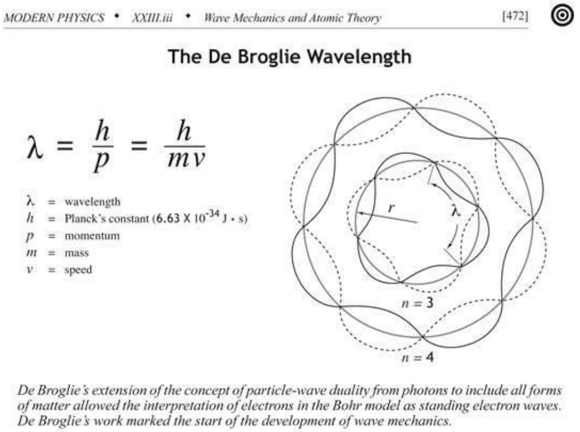
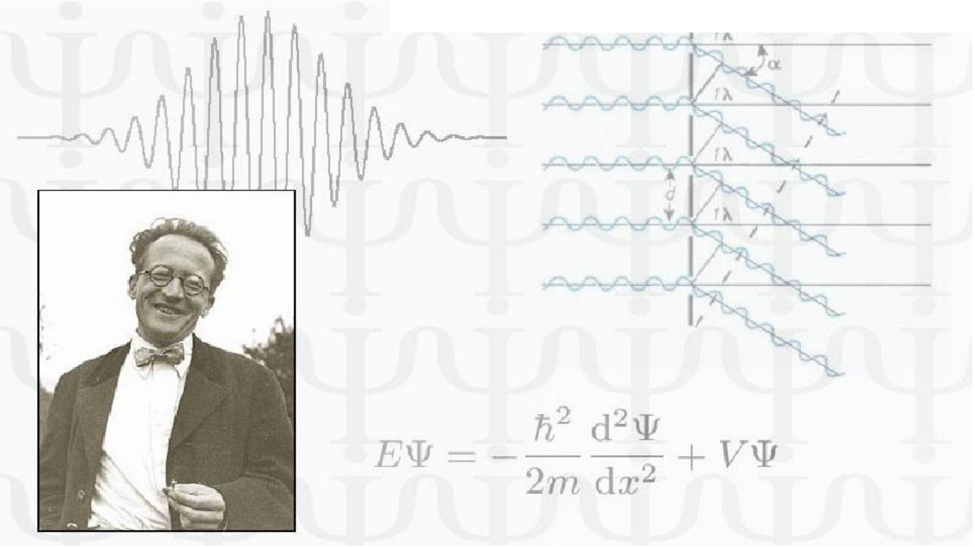
QUANTUM MECHANICS

UNIT II
Quantum Mechanics
Lecture-3





WAVE VELOCITY OR PHASE VELOCITY

- The velocity of advancement of a monochromatic wave (i.e., a wave of single frequency and wavelength) in a medium is known as *wave velocity, or phase velocity.*

$$v_p = \frac{\omega}{k}$$

- The velocity of propagation of planes of constant phase through a medium is known as *wave velocity, or phase velocity.*

NEED FOR WAVE PACKET REPRESENTATION

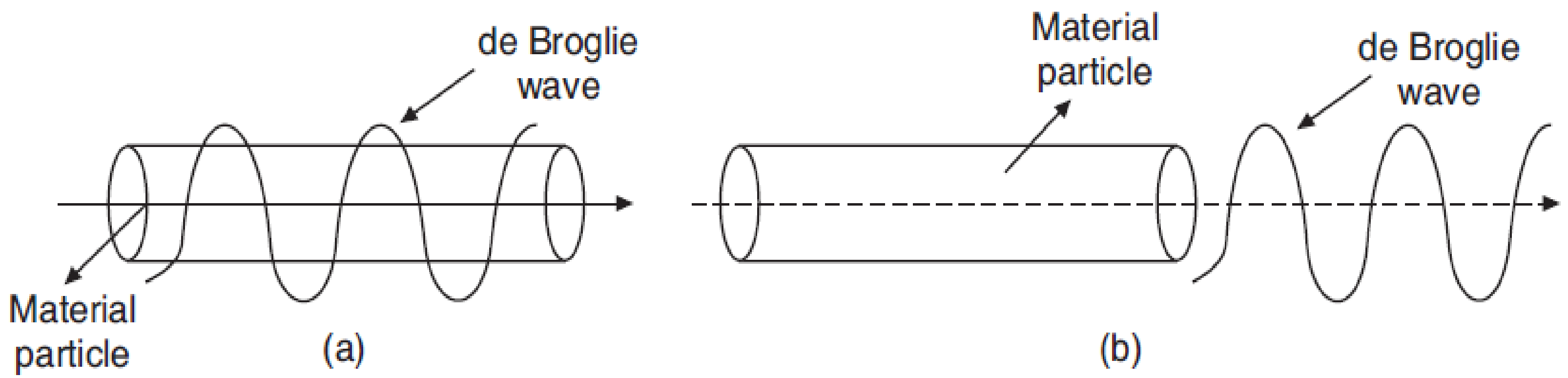
- For the material particle (including electron and proton), *u is always less than c.*

- *It means that, according to $v=c^2/u$, the phase velocity of the wave associated with the material particle is always greater than c.*

- *Now, it can be concluded that the particle and its corresponding de Broglie wave cannot travel together.*

- *Hence, the particle should be left behind to its de Broglie wave.*

NEED FOR WAVE PACKET REPRESENTATION



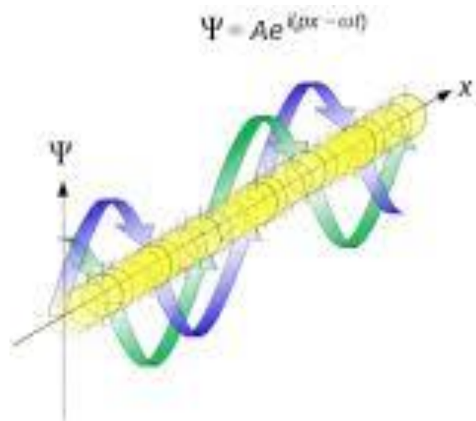
(a) Material particle and de Broglie wave are moving together and **(b)** material particle is left behind because $\upsilon_{\text{particle}} < c$ and $\upsilon_p > c$

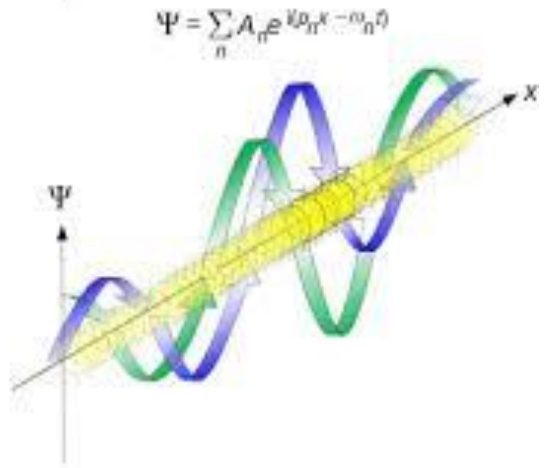
NEED FOR WAVE PACKET REPRESENTATION

- From the earlier discussion, it seems that the particle will not be able to keep pace with the associated de Broglie wave.

- However, it is not so. In order to explain the conflicting idea of velocity relationship between material particle and its de Broglie wave, Schrödinger himself introduced the idea of wave packet.

- According to the idea of wave packet, a moving material particle is equivalent to a wave packet (number of waves) instead of a single wave.

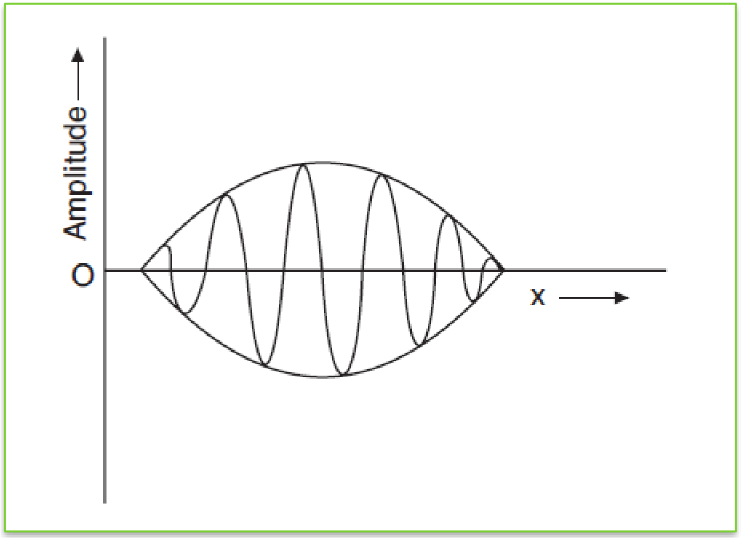


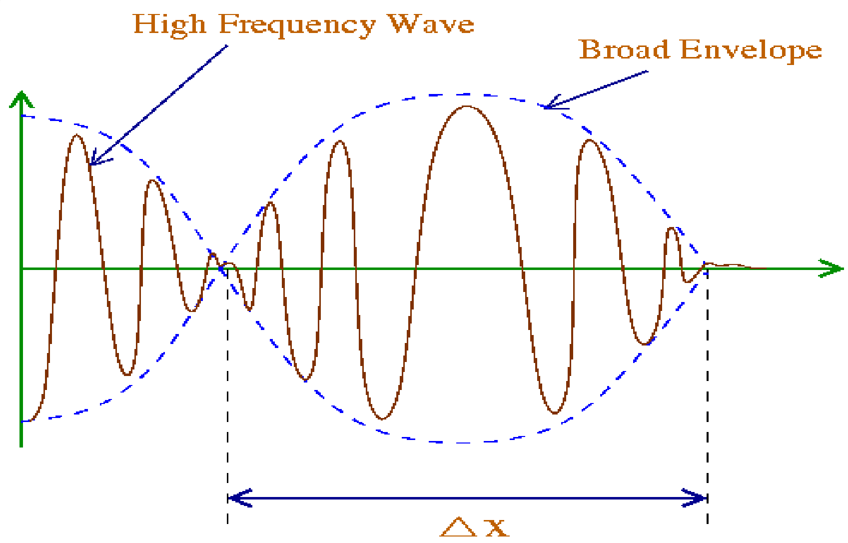


Wave Packet

- *A wave packet is the resultant of a group of waves, slightly differing in velocity and wavelength, with such phase and amplitude that they interfere constructively over a small region of space where the particle can be located.*

- *Outside this space, they interfere destructively so that the amplitude reduces to zero.*

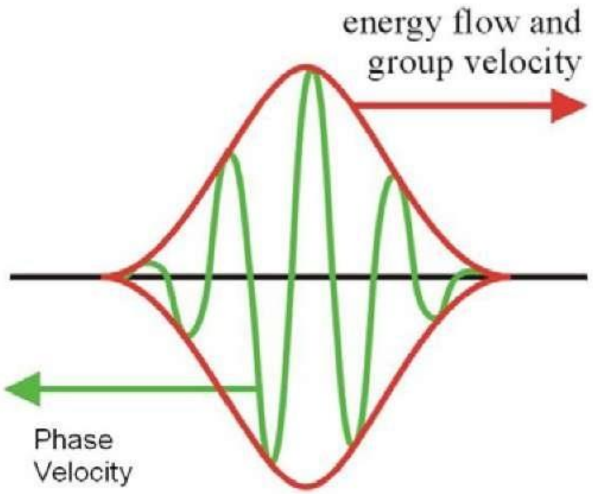




GROUP VELOCITY

The velocity with which a wave packet (or the group of waves) associated with the moving particle travels is called the *group velocity, whereas the velocity with which the individual waves comprising the wave packet travel is referred to as wave velocity, or phase velocity.*

$$\upsilon_g = \frac{d\omega}{dk}$$



EXPRESSION FOR GROUP VELOCITY

$$y_1 = a \sin(\omega_1 t - k_1 x)$$
$$y_2 = a \sin(\omega_2 t - k_2 x)$$

From the principle of superposition, the resultant displacement of these waves can be given as

$$y = y_1 + y_2$$
$$= a \sin(\omega_1 t - k_1 x) + a \sin(\omega_2 t - k_2 x)$$
$$= 2a \cos\left[\left\{\frac{\omega_1 - \omega_2}{2}\right\}t - \left\{\frac{k_1 - k_2}{2}\right\}x\right] \times \sin\left[\left\{\frac{\omega_1 + \omega_2}{2}\right\}t - \left\{\frac{k_1 + k_2}{2}\right\}x\right]$$

$$\left(\text{because } \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)\right)$$

EXPRESSION FOR GROUP VELOCITY

If ω_1 and ω_2 are very close to each other, then $(\omega_1 + \omega_2)/2 = \omega$ and $(k_1 + k_2)/2 = k$ (say). Hence, the above equation may reduce to the following form:

$$y = 2a \cos\left[\left(\frac{d\omega}{2}\right)t - \left(\frac{dk}{2}\right)x\right] \sin[\omega t - kx]$$

$$A = 2a \cos \frac{d\omega}{2}\left[t - \frac{dk}{d\omega}x\right]$$

$$A = 2a \cos \frac{d\omega}{2}\left[t - \frac{x}{\upsilon_g}\right]$$

where υ_g is the group velocity. This is given as

$$\upsilon_g = \frac{d\omega}{dk}$$

or

$$\upsilon_g = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

RELATION BETWEEN GROUP VELOCITY AND WAVE VELOCITY

From the definition of phase velocity, we know that

$$\upsilon_p = \frac{\omega}{k}$$

or $$\omega = k\upsilon_p$$

Differentiating the above equation, we get

$$d\omega = k d\upsilon_p + \upsilon_p\, dk$$

or $$\frac{d\omega}{dk} = \upsilon_p + k\frac{d\upsilon_p}{dk}$$

$$\upsilon_g = \frac{d\omega}{dk}$$

$$= \upsilon_p + k\frac{d\upsilon_p}{dk}$$

or $$\upsilon_g = \upsilon_p + k\frac{d\upsilon_p}{d\lambda}\cdot\frac{d\lambda}{dk}$$

Since $k = 2\pi/\lambda$, hence, $d\lambda/dk = -2\pi/k^2$.

$$\upsilon_g = \upsilon_p - \lambda\frac{d\upsilon_p}{d\lambda}$$

RELATION BETWEEN GROUP VELOCITY AND PARTICLE VELOCITY

$$\upsilon_g = \upsilon_p - \lambda \frac{d\upsilon_p}{d\lambda}$$

$$= \lambda^2 \left[\frac{\upsilon_p}{\lambda^2} - \frac{1}{\lambda} \frac{d\upsilon_p}{d\lambda} \right]$$

$$= -\lambda^2 \frac{d}{d\lambda} \left(\frac{\upsilon_p}{\lambda} \right)$$

$$= -\lambda^2 \frac{d}{d\lambda} \ldots$$

$$\frac{1}{\upsilon_g} = -\frac{1}{\lambda^2} \frac{d\lambda}{d\nu}$$

$$\frac{1}{\upsilon_g} = \frac{d}{d\lambda} \left(\frac{1}{\lambda} \right)$$

$$\frac{1}{\upsilon_g} = \frac{1}{h} \frac{d}{d\nu} [2m(E-V)]^{1/2}$$

$$= \frac{1}{h} \frac{1}{d\nu} \; [2m\,(h\nu - V)]^{1/2}$$

$$= \frac{1}{h} \cdot \frac{1}{2} \; [2m\,(h\nu - V)]^{-1/2} \cdot 2mh$$

$$\ldots \frac{m}{\ldots(E-V)]^{1/2}}$$

$$= \left[\frac{m}{\{2(E-V)\}} \right]^{1/2}$$

$$= \frac{1}{\upsilon_{\text{particle}}}$$

$$\upsilon_g = \upsilon_{\text{particle}}$$

Group velocity = Particle velocity

GROUP VELOCITY OF de BROGLIE WAVES

According to the relativistic consideration

$$m = \frac{m_0}{\sqrt{1-\upsilon^2/c^2}}$$

We know that

$$\omega = 2\pi\nu = \frac{2\pi E}{h} = \frac{2\pi mc^2}{h}$$

and $$k = \frac{2\pi}{\lambda} = \frac{2\pi m\upsilon}{h} \qquad \left[\because\ \lambda = \frac{h}{m\upsilon}\right]$$

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{1-\upsilon^2/c^2}}$$

and $$k = \frac{2\pi m_0 \upsilon}{h\sqrt{1-\upsilon^2/c^2}}$$

GROUP VELOCITY OF de BROGLIE WAVES

Differentiating ω and k with respect to v

$$\frac{d\omega}{d\upsilon} = \frac{2\pi m_0 \upsilon}{h[1-\upsilon^2/c^2]^{3/2}}$$

$$\frac{dk}{d\upsilon} = \frac{2\pi m_0}{h[1-\upsilon^2/c^2]^{3/2}}$$

$$\upsilon_g = \frac{d\omega}{dk} = \frac{d\omega/d\upsilon}{dk/d\upsilon}$$

$$= 2\,\pi m_0 \upsilon / 2\pi m_0$$

$$\upsilon_g = \upsilon$$

The phase velocity of sea-water waves is given by $\sqrt{(g\lambda/2\pi)}$. Calculate the group velocity of these waves. The wavelength is 680 m. Take $g = 9.8$ m/s^2.

Solution

The relation between group velocity (υ_g) and phase velocity υ_p can be given as

$d\upsilon_p$

$$= \frac{1}{2}\sqrt{\left(\frac{g\lambda}{2\pi}\right)}\frac{1}{\lambda}$$

Giver

$$= \frac{\upsilon_p}{2\lambda}$$

or

Now, by putting the value of $d\upsilon_p/d\lambda$ in Eq. (1), we get

$$\upsilon_g = \upsilon_p - \frac{\upsilon_p}{2\lambda}$$

$$= \frac{\upsilon_p}{2}$$

$$= \frac{1}{2}\left[\frac{g\lambda}{2\pi}\right]^{1/2}$$

$$= \frac{1}{2}\left[\frac{9.8\times 680}{2\times 3.14}\right]^{1/2}$$

$$= 16.29 \text{ m/s}$$

Example

Show that the de Broglie wave velocity is a function of wavelength even in free space.

Solution

The de Broglie wavelength of a wave associated with a particle of mass m moving with velocity υ is given by

$$\lambda = \frac{h}{m\upsilon}$$

$$= \frac{h}{\sqrt{\quad}}$$

$$= \frac{h}{m_0 c}\left[\frac{\upsilon_p^2}{c^2} - 1\right]^{1/2}$$

or $$\left(\frac{\lambda m_0 c}{h}\right)^2 = \frac{\upsilon_p^2}{c^2} - 1$$

$$\frac{\upsilon_p^2}{c^2} = 1 + \left(\frac{\lambda m_0 c}{h}\right)^2$$

or $$\upsilon_p = c\sqrt{1 + \left(\frac{m_0 c \lambda}{h}\right)^2}$$

It is clear from the above expression that for a particle of mass m, the wave velocity is always greater than c and it is a function of λ in free space.

Distinguish between the group velocity (υ_g) and the phase velocity (υ_p) of a wave packet and show that $\upsilon_p \upsilon_g = c^2$.

Solution

According to the de Broglie hypothesis, the wavelength of a wave associated with a particle of mass m moving with velocity υ is given as

$$\lambda = \frac{h}{m\upsilon} = \frac{h}{p}$$

The propagation constant k is defined as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi m\upsilon}{h} \quad (1)$$

If a particle has its energy E and the frequency of a wave associated with it is ν, then we can write

$$E = h\nu \quad \text{and} \quad \omega = 2\pi\nu = \frac{2\pi E}{h}$$

According to the mass energy equivalence relation, we know that

$$E = mc^2$$

Hence,

$$\omega = \frac{2\pi mc^2}{h} \quad (2)$$

We know that

$$\upsilon_p = \frac{\omega}{k} = \frac{2\pi mc^2 / h}{2\pi m\upsilon / h}$$

$$\upsilon_p = \frac{c^2}{\upsilon} \quad (3)$$

The particle is moving with a velocity υ. Since the particle velocity is equal to the group velocity, i.e., $\upsilon_g = \upsilon_{\text{particle}}$, so $\upsilon = \upsilon_g$.

Now, putting this value in Eq. (3), we get

$$\upsilon_p = \frac{c^2}{\upsilon_g}$$

or

$$\upsilon_p \upsilon_g = c^2$$

Assignment Based on this Lecture

- Define phase velocity and obtain the expression for phase velocity.
- Discuss the need for wave packet representation.
- Define group velocity and obtain the expression for group velocity.
- Obtain the relation between phase velocity and group velocity
- Prove that the particle always travel with group velocity.