Electromagnetic Field Theory

Unit I-II

By

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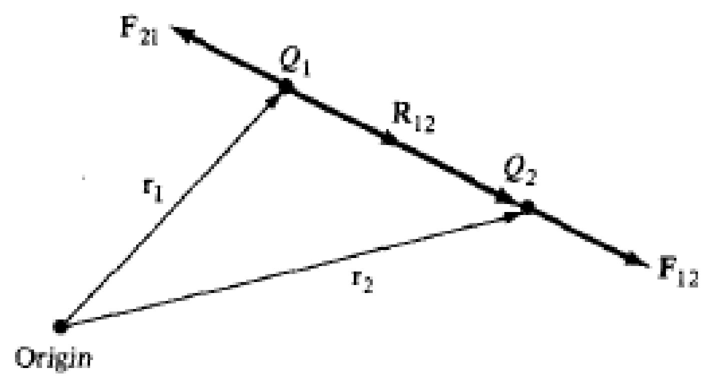
Coulomb's Law

It states that the force F between two point charges Q_1 and Q_2 is

$$F = \frac{kQ_1Q_2}{R^2}$$

In Vector form



$$\mathbf{F}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_o R^3}\mathbf{R}_{12}$$

Or $$\mathbf{F}_{12} = \frac{Q_1Q_2\,(\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

If we have more than two point charges

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}}$$

Electric Field Intensity

Electric Field Intensity is the force per unit charge when placed in the electric field

$$E = \frac{F}{Q}$$

In Vector form

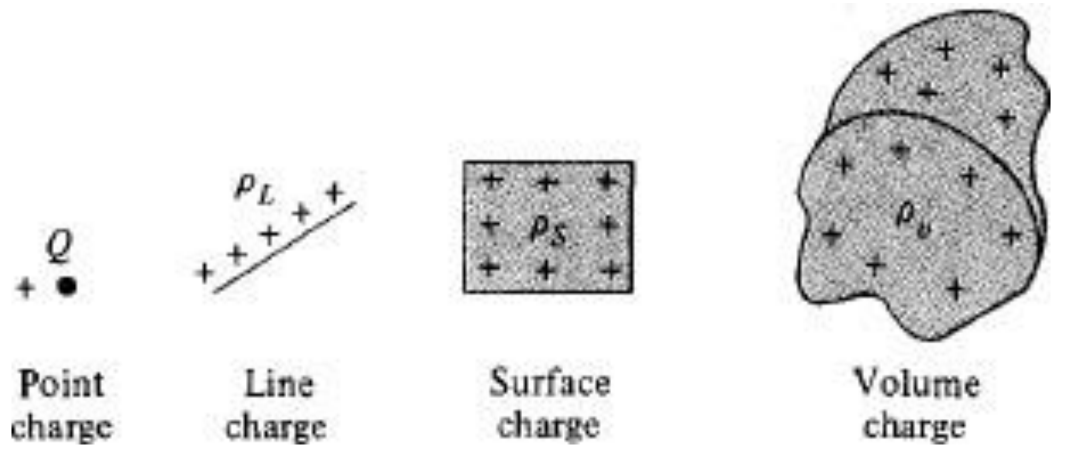
$$\mathrm{E} = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}$$

If we have more than two point charges

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}$$

Electric Field due to Continuous Charge Distribution

If there is a continuous charge distribution say along a line, on a surface, or in a volume



The charge element dQ and the total charge Q due to these charge distributions can be obtained by

$$dQ = \rho_L \, dl \rightarrow Q = \int_L \rho_L \, dl \qquad \text{(line charge)}$$

$$dQ = \rho_S \, dS \rightarrow Q = \int_S \rho_S \, dS \qquad \text{(surface charge)}$$

$$dQ = \rho_v \, dv \rightarrow Q = \int_v \rho_v \, dv \qquad \text{(volume charge)}$$

The electric field intensity due to each charge distribution ρ_L, ρ_S and ρ_V may be given by the summation of the field contributed by the numerous point charges making up the charge distribution.

$$\mathbf{E} = \int \frac{\rho_L \, dl}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(line charge)}$$

$$\mathbf{E} = \int \frac{\rho_S \, dS}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(surface charge)}$$

$$\mathbf{E} = \int \frac{\rho_v \, dv}{4\pi\varepsilon_o R^2} \mathbf{a}_R \qquad \text{(volume charge)}$$

Electric Flux Density

The electric field intensity depends on the medium in which the charges are placed.

Suppose a vector field D independent of the medium is defined by

$$D = \varepsilon_o E$$

The electric flux ψ in terms of D can be defined as

$$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$$

The vector field D is called the electric flux density and is measured in coulombs per square meter.

Electric Flux Density

For an infinite sheet the electric flux density D is given by

$$\mathbf{D} = \frac{\rho_S}{2}\,\mathbf{a}_n$$

For a volume charge distribution the electric flux density D is given by

$$\mathbf{D} = \int \frac{\rho_v \, dv}{4\pi R^2}\,\mathbf{a}_R$$

In both the above equations D is a function of charge and position only (independent of medium)

Gauss Law

It states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.

$$\psi = Q_{enc}$$

$$\Psi = \oint d\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$\text{Total charge enclosed } Q = \int \rho_v \, dv$$

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv \qquad \text{(i)}$$

Using Divergence Theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{D}\, dv \qquad \text{(ii)}$$

Comparing the two volume integrals in (i) and (ii)

$$\rho_v = \nabla \cdot \mathbf{D}$$

This is the first Maxwell's equation.

It states that the volume charge density is the same as the divergence of the electric flux density.

Electric Potential

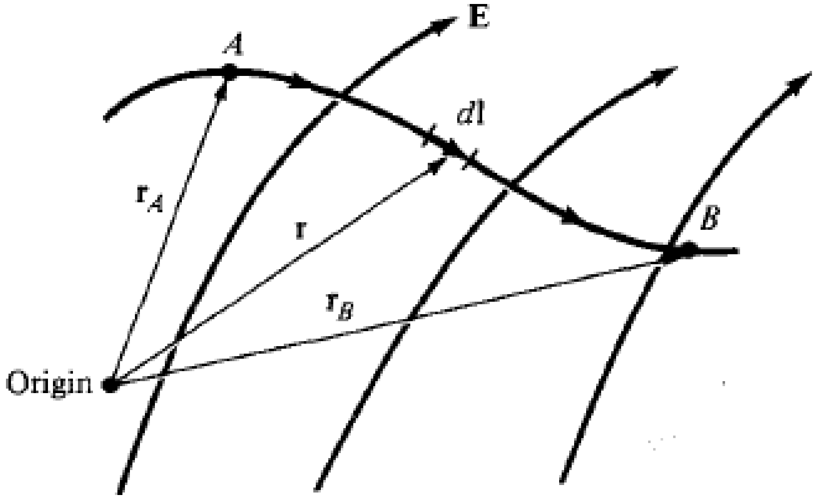
Electric Field intensity, E due to a charge distribution can be obtained from Coulomb's Law.

or using Gauss Law when the charge distribution is symmetric.

We can obtain E without involving vectors by using the electric scalar potential V.

From Coulomb's Law the force on point charge Q is

$$\vec{F} = Q\vec{E}$$



The work done in displacing the charge by length dl is

$$dW = -\vec{F}.dl \ = -Q\vec{E}.dl$$

The negative sign indicates that the work is being done by an external agent.

The total work done or the potential energy required in moving the point charge Q from A to B is

$$W = -Q\int_{A}^{B} \vec{E}.dl$$

Dividing the above equation by Q gives the potential energy per unit charge.

$$\frac{W}{Q} = -\int_{A}^{B} \vec{E}.dl \quad = V_{AB}$$

V_{AB} is known as the potential difference between points A and B.

1. If V_{AB} is negative, there is loss in potential energy in moving Q from A to B (work is being done by the field), if V_{AB} is positive, there is a gain in potential energy in the movement (an external agent does the work).
2. It is independent of the path taken. It is measured in Joules per Coulomb referred as Volt.

The potential at any point due to a point charge Q located at the origin is

$$V = \frac{Q}{4\pi\varepsilon_o r}$$

The potential at any point is the potential difference between that point and a chosen point at which the potential is zero.

Assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point.

$$V = -\int_{\infty}^{r} \vec{E}.d\vec{l}$$

If the point charge Q is not at origin but at a point whose position vector is $\vec{r}\,'$, the potential $V(\vec{r}\,')$ at $\vec{r}\,'$ becomes

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_o |\vec{r} - \vec{r}\,'|}$$

For n point charges Q_1, Q_2, Q_3…..Q_n located at points with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3.....\vec{r}_n$ the potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{\square 1}{4\pi\varepsilon_o}\sum_{k=1}^{n}\frac{Q_k}{|\vec{r} - \vec{r}_k|}$$

If there is continuous charge distribution instead of point charges then the potential at $\vec{r}$ becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_L \frac{\rho_L(\mathbf{r}')dl'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(line charge)}$$

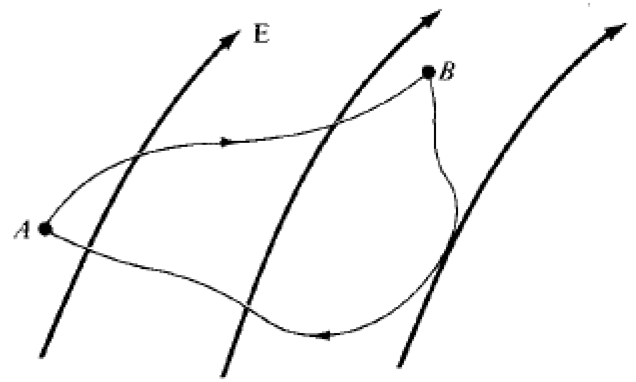
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_S \frac{\rho_S(\mathbf{r}')dS'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(surface charge)}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o}\int_v \frac{\rho_v(\mathbf{r}')dv'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{(volume charge)}$$

Relationship between E and V

The potential difference between points A and B is independent of the path taken

$$V_{AB} = -V_{BA}$$



$$V_{AB} = -\int_A^B \vec{E}.d\vec{l} \quad \text{and} \quad V_{BA} = \int_B^A \vec{E}.d\vec{l}$$

$$V_{AB} + V_{BA} = \oint \vec{E}.d\vec{l} = 0$$

$$\oint \vec{E}.d\vec{l} = 0 \qquad \text{(i)}$$

It means that the line integral of $\vec{E}$ along a closed path must be zero.

Physically it means that no net work is done in moving a charge along a closed path in an electrostatic field.

Applying Stokes's theorem to equation (i)

$$\oint \vec{E}.d\vec{l} = \int (\vec{\nabla} \times \vec{E}).d\vec{S} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad \text{(ii)}$$

Equation (i) and (ii) are known as Maxwell's equation for static electric fields.

Equation (i) is in integral form while equation (ii) is in differential form, both depicting conservative nature of an electrostatic field.

Also

$$\vec{E} = -\nabla V$$

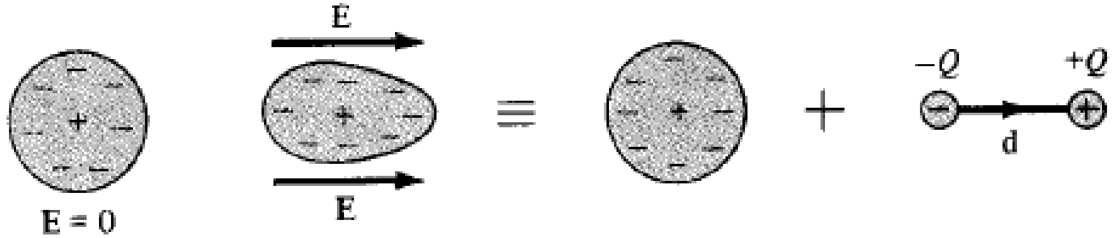
It means Electric Field Intensity is the gradient of V.

The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases.

Polarization in Dielectrics

Consider an atom of the dielectric consisting of an electron cloud (-Q) and a positive nucleus (+Q).

When an electric field $\vec{E}$ is applied, the positive charge is displaced from its equilibrium position in the direction of $\vec{E}$ by $\vec{F}_+ = Q\vec{E}$ while the negative charge is displaced by $\vec{F}_- = Q\vec{E}$ in the opposite direction.



A dipole results from the displacement of charges and the dielectric is polarized. In polarized the electron cloud is distorted by the applied electric field.

This distorted charge distribution is equivalent to the original distribution plus the dipole whose moment is

$$\vec{p} = Q\vec{d}$$

where $\vec{d}$ is the distance vector between -Q to +Q.

If there are N dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field

$$Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N = \sum_{k=1}^{N} Q_k\mathbf{d}_k$$

For the measurement of intensity of polarization, we define polarization $\vec{P}$ (coulomb per square meter) as dipole moment per unit volume

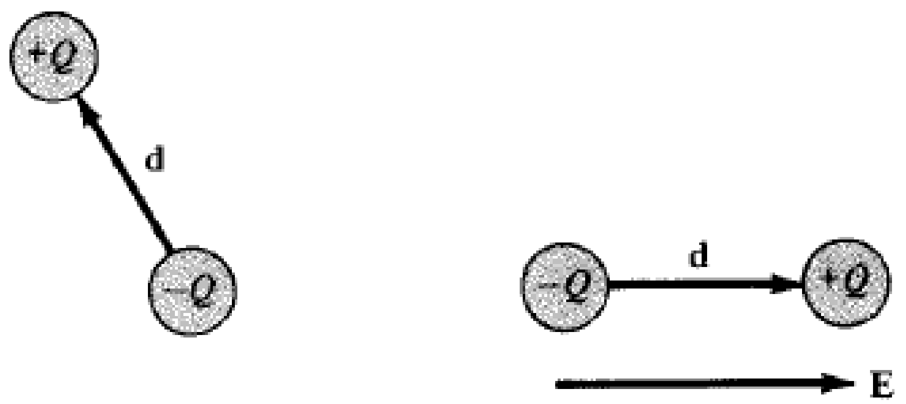
$$\mathbf{P} = \frac{\lim\limits_{\Delta v \to 0} \sum_{k=1}^{N} Q_k\mathbf{d}_k}{\Delta v}$$

The major effect of the electric field on the dielectric is the creation of dipole moments that align themselves in the direction of electric field.

This type of dielectrics are said to be non-polar. eg: H_2, N_2, O_2

Other types of molecules that have in-built permanent dipole moments are called polar. eg: H_2O, HCl

When electric field is applied to a polar material then its permanent dipole experiences a torque that tends to align its dipole moment in the direction of the electric field.

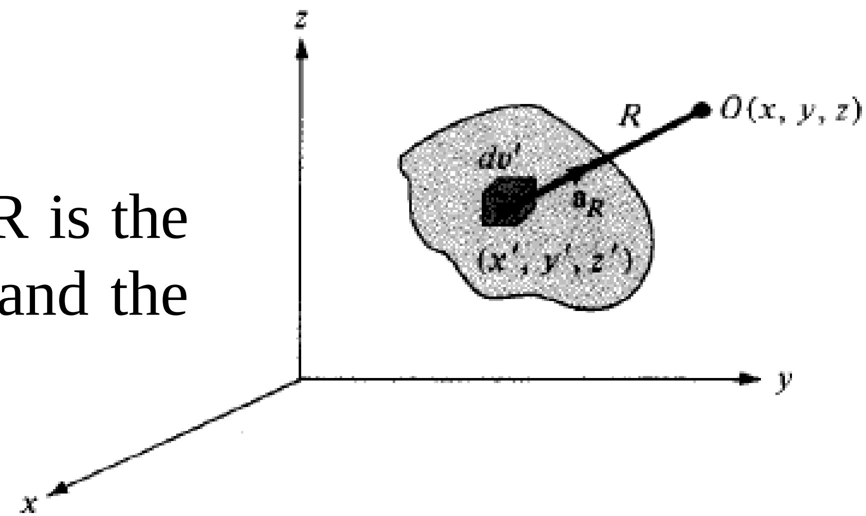


Field due to a Polarized Dielectric

Consider a dielectric material consisting of dipoles with Dipole moment $\vec{P}$ per unit volume.

The potential dV at an external point O due to $\vec{P}dv'$

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R \, dv'}{4\pi\varepsilon_o R^2} \qquad \text{(i)}$$



where $R^2 = (x-x')^2+(y-y')^2+(z-z')^2$ and R is the distance between volume element dv' and the point O.

But
$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right)$$

Applying the vector identity
$$\nabla' \cdot f\mathbf{A} = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f$$

$$\mathbf{A} \cdot \nabla' f = \nabla' \cdot f\mathbf{A} - f\nabla' \cdot \mathbf{A}$$

$$\frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} = \nabla' \cdot \frac{\mathbf{P}}{R} - \frac{\nabla' \cdot \mathbf{P}}{R}$$

Put this in (i) and integrate over the entire volume v' of the dielectric

$$V = \int_{v'} \frac{1}{4\pi\varepsilon_o}\left[\nabla' \cdot \frac{\mathbf{P}}{R} - \frac{1}{R}\nabla' \cdot \mathbf{P}\right] dv'$$

Applying Divergence Theorem to the first term

$$V = \int_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{4\pi\varepsilon_o R}\, dS' + \int_{v'} \frac{-\nabla' \cdot \mathbf{P}}{4\pi\varepsilon_o R}\, dv' \qquad \text{(ii)}$$

where $\mathbf{a}_n$' is the outward unit normal to the surface dS' of the dielectric

The two terms in (ii) denote the potential due to surface and volume charge distributions with densities

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$
$$\rho_{pv} = -\nabla \cdot \mathbf{P}$$

where ρ_{ps} and ρ_{pv} are the bound surface and volume charge densities.

Bound charges are those which are not free to move in the dielectric material.

Equation (ii) says that where polarization occurs, an equivalent volume charge density, ρ_{pv} is formed throughout the dielectric while an equivalent surface charge density, ρ_{ps} is formed over the surface of dielectric.

The total positive bound charge on surface S bounding the dielectric is

$$Q_b = \oint \mathbf{P} \cdot d\mathbf{S} = \int \rho_{ps}\, dS$$

while the charge that remains inside S is

$$-Q_b = \int_v \rho_{pv}\, dv = -\int_v \nabla \cdot \mathbf{P}\, dv$$

Total charge on dielectric remains zero.

$$\text{Total charge} = \oint_S \rho_{ps}\, dS + \int_v \rho_{pv}\, dv = Q_b - Q_b = 0$$

When dielectric contains free charge

If ρ_v is the free volume charge density then the total volume charge density ρ_t

$$\rho_t = \rho_v + \rho_{pv} = \nabla \cdot \varepsilon_o \mathbf{E}$$

Hence

$$\begin{aligned} \rho_v &= \nabla \cdot \varepsilon_o \mathbf{E} - \rho_{pv} \\ &= \nabla \cdot (\varepsilon_o \mathbf{E} + \mathbf{P}) \\ &= \nabla \cdot \mathbf{D} \end{aligned}$$

Where

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

The effect of the dielectric on the electric field $\vec{E}$ is to increase $\vec{D}$ inside it by an amount $\vec{P}$.

The polarization would vary directly as the applied electric field.

$$\mathbf{P} = \chi_e \varepsilon_o \mathbf{E}$$

Where χ_e is known as the electric susceptibility of the material

It is a measure of how susceptible a given dielectric is to electric fields.

Dielectric Constant and Strength

We know that

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$$

Thus

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

or

$$\mathbf{D} = \varepsilon \mathbf{E}$$

where $\varepsilon = \varepsilon_o \varepsilon_r$

and

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$

where ϵ is the permittivity of the dielectric, ϵ_o is the permittivity of the free space and ϵ_r is the dielectric constant or relative permittivity.

No dielectric is ideal. When the electric field in a dielectric is sufficiently high then it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting.

When a dielectric becomes conducting then it is called dielectric breakdown. It depends on the type of material, humidity, temperature and the amount of time for which the field is applied.

The minimum value of the electric field at which the dielectric breakdown occurs is called the dielectric strength of the dielectric material.

or

The dielectric strength is the maximum value of the electric field that a dielectric can tolerate or withstand without breakdown.

Continuity Equation and Relaxation Time

According to principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

The current I_{out} coming out of the closed surface

$$I_{out} = \oint \mathbf{J} \cdot d\mathbf{S} = \frac{-dQ_{in}}{dt} \qquad \text{(i)}$$

where Q_{in} is the total charge enclosed by the closed surface.

Using divergence theorem

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{J} \, dv$$

But

$$\frac{-dQ_{in}}{dt} = -\frac{d}{dt} \int_v \rho_v \, dv = -\int_v \frac{\partial \rho_v}{\partial t} \, dv$$

Equation (i) now becomes

$$\int_v \nabla \cdot \mathbf{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

or $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad \text{(ii)}$$

This is called the continuity of current equation.

Effect of introducing charge at some interior point of a conductor/dielectric

According to Ohm’s law

$$\mathbf{J} = \sigma \mathbf{E}$$

According to Gauss’s law

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}$$

Equation (ii) now becomes

$$\nabla \cdot \sigma \mathbf{E} = \frac{\sigma \rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$$

or $$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

This is homogeneous liner ordinary differential equation. By separating variables we get

$$\frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\varepsilon}\partial t$$

Integrating both sides

$$\ln \rho_v = -\frac{\sigma t}{\varepsilon} + \ln \rho_{vo}$$

where $\ln \rho_{vo}$ is a constant of integration

$$\rho_v = \rho_{vo} e^{-t/T_r} \qquad \text{(iii)}$$

where

$$T_r = \frac{\varepsilon}{\sigma}$$

ρ_{vo} is the initial charge density (i.e., ρ_v at $t = 0$)

Equation (iii) shows that as a result of introducing charge at some interior point of the material there is a decay of the volume charge density ρ_v.

The time constant T_r is known as the relaxation time or the relaxation time.

Relaxation time is the time in which a charge placed in the interior of a material to drop to e^{-1} = 36.8 % of its initial value.

For Copper T_r = 1.53 x 10^{-19} sec (short for good conductors)

For fused Quartz T_r = 51.2 days (large for good dielectrics)

Boundary Conditions

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called boundary conditions

These conditions are helpful in determining the field on one side of the boundary when the field on other side is known.

We will consider the boundary conditions at an interface separating

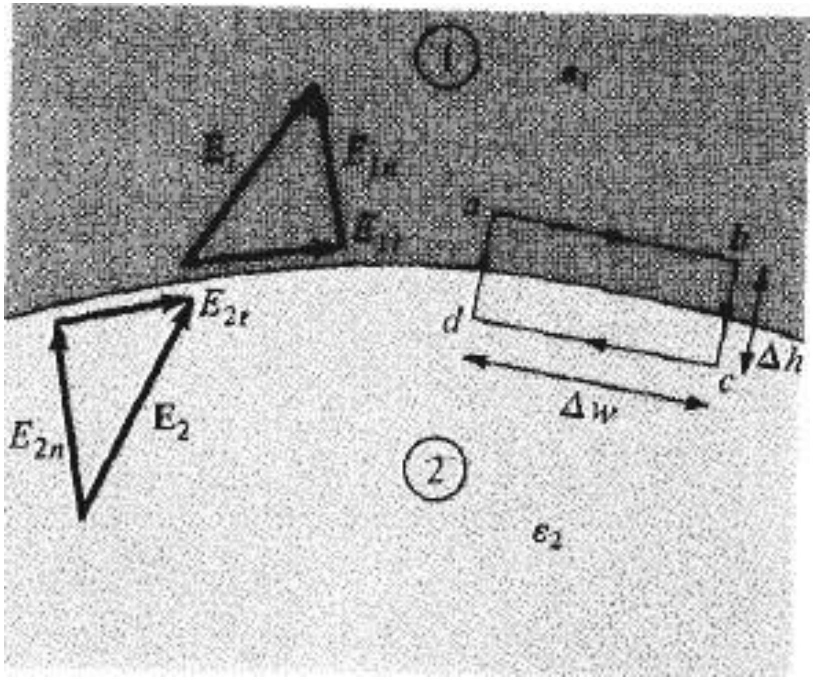
1. Dielectric (ϵ_{r1}) and Dielectric (ϵ_{r2})
2. Conductor and Dielectric
3. Conductor and free space

For determining boundary conditions we will use Maxwell's equations

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{and} \quad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

Boundary Conditions (Between two different dielectrics)

Consider the **E** field existing in a region consisting of two different dielectrics characterized by $\epsilon_1 = \epsilon_0 \epsilon_{r1}$ and $\epsilon_2 = \epsilon_0 \epsilon_{r2}$



$\mathbf{E_1}$ and $\mathbf{E}_2$ in the media 1 and 2 can be written as

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n} \quad \text{and} \quad \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$

But $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

Assuming that the path abcda is very small with respect to the variation in **E**

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}$$

As $\Delta h \longrightarrow 0$

$$E_{1t} = E_{2t}$$

Thus the tangential components of **E** are the same on the two sides of the boundary. **E** is continuous across the boundary.

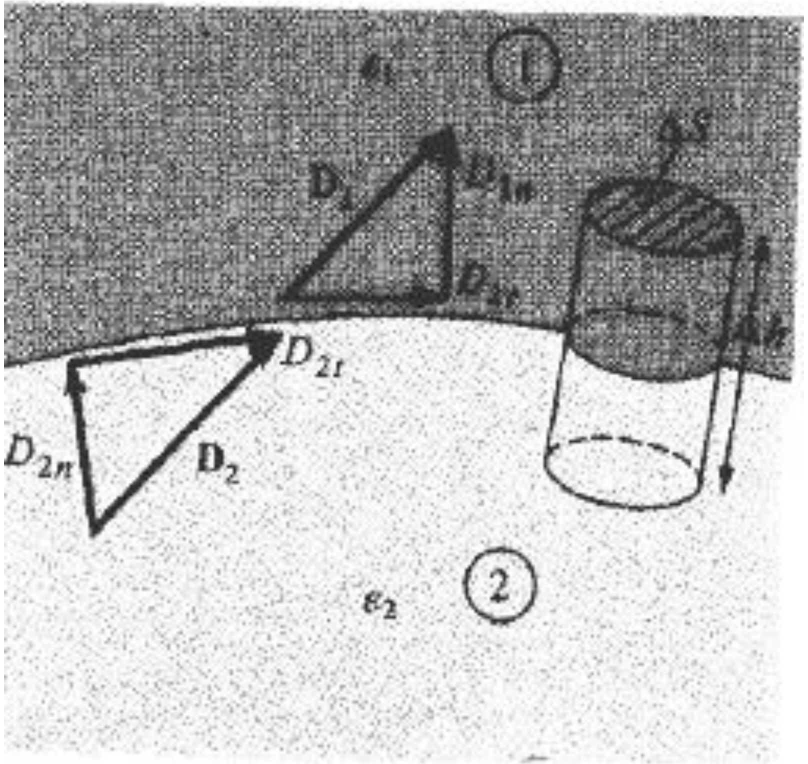
But $$\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{D}_t + \mathbf{D}_n$$

Thus
$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

or
$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

Here $\mathbf{D}_t$ undergoes some change across the surface and is said to be discontinuous across the surface.

Applying $\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$



Putting $\Delta h \rightarrow 0$ gives

$$\Delta Q = \rho_S \, \Delta S = D_{1n} \, \Delta S - D_{2n} \, \Delta S$$

$$D_{1n} - D_{2n} = \rho_S$$

Where ρ_s is the free charge density placed deliberately at the boundary

If there is no charge on the boundary i.e. $\rho_s = 0$ then

$$D_{1n} = D_{2n}$$

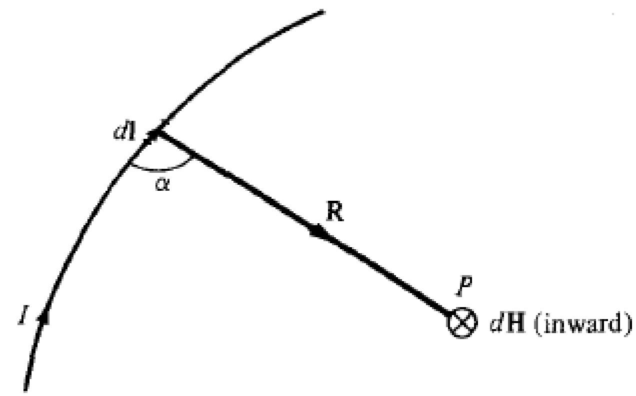
Thus the normal components of **D** is continuous across the surface.

Biot-Savart's Law

It states that the magnetic field intensity d**H** produce at a point P by the differential current element Idl is proportional to the product Idl and the sine of angle α between the element and line joining P to the element and is inversely proportional to the square of distance **R** between P and the element.

$$dH \propto \frac{I\,dl \sin\alpha}{R^2} \quad \text{or} \quad dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$

$$d\mathbf{H} = \frac{I\,d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{I\,d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$



The direction of d**H** can be determined by the right hand thumb rule with the right hand thumb pointing in the direction of the current, the right hand fingers encircling the wire in the direction of d**H**

Ampere's circuit Law

The line integral of the tangential component of **H** around a close path is the same as the net current I_{inc} enclosed by the path.

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}$$

Application of Ampere's law : Infinite Sheet Current

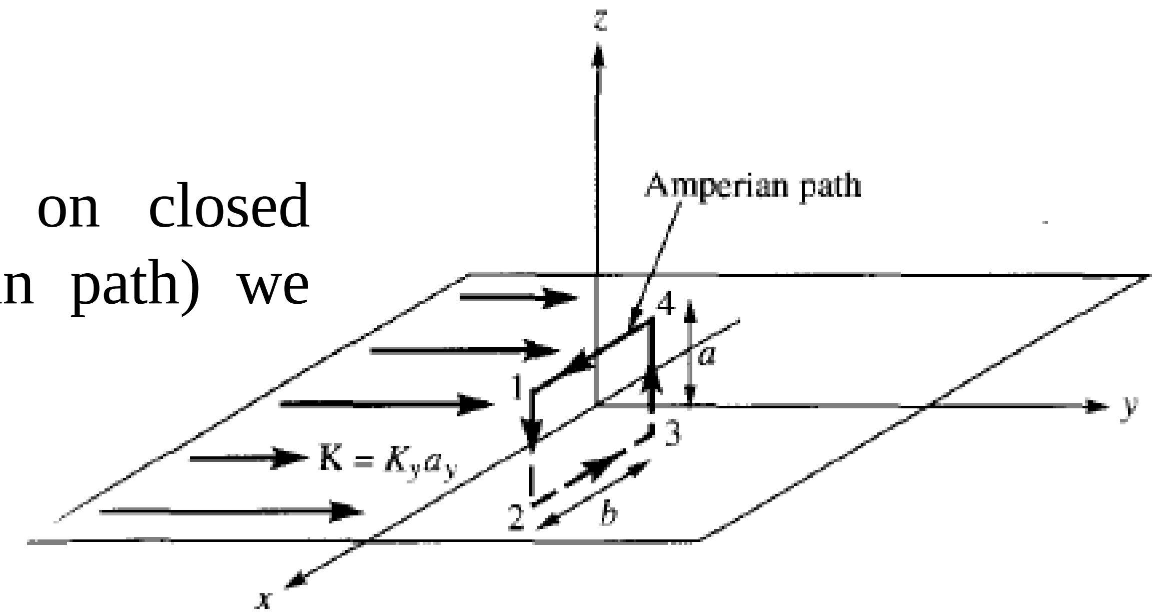
Consider an infinite current sheet in z = 0 plane.

If the sheet has a uniform current density then

$$\vec{K} = K_y \ \hat{a}_y$$

Applying Ampere's Law on closed rectangular path (Amperian path) we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b \quad \text{(i)}$$



To solve integral we need to know how **H** is like

We assume the sheet comprising of filaments d**H** above and below the sheet due to pair of filamentary current.

The resultant $d\mathbf{H}$ has only an x-component.

Also **H** on one side of sheet is the negative of the other.

Due to infinite extent of the sheet, it can be regarded as consisting of such filamentary pairs so that the characteristic of **H** for a pair are the same for the infinite current sheets

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases} \qquad \text{(ii)}$$

where H_o is to be determined.

Evaluating the line integral of **H** along the closed path

$$\oint \mathbf{H} \cdot d\mathbf{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l}$$

$$= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b)$$

$$= 2H_o b \qquad \text{(iii)}$$

Comparing (i) and (iii), we get

$$H_o = \frac{1}{2} K_y \qquad \text{(iv)}$$

Using (iv) in (ii), we get

$$\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$$

Generally, for an infinite sheet of current density **K**

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n$$

where a_n is a unit normal vector directed from the current sheet to the point of interest.

Magnetic Flux Density

The magnetic flux density **B** is similar to the electric flux density **D**
Therefore, the magnetic flux density **B** is related to the magnetic field intensity **H**

$$\mathbf{B} = \mu_o \mathbf{H}$$

where μ_o is a constant and is known as the permeability of free space.

Its unit is Henry/meter (H/m) and has the value

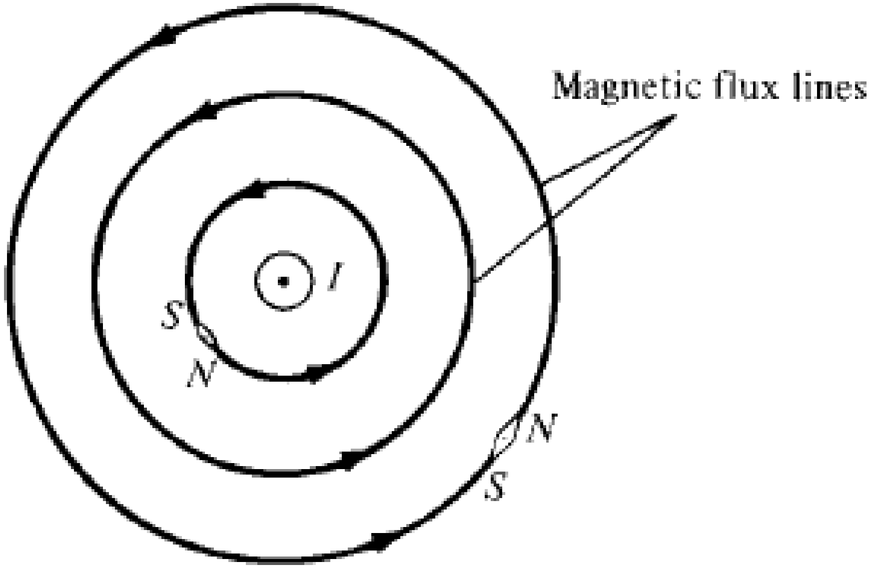
$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

where the magnetic flux ψ is in webers (Wb) and the magnetic flux density is in weber/ square meter or Teslas.

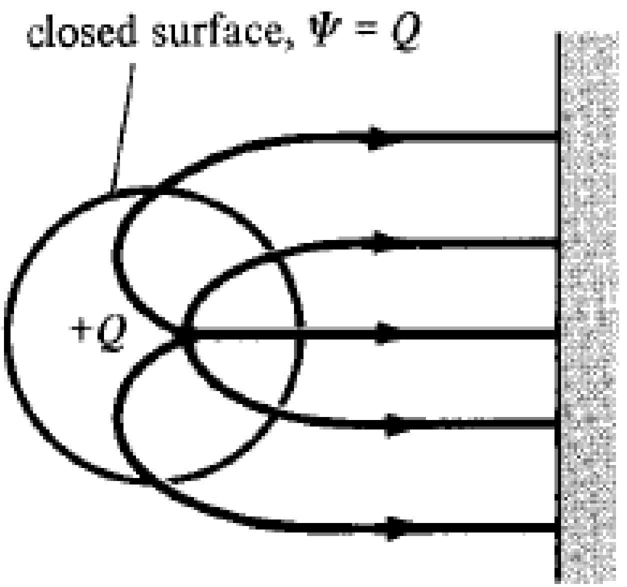
Magnetic flux lines due to a straight wire with current coming out of the page



Each magnetic flux line is closed with no beginning and no end and are also not crossing each other.

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed.

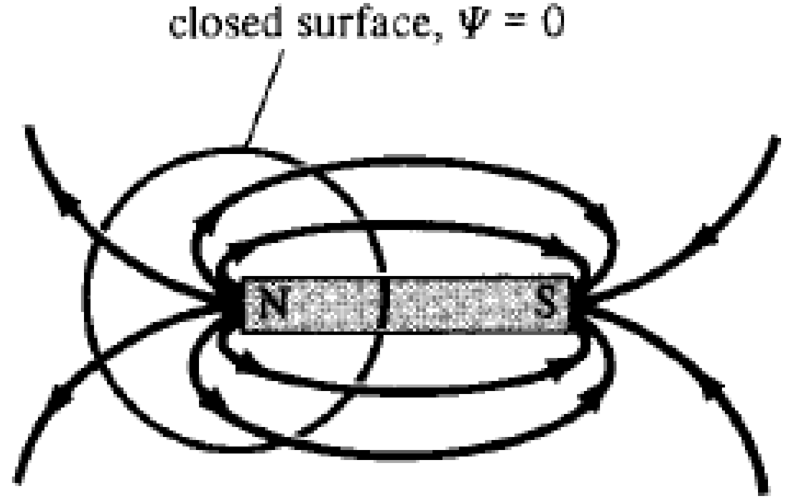
$$\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q$$



Thus it is possible to have an isolated electric charge.

Also the electric flux lines are not necessarily closed.

Magnetic flux lines are always close upon themselves,.



So it is not possible to have an isolated magnetic pole (or magnetic charges)

An isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero.

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

This equation is known as the law of conservation of magnetic flux or Gauss's Law for Magnetostatic fields.

Magnetostatic field is not conservative but magnetic flux is conserved.

Applying Divergence theorem, we get

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0$$

or $$\nabla \cdot \mathbf{B} = 0$$

This is Maxwell's fourth equation.

This equation suggests that magnetostatic fields have no source or sinks.
Also magnetic flux lines are always continuous.

Faraday's law

According to Faraday a time varying magnetic field produces an induced voltage (called electromotive force or emf) in a closed circuit, which causes a flow of current.

The induced emf (V_{emf}) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is Faraday's Law and can be expressed as

$$V_{\text{emf}} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$$

where N is the number of turns in the circuit and ψ is the flux through each turn.

The negative sign shows that the induced voltage acts in such a way to oppose the flux producing in it. This is known as Lenz's Law.

Transformer and Motional EMF

For a circuit with a single turn (N = 1)

$$V_{\text{emf}} = -\frac{d\Psi}{dt}$$

In terms of **E** and **B** this can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{(i)}$$

where ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by a closed path L..

The equation says that in time-varying situation, both electric and magnetic fields are present and are interrelated.

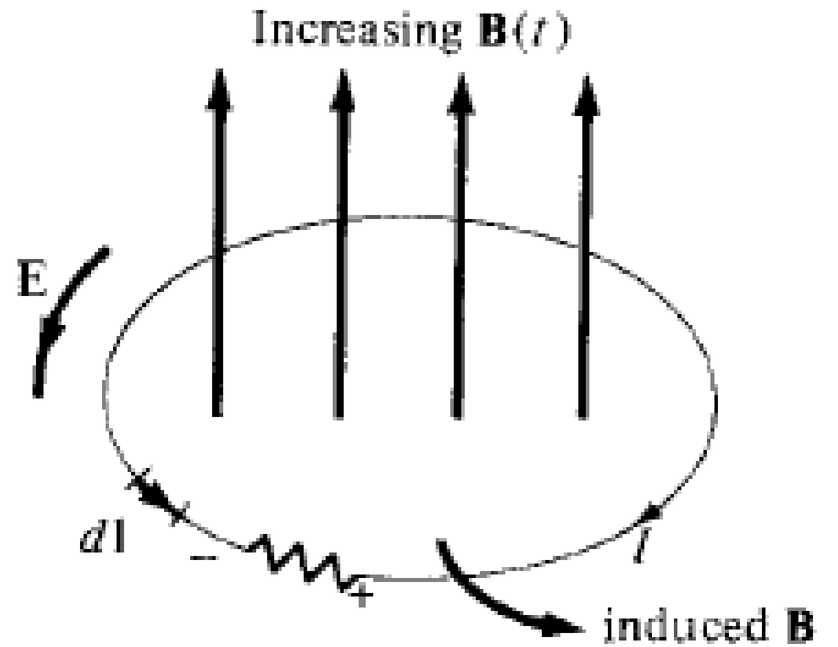
The variation of flux with time may be caused in three ways.

1. By having a stationary loop in a time-varying **B** field.
2. By having a time-varying loop area in a static **B** field.
3. By having a time-varying loop area in a time-varying **B** field.

Stationary loop in a time-varying B field (Transformer emf)

Consider a stationary conducting loop in a time-varying magnetic **B** field. The equation (i) becomes

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$



This emf induced by the time-varying current in a stationary loop is often referred to as transformer emf in power analysis since it is due to the transformer action.

By applying Stokes's theorem to the middle term, we get

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is one of the Maxwell's equations for time-varying fields.

It shows that the time-varying field is not conservative.

$$\nabla \times \mathbf{E} \neq 0$$

2. Moving loop in static B field (Motional emf)

When a conducting loop is moving in a static **B** field, an emf is introduced in the loop.

The force on a charge moving with uniform velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B}$$

The motional electric field $\mathbf{E}_m$ is defined as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B}$$

Consider a conducting loop moving with uniform velocity **u,** the emf induced in the loop is

$$V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(i)}$$

This kind of emf is called the motional emf or flux-cutting emf. Because it is due to the motional action. eg,. Motors, generators

By applying Stokes's theorem to equation (i), we get

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})$$

3. Moving loop in time-varying field

Consider a moving conducting loop in a time-varying magnetic field

Then both transformer emf and motional emf are present.

Thus the total emf will be the sum of transformer emf and motional emf

$$V_{\mathrm{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

also

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B})$$

Displacement Current

For static EM fields

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad \text{(i)}$$

But the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \qquad \text{(ii)}$$

But the continuity of current requires

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \qquad \text{(iii)}$$

Equation (ii) and (iii) are incompatible for time-varying conditions

So we need to modify equation (i) to agree with (iii)

Add a term to equation (i) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \qquad \text{(iv)}$$

where $\mathbf{J}_d$ is to defined and determined.

Again the divergence of the curl of a vector field is zero. So

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \qquad \text{(v)}$$

In order for equation (v) to agree with (iii)

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

or $$\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt} \qquad \text{(vi)}$$

Putting (vi) in (iv), we get

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

This is Maxwell's equation (based on Ampere Circuital Law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as displacement current density and **J** is the conduction current density $\mathbf{J} = \sigma \mathbf{E}$.

Maxwell's Equations in Final Form

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\dfrac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law