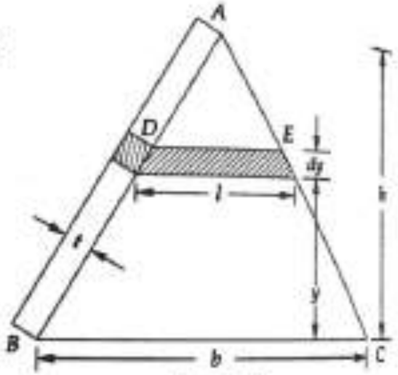


Fig. 8.35

Consider an elemental strip (assumed rectangle) of width l and depth dy located at distance y from the base line.

mass of the elemental strip $dm = \rho\, l\, t\ dy$

mass moment of inertia of the strip about base

$$= dm\, y^2 = \rho\, l\, t\, y^2 dy$$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express l in terms of y. For that we have the following correlation from the similarity of triangles ADE and ABC,

$$\frac{l}{b} = \frac{h-y}{h}, \quad l = b\,\frac{h-y}{h}$$

$\therefore$ mass moment of inertia of the triangular plate about base line

$$I_{base} = \int_0^h \rho b\left(\frac{h-y}{h}\right) t\, y^2 dy$$

$$= \frac{\rho bt}{h} \int_0^h (h-y) y^2 dy$$

$$= \frac{\rho bt}{h}\left|\frac{h\,y^3}{3} - \frac{y^4}{4}\right|_0^h = \frac{\rho bt}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{\rho bt\,h^3}{12}$$

$$= \frac{\rho bht}{2} \times \frac{h^2}{6} = \frac{1}{6} M h^2 \qquad ...(8.25)$$

8.4.4. Circular lamina

Figure 8.36 shows a thin circular plate of radius R and uniform thickness t. If ρ is the density of the plate material, then

mass of the plate M = density × volume

= density × (area × thickness) = $\rho \pi R^2 t$

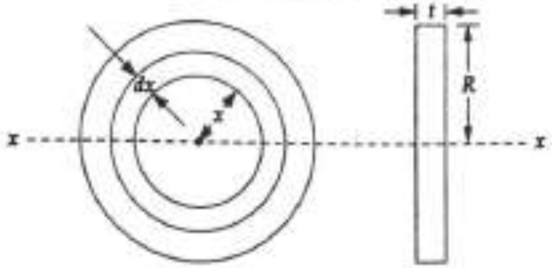


Fig. 8.36

Consider an elementary ring of radius r and width dr.

mass of elemental ring $dm = \rho\,[\pi(r + dr)^2 - \pi r^2]t$

$= \rho\,(2\pi r\,dr)\,t = 2\pi t \rho r\,dr$

Mass moment of inertia of this elementary ring about the polar axis zz

$= dm\,r^2 = 2\pi t \rho r^3 dr$

Mass moment of inertia of the circular plate about polar axis zz

$$= 2\pi t \rho \int_0^R r^3\,dr = 2\pi t \rho \left|\frac{r^4}{4}\right|_0^R$$

$$= \rho \pi R^2 t \times \frac{R^2}{2} = \frac{1}{2} M R^2 \qquad ...(8.26)$$

where $M = \rho \pi R^2 t$ is the mass of the circular lamina

Invoking the theorem of perpendicular axis, the mass moment of inertia of a circular lamina about xx or yy axis is

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{1}{4} M R^2 \qquad ...(8.27)$$

8.4.5. Solid sphere

Figure 8.37 shows a solid sphere of radius R with O as centre. If ρ is the density of the material of the sphere, then

mass of the sphere = density × volume

$$= \rho \times \frac{4}{3} \pi R^3$$

Let attention be focussed on a thin disc AB of thickness dx and at radius x from the centre.

radius of the disc $y = \sqrt{R^2 - x^2}$

mass of the disc dm

$$= \rho \times \pi y^2 dx$$
$$= \rho\pi (R^2 - x^2)\, dx$$

Mass moment of inertia of this elementary disc about the polar axis zz

$$dm\, y^2 = \rho\pi (R^2 - x^2)\, dx \times (R^2 - x^2)$$
$$= \rho\pi (R^2 - x^2)^2 dx$$
$$= \rho\pi (R^4 + x^4 - 2R^2x^2)\, dx$$

The mass moment of inertia of the whole sphere can be worked out by integrating the above expression between the limits $-R$ to R.

∴ Mass moment of inertia of the sphere about polar axis zz

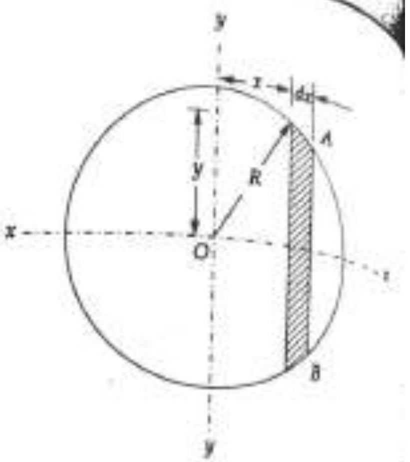


Fig. 8.37

$$I_{zz} = \rho\pi \int_{-R}^{R} (R^4 + x^4 - 2R^2x^2)\, dx$$

$$I_{zz} = \rho\pi \left| R^4 x + \frac{x^5}{5} - 2R^2 \frac{x^3}{3} \right|_{-R}^{R} = \frac{16\rho\pi R^5}{15} = \frac{4}{5} MR^2 \quad \text{...(8.29)}$$

where $M = \frac{4}{3}\rho\pi R^3$ is the mass of the solid sphere

Invoking the theorem of perpendicular axis, the mass moment of inertia of a solid sphere about xx or yy axis is,

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{2}{5} MR^2$$

8.4.6. Solid cylinder

Figure 8.38 shows a solid cylinder of radius R and height h. If ρ is the density of the material of the cylinder, then

mass of the cylinder = density × volume

$$M = \rho \times \pi R^2 h$$

Consider a thin disc of thickness dy located at distance y from the centroidal axis xx.

mass of the elemental disc, $dm = \rho \times \pi R^2 dy$

It may be recalled that mass moment of inertia of a circular lamina about its diametral axis is given by

$$= \frac{1}{4} MR^2$$

∴ mass moment of inertia of the elemental disc about its diameteral axis is

$$I_{dia} = \frac{1}{4} dm\, R^2$$

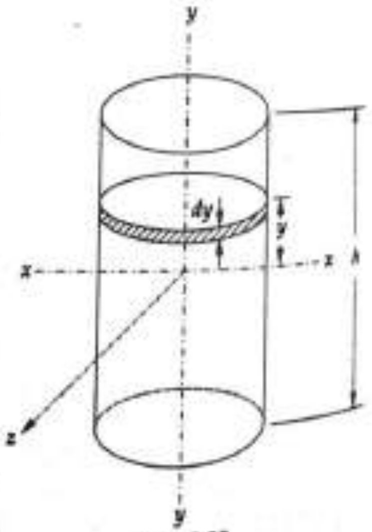


Fig. 8.38

invoking parallel axis theorem, the mass moment of inertia of elemental disc about axis xx is

$$I_{xx} = I_{disc} + (dm)\, y^2$$

$$= \frac{1}{4} dm\, R^2 + dm\, y^2$$

$$= \frac{1}{4} (\rho\pi R^2\, dy) R^2 + (\rho\pi R^2\, dy) y^2$$

$$= \frac{1}{4} \rho\pi R^4\, dy + \rho\pi R^2 y^2\, dy$$

The mass moment of inertia of the entire solid cylinder can be worked out by integrating the above expression between the limits $-\frac{h}{2}$ to $\frac{h}{2}$. Thus,

$$I_{xx} = \frac{1}{4} \rho\pi R^4 \int_{-\frac{h}{2}}^{\frac{h}{2}} dy + \rho\pi R^2 \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy$$

$$= \frac{1}{4} \rho\pi R^4 h + \frac{1}{12} \rho\pi R^2 h^3$$

$$= \rho\pi R^2 h \left(\frac{R^2}{4} + \frac{h^2}{12} \right) = M \left(\frac{R^2}{4} + \frac{h^2}{12} \right) = \frac{1}{12} M (3R^2 + h^2) \quad \text{...(8.29 a)}$$

where $M = \rho\pi R^2 h$ is the mass of the cylinder.

Similarly

$$I_{yy} = \frac{1}{12} M (3R^2 + h^2) \quad \text{...(8.29 b)}$$

and

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{6} M (3R^2 + h^2) \quad \text{...(8.30)}$$

Note : For a thin cylinder, $R = 0$. That gives :

$$I_{xx} = I_{yy} = \frac{1}{12} Mh^2 = \frac{1}{12} Ml^2$$

For a thin disc, $h = 0$. That gives :

$$I_{xx} = I_{yy} = \frac{1}{4} MR^2 \quad \text{and} \quad I_{zz} = \frac{1}{2} MR^2$$

8.4.7. Right circular cone

Consider a solid cone of height h and radius R. If ρ is the density of the material of the cone, then

mass of the cone M = density × volume

$$= \rho \times \frac{1}{3} \pi R^2 h$$

Consider an element of thickness dy and radius r at distance y from the apex A

mass of the elemental strip, $dm = \rho\pi r^2 dy$

mass moment of inertia of the elemental strip about axis yy

$$= \frac{1}{2} \times \text{mass moment of inertia about polar axis}$$

$$= \frac{1}{2} dm\, r^2 = \frac{1}{2} (\rho \pi r^2 dy) r^2$$

$$= \frac{1}{2} \rho \pi r^4 dy$$

Since the integration is to be done with respect to y within the limits 0 to h, it is necessary to express r in terms of y. For that we have the following correlation from the similarity of triangles ADE and ABC

$$\frac{r}{R} = \frac{y}{h}, \quad r = R\frac{y}{h}$$

∴ mass moment of inertia of the cone about axis yy

$$I_{yy} = \int_0^h \frac{1}{2} \rho \pi \left(R\frac{y}{h} \right)^4 dy$$

$$= \frac{\rho \pi R^4}{2h^4} \left| \frac{y^5}{5} \right|_0^h$$

$$= \frac{\rho \pi R^4 h}{10} = \frac{\rho \pi R^2 h}{3} \times \frac{3}{10} R^2 = \frac{3}{10} M R^2 \quad \text{...(8.31)}$$

where $M = \frac{1}{3} \pi \rho R^2 h$ is the mass of the right circular cone.

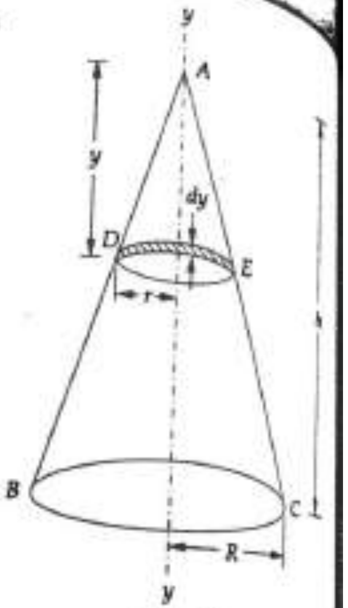


Fig. 8.39

EXAMPLE 8.15

Would you imagine that the moment of inertia of the earth around its own axis is negligible fraction of its moment of inertia about the axis of rotation around the sun? Take mean radius of the earth as 6,371 km and the mean radius of rotation around the sun as 149.7×10^6 km.

Solution : (*b*) Moment of inertia of the earth about its axis,

$$I_1 = \frac{2}{5} M R^2 = \frac{2}{5} M (6371)^2 = 16.23 \times 10^6 M$$

Moment of inertia of the earth about the axis of rotation around the sun,

$$I_2 = I_1 + Md^2 = 16.23 \times 10^6 M + M \times (149.7 \times 10^6)^2$$

$$= 16.23 \times 10^6 M + 22410.09 \times 10^{12} M$$

$$= 22410.09002 \times 10^{12} M$$

$$\text{Ratio } \frac{I_1}{I_2} = \frac{16.23 \times 10^6}{22410.09002 \times 10^{12}} = 7.24 \times 10^{-10}$$

Since the ratio is negligible, the moment of inertia of the earth around its own axis can be imagined to be a negligible fraction of its moment of inertia about the axis of rotation around the sun.