Subject Name-
ADVANCE QUANTUM MECHANICS
Subject Code- MPM-221

Teacher Name

Dr. Abhishek Kumar Gupta

Department of Physics and Material Science

Email: akgpms@mmmut.ac.in

Syllabus??

MPM-221: ADVANCE QUANTUM MECHANICS **Credit 04 (3-1-0)**

Unit I: Formulation of Relativistic Quantum Theory

Relativistic Notations, The Klein-Gordon equation, Physical interpretation, Probability current density & Inadequacy of Klein-Gordon equation, Dirac relativistic equation & Mathematical formulation, α and β matrices and related algebra, Properties of four matrices α and β, Matrix representation of $\alpha_i^{'s}$ and β, True continuity equation and interpretation.

Unit II: Covariance of Dirac Equation

Covariant form of Dirac equation, Dirac gamma (γ) matrices, Representation and properties, Trace identities, fifth gamma matrix γ^5, Solution of Dirac equation for free particle (Plane wave solution), Dirac spinor, Helicity operator, Explicit form, Negative energy states

Unit III: Field Quantization

Introduction to quantum field theory, Lagrangian field theory, Euler–Lagrange equations, Hamiltonian formalism, Quantized Lagrangian field theory, Canonical commutation relations, The Klein-Gordon field, Second quantization, Hamiltonian and Momentum, Normal ordering, Fock space, The complex Klein-Gordan field: complex scalar field

Unit IV: Approximate Methods

Time independent perturbation theory, The Variational method, Estimation of ground state energy, The Wentzel-Kramers-Brillouin (WKB) method, Validity of the WKB approximation, Time-Dependent Perturbation theory, Transition probability, Fermi-Golden Rule

Books & References:

1: Advance Quantum Mechanics by J. J. Sakurai (Pearson Education India)

2: Relativistic Quantum Mechanics by James D. Bjorken and Sidney D. Drell (McGraw-Hill Book Company; New York, 1964).

3: An Introduction to Relativistic Quantum Field Theory by S.S. Schweber (Harper & Row, New York, 1961).

4: Quantum Field Theory by F. Mandl & G. Shaw (John Wiley and Sons Ltd, 1984)

5: A First Book of Quantum Field Theory by A. Lahiri & P.B. Pal (Narosa Publishing House, New Delhi, 2000)

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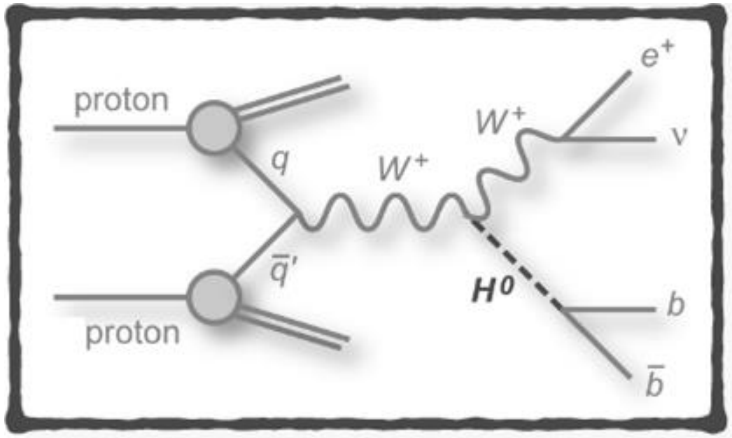
Lectures of Unit- I



Relativistic quantum mechanics (RQM)

Relativistic quantum mechanics (RQM) is formulation of quantum mechanics (QM) which is applicable to all massive particles propagating at all velocities up to those comparable to the speed of light c and can accommodate massless particles.

V=0 to c, m= 0 to V & m= infinite.

The theory has application in high energy physics, particle physics and accelerator physics, as well as atomic physics, chemistry and condensed matter physics.

- Relativistic quantum mechanics (RQM) is quantum mechanics applied with special relativity. Although the earlier formulations, like the Schrödinger picture and Heisenberg picture were originally formulated in a non-relativistic background, a few of them (e.g. the Dirac or path-integral formalism) also work with special relativity.

RQMs have beauty and features to explore depth understanding of-

The prediction of matter and antimatter, -Spin magnetic moments of elementary spin fermions,

-Fine structure, and quantum dynamics of charged particles in electromagnetic fields.

-Depth of high energy physics, particle physics and accelerator physics, as well as atomic physics, chemistry and condensed matter physics.

-The most successful (and most widely used) RQM is relativistic quantum field theory (QFT), in which elementary particles are interpreted as field quanta. A unique consequence of QFT that has been tested against other RQMs is the failure of conservation of particle number, for example in matter creation and annihilation.

Klein–Gordon equation

-The Klein–Gordon equation is a relativistic wave equation, related to the Schrödinger equation.

-It is second-order in space and time and manifestly Lorentz-covariant. It is a quantized version of the relativistic energy–momentum relation. Its solutions include a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles.

-Its theoretical relevance is similar to that of the Dirac equation. Electromagnetic interactions can be incorporated, forming the topic of s calar electrodynamics, but because common spinless particles like the pions are unstable and also experience the strong interaction (with unknown interaction term in the Hamiltonian, the practical utility is limited.

Schrödinger representation ?

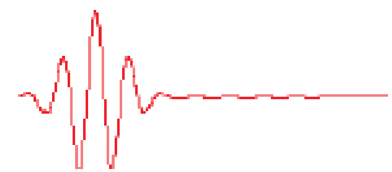
Equation

Time-dependent equation

The form of the Schrödinger equation depends on the physical situation (see below for special cases). The most general form is the time-dependent Schrödinger equation (TDSE), which gives a description of a system evolving with time:[5]:143

Time-dependent Schrödinger equation *(general)*

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$



A wave function that satisfies the nonrelativistic Schrödinger equation with $V = 0$. In other words, this corresponds to a particle traveling freely through empty space. The real part of the wave function is plotted here.

where i is the imaginary unit, $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant having the dimension of action,[6][7][note 2] Ψ (the Greek letter psi) is the state vector of the quantum system, t is time, and $\hat{H}$ is the Hamiltonian operator. The *position-space wave function* of the quantum system is nothing but the components in the expansion of the state vector in terms of the position eigenvector $|\mathbf{r}\rangle$. It is a scalar function, expressed as $\Psi(\mathbf{r}, t) = \langle \mathbf{r}|\Psi\rangle$. Similarly, the *momentum-space wave function* can be defined as $\tilde{\Psi}(\mathbf{p}, t) = \langle \mathbf{p}|\Psi\rangle$, where $|\mathbf{p}\rangle$ is the momentum eigenvector.

The most famous example is the nonrelativistic Schrödinger equation for the wave function in position space $\Psi(\mathbf{r}, t)$ of a single particle subject to a potential $V(\mathbf{r}, t)$, such as that due to an electric field.[8][note 3]

Time-dependent Schrödinger equation in position basis
(single nonrelativistic particle)

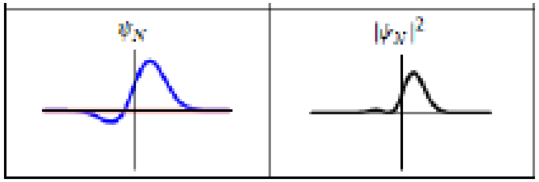
$$i\hbar \frac{\partial}{\partial t}\Psi(\mathbf{r}, t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right]\Psi(\mathbf{r}, t)$$

where m is the particle's mass, and ∇^2 is the Laplacian.

Schrödinger representation ?

Time-independent equation

The time-dependent Schrödinger equation described above predicts that wave functions can form standing waves, called stationary states.[note 4] These states are particularly important as their individual study later simplifies the task of solving the time-dependent Schrödinger equation for *any* state. Stationary states can also be described by a simpler form of the Schrödinger equation, the *time-independent Schrödinger equation* (TISE).



Each of these three rows is a wave function which satisfies the time-dependent Schrödinger equation for a harmonic oscillator. Left: The real part (blue) and imaginary part (red) of the wave function. Right: The probability distribution of finding the particle with this wave function at a given position. The top two rows are examples of stationary states, which correspond to standing waves. The bottom row is an example of a state which is *not* a stationary state. The right column illustrates why stationary states are called "stationary".

Time-independent Schrödinger equation (*general*)

$$\hat{\mathrm{H}}\,|\Psi\rangle = E|\Psi\rangle$$

where E is a constant equal to the energy level of the system. This is only used when the Hamiltonian itself is not dependent on time explicitly. However, even in this case the total wave function still has a time dependency.

In the language of linear algebra, this equation is an eigenvalue equation. Therefore, the wave function is an eigenfunction of the Hamiltonian operator with corresponding eigenvalue(s) E.

As before, the most common manifestation is the nonrelativistic Schrödinger equation for a single particle moving in an electric field (but not a magnetic field):

Time-independent Schrödinger equation (*single nonrelativistic particle*)

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$$

with definitions as above. Here, the form of the Hamiltonian operator comes from classical mechanics, where the Hamiltonian *function* is the sum of the kinetic and potential energies. That is, $H = T + V = \frac{\|\mathbf{p}\|^2}{2m} + V(x,y,z)$ for a single particle in the non-relativistic limit.

Energy–momentum relation

In physics, the energy–momentum relation, or relativistic dispersion relation, is the relativistic equation relating an object's total energy to its rest (intrinsic) mass and momentum. It is the extension of mass-energy relation for objects in motion:

$$E^2 = (pc)^2 + \left(m_0 c^2\right)^2$$

This equations holds for a system, such as a particle or macroscopic body, having intrinsic rest mass m0, total energy E, and a momentum of magnitude p, where the constant c is the speed of light, assuming the special relativity case of flat spacetime.
The Dirac sea model, which was used to predict the existence of antimatter, is closely related to the energy-momentum equation.

Formulation of Relativistic Quantum Theory

- *Klein-Gordon Equation*
- *Dirac Equation: free particles*
- *Dirac Equation: interactions*

  $e^+e^- \to \mu^+\mu^-$ *cross section*

Klein Gordon Equation

Klein–Gordon equation

The Klein–Gordon equation (Klein–Fock–Gordon equation or sometimes Klein–Gordon–Fock equation) is a relativistic wave equation, related to the Schrödinger equation. It is second-order in space and time and manifestly Lorentz-covariant. It is a quantized version of the relativistic energy–momentum relation. Its solutions include a quantum scalar or pseudoscalar field, a field whose quanta are spinless particles. Its theoretical relevance is similar to that of the Dirac equation.[1] Electromagnetic interactions can be incorporated, forming the topic of scalar electrodynamics, but because common spinless particles like the pions are unstable and also experience the strong interaction (with unknown interaction term in the Hamiltonian,) the practical utility is limited.

The equation can be put into the form of a Schrödinger equation. In this form it is expressed as two coupled differential equations, each of first order in time.[3] The solutions have two components, reflecting the charge degree of freedom in relativity.[3][4] It admits a conserved quantity, but this is not positive definite. The wave function cannot therefore be interpreted as a probability amplitude. The conserved quantity is instead interpreted as electric charge, and the norm squared of the wave function is interpreted as a charge density. The equation describes all spinless particles with positive, negative, and zero charge.

Any solution of the free Dirac equation is, component-wise, a solution of the free Klein–Gordon equation.

The equation does not form the basis of a consistent quantum relativistic *one-particle* theory. There is no known such theory for particles of any spin. For full reconciliation of quantum mechanics with special relativity, quantum field theory is needed, in which the Klein–Gordon equation reemerges as the equation obeyed by the components of all free quantum fields.[nb 1] In quantum field theory, the solutions of the free (noninteracting) versions of the original equations still play a role. They are needed to build the Hilbert space (Fock space) and to express quantum field by using complete sets (spanning sets of Hilbert space) of wave functions.

Klein-Gordon Equation

For particles of rest mass m, energy and momentum are related by

$$E^2 = m^2c^4 + c^2\mathbf{p}^2. \tag{3.1}$$

If the particles can be described by a single scalar wavefunction $\phi(x)$, the prescription of non-relativistic quantum mechanics

$$\mathbf{p} \to -i\hbar\nabla, \qquad E \to i\hbar\, \partial/\partial t \tag{3.2}$$

leads to the Klein–Gordon equation (2.27):

$$(\Box + \mu^2)\phi(x) = 0 \tag{3.3}$$

Lorentz invariant Schrödinger eqn.?

With the quantum mechanical energy & momentum operators:

$$E = i\frac{\partial}{\partial t}$$

$$\vec{p} = -i\vec{\nabla}$$

recall: $p^\mu = (E, \vec{p})$ ***and*** $\partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

You simply 'derive' the Schrödinger equation from classical mechanics:

$$E = \frac{\boldsymbol{p}^2}{2m} \rightarrow i\frac{\partial}{\partial t}\phi = -\frac{1}{2m}\nabla^2\phi$$

Schrödinger equation

With the relativistic relation between E, p & m you get:

$$E^2 = \boldsymbol{p}^2 + m^2 \rightarrow \frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

Klein-Gordon equation

Free Klein-Gordon particle wave functions

With the quantum mechanical energy & momentum operators:

$$E = i\frac{\partial}{\partial t}$$

$$\vec{p} = -i\vec{\nabla}$$

recall: $\boldsymbol{p^\mu = (E, \vec{p})}$ ***and*** $\boldsymbol{\partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)}$

non-relativistic $E = \frac{p^2}{2m}$ ***yields Schrödinger equation:*** $i\frac{\partial}{\partial t}\phi = -\frac{1}{2m}\nabla^2\phi$

We 'derived' Klein-Gordon equation from relativistic $\boldsymbol{E^2 = \vec{p}^2 + m^2}$

$$\frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi \quad \textbf{or} \quad \left(\partial_\mu\partial^\mu + m^2\right)\phi = 0$$

‘Simple’ plane-wave solutions for ϕ: ??

Use 4-derivatives to make Klein-Gordon equation Lorentz invariant:

$$\frac{\partial^2}{\partial t^2}\phi = \nabla^2 \phi - m^2 \phi \rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0$$

'Simple' plane-wave solutions for ϕ: ??

1 Let the sorn of eq. K-G eqn

$$\left(\partial^{\mu}\partial_{\mu} + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0 \rightarrow (1)$$

in the form of plane wave is:

$$\phi(x) = \phi(k)\, e^{-ikx} = \phi(k)\, e^{-ik_{\mu}x^{\mu}} \rightarrow (2)$$

where 4-vectr k is defined as

$$k^{\mu} = \left(\frac{\omega}{c}, \vec{k}\right) \rightarrow (3)$$

whe ω is freq. of wave &

k is the wave no

& ∴ 4-momentum vectr could be :-

$$p^{\mu} = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right) \rightarrow (4)$$

'Simple' plane-wave solutions for ϕ: ??

2 from (1) & (2)

$$\partial^\mu \partial_\mu \phi(x) = \phi(k)\,\partial^\mu\left(\frac{\partial}{\partial x^\mu} e^{-ik_\mu x^\mu}\right)$$

$$= \phi(k)\,\partial^\mu\left(-ik_\mu e^{-ik_\mu x^\mu}\right)$$

$$= -k^\mu k_\mu\, \phi(k)\, e^{-ik_\mu x^\mu} \quad (5)$$

use above in eqn (1) then K.G. eqn becomes:-

$$\left(k^\mu k_\mu - \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0 \quad (6)$$

which can be solved for

$$k^\mu k_\mu = \frac{m^2c^2}{\hbar^2}$$

i.e. $\frac{\omega^2}{c^2} - k^2 = \frac{m^2c^2}{\hbar^2}$

$\Rightarrow \hbar^2\omega^2 - \hbar^2 k^2 c^2 = m^2c^4$

$\Rightarrow \hbar^2\omega^2 = m^2c^4 + \vec{p}^{\,2}c^2$

or $\boxed{E^2 = \vec{p}^{\,2}c^2 + m^2c^4}$ — (

Probability & current densities ??

Probability & current densities

ψ^* time 3 Inadequacy of K.G. eqn

Charge & current densities:

K. G. eqn for a free particle is:

$$\left(\Box^2 - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

or, $$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

or, $$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0 \rightarrow (13)$$

taking complex conjugate of above eqn, we get

$$\nabla^2\psi^* - \frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^* = 0 \quad (14)$$

multiplying (13) & (14) by ψ^* & ψ resp. from left & right we get

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Probability & current densities Cont..

4 taking complex conjugate of above eqn, we get

$$\nabla^2\psi^* - \frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^* = 0 \quad (14)$$

multiplying (15) & (14) by ψ^* & ψ resp. from left & right we get

$$\psi^*\nabla^2\psi - \frac{1}{c^2}\psi^*\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^*\psi = 0 \quad (15)$$

$$\psi\nabla^2\psi^* - \frac{1}{c^2}\psi\frac{\partial^2\psi^*}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^*\psi = 0 \quad (16)$$

Subtracting (15) − (16)

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* - \frac{1}{c^2}\left[\psi^*\frac{\partial^2\psi}{\partial t^2} - \psi\frac{\partial^2\psi^*}{\partial t^2}\right] = 0$$

or, $\nabla\cdot\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] - \frac{1}{c^2}\frac{\partial}{\partial t}$

$\left[\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi\right.$

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Probability & current densities Cont..

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... $\frac{\hbar}{2im}$, we get

$$\nabla \cdot \left[\frac{\hbar}{2im}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right)\right] \to \vec{J}(\bar{r},t)$$

$$+ \frac{\partial}{\partial t}\left[\frac{\hbar}{2imc^2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right)\right] = 0 \quad \to (17)$$

$\to \rho(\bar{r},t)$

put

$$\rho(\bar{r},t) = \frac{\hbar}{2imc^2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right) \to (a)$$

&

$$\vec{J}(\bar{r},t) = \frac{\hbar}{2im}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right)$$

eq^n (17) becomes.

$$\nabla \cdot \vec{J}(r,t) + \frac{\partial \rho(\bar{r},t)}{\partial t} = 0 \to (19)$$

which is well known equation of continuity.

Difficulties with Probability densities

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Difficulty with $\rho(\bar{r},t)$:-

in eq^n (19) current density $j(\bar{r},t)$ have the same form as in non-relativistic case, but the $\rho(\bar{r},t)$ can not be interpreted as position probability density in analogy with non relativistic case in which $\rho(\bar{r},t) = \psi^*\psi$, because of following reasons:-

$\because$

$$\rho(\bar{r},t) = \frac{\hbar}{2imc^2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right)$$

$$= \frac{1}{2mc^2}\left[\left(-i\hbar\frac{\partial\psi^*}{\partial t}\right)\psi\right.$$

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$$\rho(\bar{r},t) = \frac{\hbar}{2imc^2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right)$$

$$= \frac{1}{2mc^2}\left[\left(-i\hbar\frac{\partial\psi^*}{\partial t}\right)\psi + \psi^*\left(i\hbar\frac{\partial\psi}{\partial t}\right)\right]$$

$$= \frac{1}{2mc^2}\left[(E\psi^*)\psi + \psi^*(E\psi)\right]$$

$$= \frac{1}{2mc^2}\left[E\psi^*\psi + E\psi^*\psi\right]$$

$$\rho(\bar{r},t) = \frac{E}{mc^2}\left[\psi^*\psi\right]$$

$\because$ $E = \pm\sqrt{p^2c^2 + m^2c^4}$

$\Rightarrow$ $E \rightarrow$ energy of a particle can be either +ve or −ve.

$\Rightarrow$ $\rho(\bar{r},t)$ is not definitely +ve

$\Rightarrow$ it is not regarded as conventional position prob. density.

Dirac Relativistic Equation??

Dirac Relativistic Equation Cont..

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Mathematical formulation:-

$$E\psi = H\psi$$

i.e. $$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

Dirac took an eqn which, H is linear in energy & momentum.

$$H = c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2$$ Hamiltonian; its linear

i.e. in time derivative & space derivatives as well. This eqn have the form:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \{c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2\}\psi(x,t) = H\psi(x,t)$$

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

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Dirac Relativistic Equation Cont..

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$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \{c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2\}\psi(x,t) = H\psi(x,t)$$

$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2$$

Here H may have different form

$$= \beta mc^2 + c\left(\sum_{n=1}^{3} \alpha_n p_n\right)$$

$$= \beta mc^2 + \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3}\right)$$

here $p_1, p_2, p_3 \equiv \frac{\hbar}{i}\frac{\partial}{\partial x^1}, \frac{\hbar}{i}\frac{\partial}{\partial x^2}, \frac{\hbar}{i}\frac{\partial}{\partial x^3}$ are the components of the momentum, understood to be the momentum operator.

$$H = -i\hbar c(\vec{\alpha}\cdot\vec{\nabla}) + \beta mc^2$$

$$H = c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2$$

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Dirac Relativistic Equation Cont..

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∴ Dirac eqn can be written in different form :-

$$\left[\beta mc^2 + c\left(\sum_{n=1}^{3} \alpha_n p_n\right)\right]\psi(x,t) = i\hbar\frac{\partial \psi(x,t)}{\partial t}$$

or,

$$\frac{\hbar c}{i}\left(\alpha_1\frac{\partial \psi}{\partial x^1} + \alpha_2\frac{\partial \psi}{\partial x^2} + \alpha_3\frac{\partial \psi}{\partial x^3}\right) + \beta mc^2\psi = i\hbar\frac{\partial \psi}{\partial t}$$

or,

$$\left[c\,\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2\right]\psi(x,t) = i\hbar\frac{\partial \psi}{\partial t} = E\psi$$

Simply

$$\left(E - c\vec{\alpha}\cdot\vec{p} - \beta mc^2\right)\psi(x,t) = 0 \quad (5)$$

Dirac Relativistic Equation Cont..

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$$\left[c\,\vec{\alpha}\cdot(-i\hbar\vec{\nabla}) + \beta mc^2\right]\psi(r,t) = i\hbar\frac{\partial\psi}{\partial t}$$

$$= E\psi$$

Simply

$$\left(E - c\,\vec{\alpha}\cdot\vec{p} - \beta mc^2\right)\psi(r,t) = 0 \quad \rightarrow (5)$$

Dirac's purpose in casting this eqn was to explain the behaviour of the relativistically moving e^-, & so to allow the atom to be treated in a manner consistent with the relativity.

- Dirac's eqn had far deeper implications for the structure of matter & introduced new mathematical classes of objects that are now essential elements of fundamental physics.

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Dirac Relativistic Equation Cont..

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New things in Dirac's eqn is now to find out :-

4×4 Dirac matrices α_k (or α_i) & β

& 4-component wave function ψ.

operating the eqn (5) by

$(E + c\vec{\alpha}\cdot\vec{p} + \beta mc^2)$ from left,

we get

$$(E + c\vec{\alpha}\cdot\bar{p} + \beta mc^2)(E - c\vec{\alpha}\cdot\bar{p} - \beta mc^2)\psi = 0$$

or,

$$[E^2 - (c\vec{\alpha}\cdot\bar{p} + \beta mc^2)^2]\psi = 0$$

or,

$$\{E^2 - c^2(\vec{\alpha}\cdot\bar{p})^2 - \beta^2 m^2 c^4 - mc^3(\vec{\alpha}\cdot\bar{p})\beta - mc^3\beta\vec{\alpha}\cdot\bar{p}\}\psi = 0 \quad \rightarrow (6)$$

Dirac Relativistic Equation Cont..

$$[E^2 - c^2(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)^2$$
$$- \beta^2 m^2 c^4 - mc^3(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)\beta$$
$$- mc^3 \beta(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)]\psi = 0$$

or,

$$[E^2 - c^2\{\alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2 +$$
$$(\alpha_x\alpha_y + \alpha_y\alpha_x)p_x p_y +$$
$$(\alpha_y\alpha_z + \alpha_z\alpha_y)p_y p_z +$$
$$(\alpha_z\alpha_x + \alpha_x\alpha_z)p_z p_x\} - \beta^2 m^2 c^4$$
$$- mc^3\{(\alpha_x\beta + \beta\alpha_x)p_x +$$
$$(\alpha_y\beta + \beta\alpha_y)p_y +$$
$$(\alpha_z\beta + \beta\alpha_z)p_z\}]\psi = 0$$

Dirac Relativistic Equation Cont..

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$$\Big[E^2 - c^2 \{ \alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2 +$$

~~$(\alpha_x p_x)(\alpha_y p_y)$~~

$$(\alpha_x \alpha_y + \alpha_y \alpha_x) p_x p_y +$$

$$(\alpha_y \alpha_z + \alpha_z \alpha_y) p_y p_z +$$

$$(\alpha_z \alpha_x + \alpha_x \alpha_z) p_z p_x \} - \beta^2 m^2 c^4$$

$$- mc^3 \{ (\alpha_x \beta + \beta \alpha_x) p_x +$$

$$(\alpha_y \beta + \beta \alpha_y) p_y +$$

$$(\alpha_z \beta + \beta \alpha_z) p_z \} \Big] \psi = 0$$

$\rightarrow$ (8)

also, we know from K.G. eqn that

$$\left[E^2 - c^2 (p_x^2 + p_y^2 + p_z^2) - m^2 c^4\right] \psi = 0$$

$\rightarrow$ (9)

comparing (8) & (9), we can get that, ~~the~~ ~~the~~ both ~~are~~ may stand & resurrect if four-matrices $\alpha_i (i = \alpha_x, \alpha_y, \alpha_z)$ & β obey the

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Dirac Relativistic Equation Cont..

21 matrix algebra :-

(i) & (II) → They all mutually anti-commute.

(i) $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$ → (10)

{or, $\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$ if $i \neq j$ pos}

i.e.

$$\alpha_x \alpha_y + \alpha_y \alpha_x = 0;$$
$$\alpha_y \alpha_z + \alpha_z \alpha_y = 0;$$
$$\alpha_z \alpha_x + \alpha_x \alpha_z = 0$$

(ii) $\alpha_i \beta + \beta \alpha_i = 0$ → (11)

i.e.

$$\alpha_x \beta + \beta \alpha_x = 0;$$
$$\alpha_y \beta + \beta \alpha_y = 0;$$
$$\alpha_z \beta + \beta \alpha_z = 0$$

i.e. they anticommute with one another in pairs.

(iii) $\alpha_i^2 = \beta^2 = 1$ = Identity matrix. → (12)

i.e. $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

i.e. their squares are equal

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Dirac Relativistic Equation Cont..

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$$\left[E^2 + \hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\alpha_j \alpha_i + \alpha_i \alpha_j}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j}\right.$$

$$-\frac{\hbar}{i} m c^3 \sum_{i=1}^{3} (\alpha_i \beta + \beta \alpha_i) \frac{\partial \psi}{\partial x^i}$$

$$\left.- \beta^2 m^2 c^4 \psi\right] \psi = 0$$

or,

$$E^2 \psi = -\hbar^2 c^2 (\quad) + \frac{\hbar}{i} m c^3 (\quad)$$

$$+ \beta^2 m^2 c^4 \psi$$

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 (\quad) + \frac{\hbar}{i} m c^3 (\quad)$$

$$+ \beta^2 m^2 c^4 \psi$$

Dirac Relativistic Equation Cont..

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Properties of four matrices α_i & β:-

(1) The α_i & β must be Hermitian matrices.

Component of Hamiltonian appearing in σ^{th} row & τ^{th} column is:

$$\hat{H}_{\sigma\tau} = \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial}{\partial x^1} + \alpha_2 \frac{\partial}{\partial x^2} + \alpha_3 \frac{\partial}{\partial x^3}\right)_{\sigma\tau} + \beta_{\sigma\tau} mc^2$$

Since, $H_{\sigma\tau}$ is physically observable & sum of a Hermitian operator & Hermitian operator is also Hermitian operator. Therefore α_j's & β are also Hermitian operator.

Dirac Relativistic Equation Cont..

operator. Therefore α_i's & β are also Hermitian operator.

② Eigen values of α_i's & β :→

Eigen value eqn is

$$A|\psi\rangle = a|\psi\rangle$$

Now,

$$\alpha_i|\psi\rangle = \lambda|\psi\rangle$$

$$\Rightarrow \alpha_i^2|\psi\rangle = \alpha_i(\lambda|\psi\rangle)$$

$$\Rightarrow |\psi\rangle = \lambda(\alpha_i|\psi\rangle)$$

$$\Rightarrow |\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow \boxed{\lambda^2 = \pm 1}$$

→ eigenvalues of α_i's & β are real i.e. ± 1.

(3) α_i's & β are anticommutator

from eqn (10), we have

$\because \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}$

if $j \neq k$ then

$\alpha_j \alpha_k + \alpha_k \alpha_j = 0$

$\Rightarrow \{\alpha_j, \alpha_k\} = 0$

& $\alpha_i \beta + \beta \alpha_i = 0$

$\Rightarrow \{\alpha_i, \beta\} = 0$

Dirac Relativistic Equation Cont..

(4) Trace of α_i's & β matrices are zero.

from eqn (1), we have

$$\alpha_i \beta = -\beta \alpha_i$$

$$\Rightarrow \alpha_i \beta \beta = -\beta \alpha_i \beta$$

$$\Rightarrow \alpha_i = -\beta \alpha_i \beta$$

$$\therefore \text{Trace}(\alpha_i) = \text{Trace}(-\beta \alpha_i \beta)$$

$$= -\text{Trace}(\alpha_i \beta^2)$$

$\{\because Tr[ABC] = Tr[CA$

$$= -\text{Trace}(\alpha_i) \rightarrow (4a)$$

$$\Rightarrow \boxed{\text{Trace}(\alpha_i) = 0}$$

Similarly,

$$\alpha_i \beta = -\beta \alpha_i$$

$$\Rightarrow \alpha_i^2 \beta = -\alpha_i \beta \alpha_i$$

$$\Rightarrow \beta = -\alpha_i \beta \alpha_i$$

Dirac Relativistic Equation Cont..

$\Rightarrow Tr(\beta) = -\text{Trace}(\beta \alpha_i^2)$

$= -\text{Trace}\ \beta$

$\Rightarrow \boxed{\text{Trace}(\beta) = 0} \rightarrow (4b)$

eqn (4a) & (4b), shows that the trace of each of matrices α_i & β must be zero.

(5) Dimension of α_i's & β - Dirac matrices

~~Let~~ since the trace is just the sum of eigenvalues, the number of eigenvalue ± 1 must +ve & -ve eigenvalue must be equal and the α_i & β must therefore be even dimensional

Dirac Relativistic Equation Cont..

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(b) Matrix representation of α_i's & β Dirac matrices

Since we know that matrices α_i & β are s.t. their squares = unity & they anticommute with another.

As, we already know that there are 3 well known 2×2 matrices σ_x, σ_y, σ_z (which are called as Pauli spin matrices), which are independent & non-commuting matrices & given as →

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (6a)$$

$$\sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (6b)$$

$$\sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6c)$$

Dirac Relativistic Equation Cont..

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as Pauli

are independent & non-commuting matrices given as →

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (6a)$$

$$\sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (6b)$$

$$\sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6c)$$

~~Since a 2×2 matrix has~~

These 3 Pauli spin matrices are satisfying following properties:-

(i) $(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = 1$

(ii) $\sigma^1 \sigma^2 = i\sigma^3$

$\sigma^2 \sigma^3 = i\sigma^1$

$\sigma^3 \sigma^1 = i\sigma^2$

Dirac Relativistic Equation Cont..

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or, in more general –

$$\sigma^k \sigma^l = \delta^{kl} + i \epsilon^{klm} \sigma^m \quad (5a)$$

& Also if we take

$$\alpha^i = \underset{2\times 2 \text{ matrix}}{\sigma^i} \quad (i = 1,2,3 \text{ or } x,y,3)$$

$$\& \quad \beta = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly, these four matrices are linearly independent matrices.

because three matrices $\sigma_x, \sigma_y, \sigma_3$ are already independent hence only other fourth linearly independent of these matrix is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

we can see that at the above four matrix satisfied the (10) & (12) condition

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Dirac Relativistic Equation Cont..

30a we can see that ~~th~~, all the above four matrix satisfied the condition (10) & (12)

i.e. $\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}$

& $\alpha_i^2 = \beta^2 = 1$

but it can't full filled the condition of (11) i.e. $\alpha_i \beta + \beta \alpha_i = 0$.

because as β is a unit matrix & therefore it will commute with each α_i rather to anticommute with every α_i. hence β as a unit matrix not satisfying all the properties of Dirac Matrices.

(11)

Dirac Relativistic Equation Cont..

31 Now, we will show that the next simplest choice is they are 4×4 matrices rather to 3×3 matrices as we know that Dirac matrices α_i & β must be even dimentional.

$\therefore$

we will choose β matrix as.

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Since β anticommute with all the components of α_i hence α_i also should be 4×4 matrix.

Therefore; using Pauli spin matrices σ_x, σ_y & σ_z we can have $\alpha_x, \alpha_y, \alpha_z$ as follows:—

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Dirac Relativistic Equation Cont..

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There fore; using Pauli spin matrices σ_x, σ_y & σ_z we can have $\alpha_x, \alpha_y, \alpha_z$ as follows:—

$$\alpha_x = \alpha_1 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_y = \alpha_2 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

&

$$\alpha_z = \alpha_3 = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Dirac Relativistic Equation Cont..

All these 4×4 matrices are Hermitian & in abbriviate form. they can be expressed as follow.

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

where each element is a matrix of 2 rows & 2 columns.

clearly, All the above four matrices satisfies all the conditions of (10) (11) (12) of Dirac matrices by help of above eqn (62).

$$\alpha^i \alpha^j = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^i \sigma^j & 0 \\ 0 & \sigma^i \sigma^j \end{pmatrix}$$

Dirac Relativistic Equation Cont..

clearly, All the above four matrices satisfies all the conditions of (10) (11) (12) of the Dirac matrices by help of above eqn (62).

$$\alpha^i \alpha^j = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^i \sigma^j & 0 \\ 0 & \sigma^i \sigma^j \end{pmatrix}$$

$$= \delta^{ij} + i \sum_{k=1}^{3} \epsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$$\Rightarrow \boxed{\alpha^i \alpha^j = \delta^{ij} + i\, \epsilon^{ijk} \Sigma^k}$$

$$\left\{ \text{where } \Sigma^k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \right\}$$

Dirac Relativistic Equation Cont..

(7) α_i & β are linearly independent :→

Suppose β can be written as a linear combination of α_i's then :–

$$\beta = b_i \alpha_i \longrightarrow (1)$$

↳ where b_i are c-number.

$$\because \quad \alpha_j \beta + \beta \alpha_j = 0$$

$$\Rightarrow \quad \alpha_j b_i \alpha_i + b_i \alpha_i \alpha_j = 0$$

$$\Rightarrow \quad b_i (\alpha_j \alpha_i + \alpha_i \alpha_j) = 0$$

$$\Rightarrow \quad b_i (2 \delta_{ji}) = 0$$

$$\Rightarrow \quad 2 b_j = 0$$

Then, $b_j = 0 \quad (\because j = 1, 2, 3)$

i.e. β can not be written as a linear combination of α_i's.

Similarly, α_i can not be written as linear combination of α_j's & β.

Dirac Relativistic Equation Cont..

Probability Density & current

Density: True interpreting

ψ : True continuity equation

we will now check whether the Dirac equation leads to the correct probability density or not.

As we know the Dirac relativistic eqn is :→

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^{3} \alpha_k \frac{\partial \psi}{\partial x_k} + \beta m c^2 \psi \quad \text{(1)}$$

• As α_i & β are 4x4 matrices then ψ will be 4x1 column matrix

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_3 \end{bmatrix}.$$

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Dirac Relativistic Equation Cont..

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^{3} \alpha_k \frac{\partial \psi}{\partial x_k} + \beta m c^2 \psi \quad \text{--- (1)}$$

- As α & β are 4x4 matrices then ψ will be 4x1 column matrix

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_3 \end{bmatrix}.$$

Now, to construct the law of current conservation, we will introduce the hermitian conjugate wave function $\psi^* = [\psi_1^* \; \psi_2^* \; \psi_3^* \; \psi_4^*]$

The Hermitian conjugate of (1) is

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar c}{i} \sum_{k=1}^{3} \frac{\partial \psi^*}{\partial x_k} \alpha_k + m c^2 \psi^* \beta \quad \text{--- (2)}$$

{where $\alpha_k^* = \alpha_k$ & $\beta^* = \beta$ }

Dirac Relativistic Equation Cont..

multiply eqn (1) by ψ^* from left and eqn (2) by ψ from right & subtract

Subtract (1) − (2)

$$i\hbar \psi^* \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\sum_{k=1}^{3} \psi^* \alpha_k \frac{\partial \psi}{\partial x^k} + mc^2 \psi^* \beta \psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t}\psi = -\frac{\hbar c}{i}\sum_{k=1}^{3} \frac{\partial \psi^*}{\partial x^k}\alpha_k \psi + mc^2\psi^*\beta\psi$$

−

$$i\hbar\left[\psi^*\partial_t\psi + \partial_t\psi^*\psi\right] = \frac{\hbar c}{i}\sum_{k=1}^{3}\left(\psi^*\alpha_k\partial_k\psi + \partial_k\psi^*\alpha_k\psi\right)$$

{where $\partial_t = \frac{\partial}{\partial t}$ & $\partial_k = \frac{\partial}{\partial x^k}$}

$$\Rightarrow i\hbar\frac{\partial}{\partial t}(\psi^*\psi) = -i\hbar c\sum_{k=1}^{3} \frac{\partial}{\partial x^k}(\psi^*\alpha_k\psi)$$

or, $$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x^k}(c\psi^*\alpha_k\psi)$$

Dirac Relativistic Equation Cont..

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or, $\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x^k}(c\psi^* \alpha_k \psi)$

or, $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(c\psi^* \alpha_k \psi) = 0$ → (3)

~~$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(j^k) = 0$~~

or, $\frac{\partial \rho}{\partial t} + \text{div}\, \vec{j} = 0$ → (3*)

where

$\rho = \psi^* \psi$ = probability density

$$\rho = \sum_{r=1}^{4} \psi_r^* \psi_r$$

Dirac Relativistic Equation Cont..

& Probability current density

$$j^k = c\,\psi^* \alpha_k \psi$$

↳ with three components.

Covariant form, from eqn (3)

$$\frac{c}{c}\frac{\partial}{\partial t}(\psi^*\psi) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(c\psi^*\alpha_k\psi) = 0$$

or

$$\frac{\partial}{\partial (ct)}(c\psi^*\psi) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(c\psi^*\alpha_k\psi) = 0$$

or,

$$\frac{\partial}{\partial x^0}(j^0) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(j^k) = 0$$

or,

$$\sum_{\mu=0}^{4}\frac{\partial}{\partial x^\mu}(j^\mu) = 0$$

$$\Rightarrow \partial_\mu j^\mu = 0$$

Dirac Relativistic Equation Cont..

& probability current density

$$j^k = c\,\psi^* \alpha_k \psi$$

with three components.

Covariant form, from eqn (3)

$$\frac{c}{c}\frac{\partial}{\partial t}(\psi^*\psi) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(c\psi^*\alpha_k\psi) = 0$$

or

$$\frac{\partial}{\partial(ct)}(c\psi^*\psi) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(c\psi^*\alpha_k\psi) = 0$$

or,

$$\frac{\partial}{\partial x_0}(j^0) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}(j^k) = 0$$

or,

$$\sum_{\mu=0}^{4}\frac{\partial}{\partial x^\mu}(j^\mu) = 0$$

$$\Rightarrow \quad \partial_\mu j^\mu = 0$$

where $j^\mu = (j^0, j^1, j^2, j^3)$

$= (c\psi^*\psi, c\psi^*\alpha_x\psi, c\psi^*\alpha_y\psi, c\psi^*\alpha_z\psi)$

Covariance form of the Dirac Equation

Covariance form of the Dirac Equation :-

- In discussion of covariance, we will express the Dirac eqn in 4-D notation which preserves the symmetry both ct & x^i. for this we will multiply dirac eqn

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \alpha_k \frac{\partial \psi}{\partial x^k} + \beta mc^2 \psi \rightarrow (1)$$

by β/c & we will introduce a notation

$$\gamma^0 = \beta \quad \& \quad \gamma^i = \beta\alpha_i, \; i = 1,2,3 \rightarrow (1a)$$

& we will get as follows:—

$$i\hbar \beta \frac{\partial \psi}{\partial (ct)} = \frac{\hbar}{i}\left[\beta \alpha_k \frac{\partial \psi}{\partial x^k}\right] + \beta^2 mc\psi \rightarrow (2)$$

$$[\beta^2 = 1]$$

$$i\hbar \beta \frac{\partial \psi}{\partial (ct)} = -i\hbar\left[\beta\alpha_k \frac{\partial \psi}{\partial x^k}\right] + mc\psi$$

$$\Rightarrow i\hbar\left[\beta \frac{\partial \psi}{\partial x^0} + \beta\alpha_1 \frac{\partial \psi}{\partial x^1} + \beta\alpha_2 \frac{\partial \psi}{\partial x^2} + \beta\alpha_3 \frac{\partial \psi}{\partial x^3}\right] - (mc)\psi = 0$$

$$\Rightarrow i\hbar\left[\gamma^0 \frac{\partial \psi}{\partial x^0} + \gamma^1 \frac{\partial \psi}{\partial x^1} + \gamma^2 \frac{\partial \psi}{\partial x^2} + \gamma^3 \frac{\partial \psi}{\partial x^3}\right] - (mc)\psi = 0$$

$$i\hbar\left[\gamma^0 \frac{\partial \psi}{\partial x^0} + \gamma^k \frac{\partial \psi}{\partial x^k}\right] - (mc)\psi = 0$$

or,

$$i\hbar \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - (mc)\psi = 0$$

$$\text{or, } \boxed{(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0} \rightarrow (3)$$

This eqn is a covariant form because here space & time derivatives are treated on equal footing.

- To represent it in more simpler form, it is convenient to introduce Feynman dagger, or slash, notation

$$\boxed{\not{A} = \gamma^\mu A_\mu = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^0 A^0 - \gamma \cdot \vec{A}} \rightarrow (4)$$

for this particular case,

$$\boxed{\not{\partial} = \gamma^\mu \partial_\mu = \gamma^\mu \frac{\partial}{\partial x^\mu} = \frac{\gamma^0}{c}\frac{\partial}{\partial t} + \gamma \cdot \vec{\nabla} = \not{\partial}} \rightarrow (5)$$

hence eqn (3) becomes

$$\boxed{(i\hbar \not{\partial} - mc)\psi = 0} \rightarrow (6)$$

if we put $p^\mu = i\hbar \frac{\partial}{\partial x_\mu}$

$$\Rightarrow \boxed{\not{p} = \gamma^\mu p_\mu = i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} = i\hbar \not{\partial} = i\hbar \not{\partial}}$$

∴ (6) becomes

$$\boxed{(\not{p} - mc)\psi = 0} \rightarrow (7)$$

Covariance form of the Dirac Equation

In natural units, the Dirac eqn may be written as

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

↳ where ψ is a Dirac spinor

In Feynman notation, the Dirac eqn:

$$(i\not\partial - m)\psi = 0$$

ψ is a multi-component object (spinor).

$\bar{\psi} = \psi^{\dagger}\gamma^{0}$ takes care of $\bar{\gamma}^{\dagger} = -\bar{\gamma}$ & useful in taking Hermitian conjugate of the equation.

Gamma Matrices

Gamma Matrices :-

It is important to realize in Dirac eqn that, the wave function(ψ) is now 4-component column vector.

We will now, introduce detail algebra & properties of new matrices γ^μ.

Since γ matrices are defined as:-

$$\gamma^0 = \beta \quad \& \quad \gamma^j = \beta\alpha_j \; ; \; j = 1,2,3.$$

& we have already introduced that using Pauli spin matrices, the matrices α_j & β matrices are defined as

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \& \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where I denotes unit 2×2 matrix &

Then the γ - matrices are.

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\& \; \gamma^j = \beta\alpha_j = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$$

Gamma matrices :-

The gamma matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ also known as Dirac matrices, are a set of conventional matrices with specific anti commutation relns.

~~In Dirac~~

Spinors facilitate spacetime computations & are very fundamental to the Dirac Eqn for relativistic spin ½ particles.

In Dirac representation the four contravariant gamma matrices are:

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\gamma^0 \rightarrow$ is time like matrix & other three are space like matrices.

(1) follows anti reln:

Properties of Gamma Matrices

Properties of Dirac γ– matrices

① γ^0 is Hermitian while γ^j are antihermitian operator.

Proof:→ Since α_j & β both are Hermitian operators.

Hence,

$\because$ $\gamma^0 = \beta \rightarrow \gamma^0$ is Hermitian operator

& $\gamma^j = \beta\alpha_j$

$$\Rightarrow (\gamma^j)^+ = (\beta\alpha_j)^+$$
$$= \alpha_j^+ \beta^+$$
$$= \alpha_j \beta$$
$$= -\beta\alpha_j$$
$$= -\gamma^j$$

$\Rightarrow$ γ^j are antihermitian operator.

② Anti commutation Property:–

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I$$

or $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$

where $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

& $I \rightarrow 4\times4$ unit matrix

i.e. $g_{00} = -g_{11} = -g_{22} = -g_{33} = +1$

& $g_{\mu\nu} = 0$ for $\mu \neq \nu$

Properties of Gamma Matrices

Proof (ii): for $\mu = 0, \nu \neq 0$

$\gamma^0\gamma^\nu + \gamma^\nu\gamma^0 = 0$

$\because \gamma^0\gamma^\nu = \beta\beta\alpha_\nu = \beta(-\alpha_\nu\beta)$

$= -\beta\alpha_\nu\beta$

$= -\gamma^\nu\beta$

$= -\gamma^\nu\gamma^0$

$\Rightarrow \boxed{\gamma^0\gamma^\nu + \gamma^\nu\gamma^0 = 0}$

(ii) for $\mu \neq \nu \neq 0$

$\gamma^\mu\gamma^\nu = (\beta\alpha_\mu)(\beta\alpha_\nu)$

$= (\beta\alpha_\mu)(-\alpha_\nu\beta)$

$= -\beta(\alpha_\mu\alpha_\nu)\beta$

$= \beta(\alpha_\nu\alpha_\mu)\beta$

$= (\beta\alpha_\nu)(\alpha_\mu\beta)$

$= -\gamma^\nu\gamma^\mu$

$\Rightarrow \boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0}$

(iii) for $\mu = \nu$

$\Rightarrow \{\gamma^\mu\gamma^\mu\} = (\gamma^\mu)^2 + (\gamma^\mu)^2$

To proof trace identities we will use 3 main properties of trace operator:

1. $Tr(A+B) = Tr(A) + Tr(B)$
2. $Tr(\gamma A) = \gamma\, Tr(A)$
3. $Tr(ABC) = Tr(CAB) = Tr(BCA)$

(4) Trace identities

The gamma matrices obey the following trace identities.

(i) $Tr(\gamma^\mu) = 0$

i.e. γ^μ's are traceless matrices

$Tr(\gamma^0) = Tr(\beta) = Tr\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$= 0 \rightarrow$ (a)

$Tr(\gamma^i) = Tr(\gamma^i\gamma^0\gamma^0) \quad \{\because (\gamma^0)^2 = 1$

$= Tr(\gamma^0\gamma^i\gamma^0)$

$\{\because Tr(ABC) = Tr(CAB) = Tr(BCA)$

$= -Tr(\gamma^i\gamma^0\gamma^0)$ (anti commutation law)

$= -Tr(\gamma^i)$

$\Rightarrow \boxed{Tr(\gamma^i) = 0} \rightarrow$ (b)

from (a) & (b)

$\Rightarrow \boxed{Tr(\gamma^\mu) = 0}$

(3) Square of γ-matrices:

$\boxed{(\gamma^0)^2 = \beta^2 = 1} \rightarrow$ (a)

$(\gamma^i)^2 = \gamma^i\gamma^i$

$= (\beta\alpha_i)(\beta\alpha_i)$

$= -(\alpha_i\beta)(\beta\alpha_i) = -\alpha_i\beta^2\alpha_i$

$= -(\alpha_i)^2 = -1$

$\Rightarrow \boxed{(\gamma^i)^2 = -1} \rightarrow$ (b)

Properties of Gamma Matrices

More trace identities

(5) $$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

Proof:-

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = \frac{1}{2}\left[\mathrm{Tr}(\gamma^\mu \gamma^\nu) + \mathrm{Tr}(\gamma^\nu \gamma^\mu)\right]$$

$$= \frac{1}{2}\mathrm{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$= \frac{1}{2}\mathrm{Tr}(\{\gamma^\mu, \gamma^\nu\})$$

$$= \frac{1}{2}\mathrm{Tr}(2 g^{\mu\nu} I)$$

$$= \frac{1}{2} \times 2 g^{\mu\nu}\, \mathrm{Tr}(I)$$

($\mathrm{Tr}(I)$ ↳ 4)

$$= 4 g^{\mu\nu}$$

{∵ I = 4×4 unit matrix}

Properties of Gamma Matrices

6 The fifth gamma matrix, γ^5:

it is useful to define the product of 4 gamma matrices as follows:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

γ^5 has also an alternative form:

$$\gamma^5 = \frac{i}{4!}\varepsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$$

Some properties of γ^5 are:

- it is hermitian
  $(\gamma^5)^\dagger = \gamma^5$
- its eigenvalues are ± 1 because
  $(\gamma^5)^2 = I_4 = 1$
- it anticommute with the 4 gamma matrices:
  $\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$

The above Dirac matrices can be written in terms of Dirac basis. Dirac basis is defined by following matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \beta \ \& \ \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \text{ where } k = 1 \text{ to } 3 \ \& \ \sigma^k \text{ are Pauli matrices.}$$

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the rel^n. $\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\sigma_k$

(17)

Properties of Gamma Matrices

(7) Trace of γ^5:

(i) Trace of $\gamma^5 = 0$

Proof :–

$$Tr(\gamma^5) = Tr(\gamma^0\gamma^0\gamma^5)$$

$\because \gamma^0\gamma^0 = 1$

$$= -tr(\gamma^0\gamma^5\gamma^0)$$

{anti commute of γ^5 with γ^0}

$$= -tr(\gamma^0\gamma^0\gamma^5)$$

$tr(ABC) = tr(CAB)$

$$= -tr(\gamma^5)$$

$$\Rightarrow 2\,tr(\gamma^5) = 0$$

$$\Rightarrow \boxed{tr(\gamma^5) = 0}$$

Similarly, we can show that

$$\boxed{tr(\gamma^\mu\gamma^\nu\gamma^5) = 0}$$

i.e. Trace of odd no. of γ is zero. $Tr(\gamma^\mu) = 0$

i.e. Tr (odd no. of γ) $= 0$

Solution of Dirac Equation for free particles: Plane wave solution

Solution of Dirac Equation for free particle (Plane wave solution):

Dirac spinor

Dirac eqn is

$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$, put $\gamma^\mu \partial_\mu = \not\partial$

or, $(i\not\partial - m)\psi(x) = 0 \quad \rightarrow (1)$

- For quantum field theory, the Dirac eqn admits plane wave solution ~~(known as Dirac spinor; which is bi-spinor) is~~ →

$$\psi(x) = w_{\vec p}\, e^{-ipx} = u(p)\, e^{-ipx} \quad \rightarrow (2)$$
$$= u_{\vec p}\, e^{-ipx}$$

where (i) $w_{\vec p} = u(p) = u_{\vec p}$ is known as Dirac spinor related to a plane-wave with wave-vector $\vec p$, which is a column vector of type

$$w_{\vec p} = u(p) = \begin{bmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{bmatrix} = \begin{bmatrix} u_1 \\ \frac{\vec\sigma\cdot\vec p}{E_p + m} u_1 \end{bmatrix} = \begin{bmatrix} \phi \\ \frac{\vec\sigma\cdot\vec p}{E_p+m}\phi \end{bmatrix}$$

ϕ arbitrary two-spinor

$\vec\sigma$ are the Pauli matrices

E_p is the +ve square root.

$$E_p = +\sqrt{m^2 + \vec p^2}$$

& $px = p_\mu x^\mu \equiv p^0 x^0 - \vec p\cdot\vec x = Et - \vec p\cdot\vec x$ (4-position, 4-momentum)

put (2) in (1)

$$i\gamma^\mu \left\{\frac{\partial}{\partial x^\mu} e^{-ip_\mu x^\mu}\right\} u(p) - m e^{-ip_\mu x^\mu} u(p) = 0$$

$\Rightarrow i\gamma^\mu(-ip_\mu) e^{-ipx} u(p) - m(\quad) = 0$

$\Rightarrow (\gamma^\mu p_\mu - m) u(p) = 0$

or, $\boxed{(\not p - m)u(p) = 0} \rightarrow (4)$ (in natural unit here $c=1$)

So, we have to find out $u(p)$ which satisfy eqn (4) so that $\psi(x)$ can be find out.

As, $u(p)$ has 4-component, hence (4) is a system of 4-eqn linear homogeneous eqns for having components u_α, $\alpha = 1,2,3,4$ for which eqn. The non-trivial soln of eqn (4) can be find out only if $\det(\not p - m) = 0$

(10)

Solution of Dirac Equation for free particles: Continue…

$\Rightarrow (p^2 - m^2)^2 = 0$

$\Rightarrow p^2 = m^2$

or, $p_0^2 - \vec{p}^{\,2} = m^2$

or $p_0^2 = \vec{p}^{\,2} + m^2$

or, $\boxed{p_0 = \pm E(\vec{p})}$ (in natural unit)

$\{\because \; E(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}\,\} \rightarrow (5)$

$\Rightarrow$ there exists two solutions $u_+(\vec{p})$ & $u_-(\vec{p})$ corresponding to two values of energy $+E(\vec{p})$ & $-E(\vec{p})$ respectively.

- Let us suppose that $u_+(\vec{p})$ is a solution for $p_0 = +E(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$ so that $u_+(\vec{p})$ satisfied the Dirac eqn.–

$$(c\,\vec{\alpha}\cdot\vec{p} + \beta m c^2)\, u_+(\vec{p}) = E(\vec{p})\, u_+(\vec{p})$$

($c = 1$ in nat. unit)

$$\rightarrow (\vec{\alpha}\cdot\vec{p} + \beta m)\, u_+(\vec{p}) = E(\vec{p})\, u_+(\vec{p}) \rightarrow (6)$$

Let us write

$u_+ = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where u_1 & u_2 both have two components and adopt the value of matrices $\vec{\alpha}$ & β as:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \;\&\; \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Hence eqn written as

$$\left[\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\vec{p} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} m\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = E(\vec{p})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow \left[\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}\right]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} E(\vec{p})\, u_1 \\ E(\vec{p})\, u_2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} E(\vec{p})\, u_1 \\ E(\vec{p})\, u_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} m u_1 + (\vec{\sigma}\cdot\vec{p})\, u_2 \\ (\vec{\sigma}\cdot\vec{p})\, u_1 - m u_2 \end{pmatrix} = \begin{pmatrix} E(\vec{p})\, u_1 \\ E(\vec{p})\, u_2 \end{pmatrix}$$

Solution of Dirac Equation for free particles: Continue

$\Rightarrow \quad (\bar{\sigma}\cdot\bar{p})\,u_2 + m u_1 = E(\bar{p})\,u_1 \rightarrow (7a)$

& $\quad (\bar{\sigma}\cdot\bar{p})\,u_1 - m u_2 = E(\bar{p})\,u_2 \rightarrow (7b)$

(Coupled eqn.)

From (7b)

$$(\bar{\sigma}\cdot\bar{p})\,u_1 = [E(\bar{p}) + m]\,u_2$$

$$\Rightarrow \boxed{u_2 = \left[\frac{\bar{\sigma}\cdot\bar{p}}{E(\bar{p})+m}\right] u_1} \rightarrow (8)$$

put this in (7a) & get

$$\left[\frac{(\bar{\sigma}\cdot\bar{p})(\bar{\sigma}\cdot\bar{p})}{E(\bar{p})+m} + m\right] u_1 = E(\bar{p})\,u_1$$

$$\Rightarrow \left[\frac{\bar{p}^2}{E(\bar{p})+m} + m\right] u_1 = E(\bar{p})\,u_1 \rightarrow (8a)$$

Since we know the identities Pauli matrices

$$(\vec{\sigma}\cdot\bar{a})(\bar{\sigma}\cdot\bar{b}) = \bar{a}\cdot\bar{b} + i\,\bar{\sigma}\cdot(\bar{a}\times\bar{b})$$

for any two vectr $\bar{a}$ & $\bar{b}$

put $\bar{a} = \bar{b} = \bar{p}$

get $(\bar{\sigma}\cdot\bar{p})^2 = \bar{p}^2 + 0 = \bar{p}^2$

In 8(a) ~~convert in to~~ put

$$\bar{p}^2 = E^2(p) - m^2 = [E(\bar{p}) - m][E(\bar{p}) + m]$$

$\{\because\ E^2(p) = \bar{p}^2 + m^2\}$

we get

$$[E(\bar{p}) - m + m]\,u_1 = E(\bar{p})\,u_1$$

$= R.H.S$

$\Rightarrow$ L.H.S identically satisfied with R.H.S.

Therefore $\Rightarrow$ there are two linearly independent +ve energy solutions for each momentum $\bar{p}$ i.e.,

which correspond to, e.g., choosing

$u_1 = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

~~$u_2 = \ldots$~~

Let us choose that soln of corresponding to $u_1 =$

Solution of Dirac Equation for free particles: Continue

To add the contribution of spin of the particle parallel to the direction of motion, we will also check somewhat differently how the operator corresponding to spin of particle (i.e. Helicity operator $S(\bar{P})$ or simply the helicity of the particle behave with.. Hamiltonian operator

$$\bar{H} = c\bar{\alpha}\cdot\bar{p} + \beta mc^2 .$$

we will find the, the Hamiltonian operator commutes with $S(\bar{P})$ where.

$$S(\bar{P}) = \frac{\bar{\Sigma}\cdot\bar{P}}{|\bar{P}|} \text{ , where } \bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix} = \bar{\Sigma}\cdot\hat{P}$$

is helicity operator or simply helicity of the particle; & physically it corresponds to the spin of the particle parallel to the direction of motion.

$$S(\bar{P}) = \bar{\Sigma}\cdot\hat{n} \text{ , where } \hat{n} = \frac{\bar{P}}{|\bar{P}|}$$

Now, here we will mention some properties of $S(\bar{P})$ and we will show that : since ..

(i) $S(\bar{P})$ commute with $\bar{H}$ operator i.e. $[S(\bar{P}), H] = 0$ &

(ii) $S^2(\bar{P}) = 1$ & (iii) eigen values of $S(\bar{P}) = \pm 1$

Hence the solutions of the Dirac eqn can be therefore chosen to be simultaneous eigen functions of $\bar{H}$ & $S(\bar{P})$. & also since EV's of $S(\bar{P}) = \pm 1$, ∴ for a given momentum & sign of the energy, the solutions can therefore be classified according to the eigenvalues (EV's) +1 or −1 of $S(\bar{P})$..

. Therefore, +ve energy sol^{n}s can be classify according to eigen values of Helicity operator. i.e. for a given $\bar{P}$ & +E, $S(\bar{P}) = \pm 1$.

i.e. sol^{n}s.

	Energy	Helicity
u_+^+ (+ → helicity, + → Energy)	+E	+1
u_+^-	+E	−1

Solution of Dirac Equation for free particles: Continue

A similar classification can be made for the -ve energy solns for which $p_0 = -E(\bar p) = -\sqrt{p^2+m^2}$ &

& here also, for a given momentum $\bar p$, there are again two linearly independent solutions which correspond to the eigen value +1 & -1 of $S(p)$.

i.e.

Solution	Energy	Helicity
u^{+}_{-} ← helicity, ← energy	$-E$	$+1$
u^{-}_{-} ← helicity, ← energy	$-E$	-1

Summarizing above, for a given 4-momentum p, there are four linearly independent solutions of the Dirac equation characterized by

$p_0 = \pm E(\bar p)$ &

$S(\bar p) = \pm 1$

∴ the 4-component spinor, for $p_0 = +E(\bar p)$, i.e. for +ve energy, is:

$$u_+(\bar p) = \begin{bmatrix} u_1 \\ \left(\frac{\bar\sigma\cdot\bar p}{E+m}\right) u_1 \end{bmatrix}.$$

& for $p_0 = -E(p)$ i.e. for -ve energy

$$u_-(\bar p) = \begin{bmatrix} -\left(\frac{\bar\sigma\cdot\bar p}{E+m}\right) u_2 \\ u_2 \end{bmatrix}.$$

Hence, these can be tabulated as to the representation

$u^{\pm}_{\pm}$ ← helicity, ← sign of energy

Soln	Energy	Helicity
u^{+}_{+}	$+E$	$+1$
u^{-}_{+}	$+E$	-1
u^{+}_{-}	$-E$	$+1$
u^{-}_{-}	$-E$	-1

Solution of Dirac Equation for free particles: Continue

Explicite form of two linearly independant solutions :

An explicit form for two linearly independent sol^ns for +ve energy & momentum $\bar{p}$ is given by.

$$u_+^{(1)}(\bar{p}) = N(\bar{p})\begin{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ \dfrac{\bar{\sigma}\cdot\bar{p}}{E(\bar{p})+m}\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix} \rightarrow (13a)$$

&

$$u_+^{(2)}(\bar{p}) = N(\bar{p})\begin{bmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ \dfrac{\bar{\sigma}\cdot\bar{p}}{E(\bar{p})+m}\begin{pmatrix}0\\1\end{pmatrix}\end{bmatrix} \rightarrow (13b)$$

where $N(\bar{p})$ is normalization constant determined by the requirement that :-

$$u^* u = 1.$$

use (13a) in this cond^n

$$u^* u = 1$$

$$\Rightarrow N^2(p)\left[1 \quad 0 \quad \frac{\bar{\sigma}\cdot\bar{p}}{E(p)+m} \quad 0\right]\begin{bmatrix}1\\0\\ \frac{\bar{\sigma}\cdot\bar{p}}{E(p)+m}\\0\end{bmatrix} = 1$$

$$\Rightarrow N^2(p)\left(1+0+\frac{(\bar{\sigma}\cdot\bar{p})^2}{[E(p)+m]^2}+0\right) = 1$$

$$\Rightarrow N^2(p)\left[1+\frac{\bar{p}^2}{(E+m)^2}\right] = 1 \qquad [\because (\bar{\sigma}\cdot\bar{p})^2 = \bar{p}^2]$$

$$\Rightarrow N^2(p)\left[1+\frac{E^2-m^2}{(E+m)^2}\right] = 1 \qquad \left\{\because E^2 = \bar{p}^2+m^2 \Rightarrow \bar{p}^2 = E^2-m^2\right\}$$

$$\Rightarrow N^2(p)\left[\frac{E+m+E-m}{E+m}\right] = 1$$

$$\Rightarrow N^2(p)\left(\frac{2E}{E+m}\right) = 1$$

$$\Rightarrow N(p) = \left(\frac{E+m}{2E}\right)^{1/2} \rightarrow (14)$$

Put this in (13a) & (13b) & get the explicit form of linearly independent sol^ns for two +ve energy & momentum $\bar{p}$. (Put & find out)

It may be noted that these two solutions (13a) & (13b) are orthogonal to each other, i.e.,

$$u_+^{(r)*}(p)\, u_+^{(s)}(p) = \delta_{rs}, \quad r,s = 1,2 \rightarrow (15)$$

Solution of Dirac Equation for free particles: Continue

The above solns (13a & 13b) are not eigenfunctions of $S(\bar{p})$. Positive energy solns. corresponding to definite helicity are obtained by ~~the noting that~~ considering the eigenvalue eqn as follows:

$$S(\bar{p})\, u_+^{(\pm)}(\bar{p}) = \pm\, u_+^{(\pm)}(\bar{p}) \rightarrow (16)$$

In eqn (16) put followings:

(i) $$s(\bar{p}) = \frac{\bar{\Sigma}\cdot\bar{p}}{|\bar{p}|} = \bar{\Sigma}\cdot\frac{\bar{p}}{|\bar{p}|} = \bar{\Sigma}\cdot\hat{n} = \begin{pmatrix} \bar{\sigma}\cdot\bar{n} & 0 \\ 0 & \bar{\sigma}\cdot\bar{n} \end{pmatrix} \rightarrow (17)$$

where $\bar{n}$ is the unit vect in the directn of $\bar{p}$ & $\bar{n} = \frac{\bar{p}}{|\bar{p}|}$

(ii) $$u_+^{(\pm)}(\bar{p}) = \begin{pmatrix} u_1^{(\pm)} \\ \frac{\bar{\sigma}\cdot\bar{p}}{E+m} u_1^{(\pm)} \end{pmatrix} = \begin{pmatrix} u_1^{(\pm)} \\ u_2^{(\pm)} \end{pmatrix} \rightarrow (18)$$

$$= \begin{pmatrix} u_1^{(\pm)} \\ |\bar{p}|\frac{\bar{\sigma}\cdot\bar{n}}{E+m} u_1^{(\pm)} \end{pmatrix} \quad \left\{ \text{where } \frac{\bar{\sigma}\cdot\bar{p}}{E+m} u_1^{(\pm)} = u_2^{(\pm)} \right\} \rightarrow (20)$$

$\left\{ \because \frac{\bar{p}}{|\bar{p}|} = \bar{n} \right\}$

put (17) & (18) in (16) soi

$$\begin{pmatrix} \bar{\sigma}\cdot\bar{n} & 0 \\ 0 & \bar{\sigma}\cdot\bar{n} \end{pmatrix} \begin{pmatrix} u_1^{(\pm)} \\ u_2^{(\pm)} \end{pmatrix} = \pm \begin{pmatrix} u_1^{(\pm)} \\ u_2^{(\pm)} \end{pmatrix}$$

{ where, $u_1^{(\pm)}$ & $u_2^{(\pm)}$ are the upper & lower component respectively of $u_+^{(\pm)}$. }

or,

$$(\bar{\sigma}\cdot\bar{n})\, u_1^{(\pm)} = \pm\, u_1^{(\pm)} \rightarrow (19a)$$

$$(\bar{\sigma}\cdot\bar{n})\, u_2^{\pm} = \pm\, u_2^{(\pm)} \rightarrow (19b)$$

Let us first solve eqn (19a) for +ve helicity only, & hence evaluating only for $u_1^{(+)}$.

$\therefore$ (19a) becomes

$$(\bar{\sigma}\cdot\bar{n})\, u_1^{(+)} = +\, u_1^{(+)} \rightarrow (21)$$

put here; (Pauli's matrix)

$$\bar{\sigma}\cdot\bar{n} = \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} n_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} n_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_3$$

$$= \begin{pmatrix} 0 & n_1 \\ n_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -in_2 \\ in_2 & 0 \end{pmatrix} + \begin{pmatrix} n_3 & 0 \\ 0 & -n_3 \end{pmatrix}$$

$$= \begin{bmatrix} (n_3) & (n_1 - in_2) \\ (n_1 + in_2) & -n_3 \end{bmatrix} \rightarrow (22a)$$

& also let us choose

$$u_1^{(+)} = \begin{pmatrix} A \\ B \end{pmatrix} \rightarrow (22b)$$

where A & B are constants need to be determined here.

put [22 (a) & (b)] in (21) & we get

Solution of Dirac Equation for free particles: Continue

$$\begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

or, (i) $n_3 A + (n_1 - i n_2) B = A$

$\Rightarrow (n_1 - i n_2) B = (1 - n_3) A$

$\Rightarrow \frac{A}{B} = \left(\frac{n_1 - i n_2}{1 - n_3}\right)$

& (ii)

$(n_1 + i n_2) A - n_3 B = B$

$\Rightarrow (n_1 + i n_2) A = (1 + n_3) B$

$\Rightarrow \frac{A}{B} = \left(\frac{n_1 + i n_2}{1 + n_3}\right)$

$\frac{A}{B} = \left(\frac{1 + n_3}{n_1 + i n_2}\right)$

Hence both given

$$\frac{A}{B} = \left(\frac{n_1 - i n_2}{1 - n_3}\right) = \left(\frac{1 + n_3}{n_1 + i n_2}\right)$$

Let we choose, $\frac{A}{B} = \frac{1 + n_3}{n_1 + i n_2}$,

Since, $A = \left(\frac{n_1 - i n_2}{-n_3 + 1}\right) B$

& $B = \left(\frac{n_1 + i n_2}{n_3 + 1}\right) A$

Hence normalized $u_1^{(+)}$ are given by

$$u_1^{(+)} = \frac{1}{\sqrt{2(n_3+1)}} \begin{pmatrix} n_3 + 1 \\ n_1 + i n_2 \end{pmatrix} \quad (23a)$$

Similarly, taking (-ve) helicity one may derive $u_1^{(-)}$ as to be:

$$u_1^{(-)} = \frac{1}{\sqrt{2(n_3+1)}} \begin{pmatrix} -n_1 + i n_2 \\ n_3 + 1 \end{pmatrix} \quad (23b)$$

Therefore, A normalized +ve energy eigenfunction with helicity +1 is given by:

$$u_+^{(+)}(p) = \frac{1}{\sqrt{2(n_3+1)}} \sqrt{\frac{E(p)+m}{2E(p)}} \begin{bmatrix} n_3 + 1 \\ n_1 + i n_2 \\ \frac{|p|}{E(p)+m} \begin{pmatrix} n_3 + 1 \\ n_1 + i n_2 \end{pmatrix} \end{bmatrix}$$

A similar classification can be done for the (-ve) energy solutions for which $E(p) = -\sqrt{p^2 + m^2}$ & for a given momentum, there are again two linearly independent solutions.

Solution of Dirac Equation for free particles: Continue

So, for a given momentum there are 4 - linearly independed solm for Dirac equation.

These are characterized by

$\pm E(p)$ & $s(p) = \pm 1$.

Solution of Dirac Equation for free particles: Plane wave solution

Solution of Dirac Equation for free particle (Plane wave solution):

Dirac spinor

Dirac eqn is

$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$, put $\gamma^\mu \partial_\mu = \not\partial$

or, $(i\not\partial - m)\psi(x) = 0$ → (1)

• For quantum field theory, the Dirac eqn admits plane wave solution ~~(known as Dirac spinor; which is bi spinor) is~~ :→

$$\psi(x) = w_{\vec{p}}\, e^{-ipx}$$
$$= u(p)\, e^{-ipx} \quad \to (2)$$
$$\text{or, } = u_{\vec{p}}\, e^{-ipx}$$

or,

where (i) $w_{\vec{p}} = u(p) = u_{\vec{p}}$ is known as Dirac spinor related to a plane-wave with wave-vector $\vec{p}$, which is a column vector of type

$$w_{\vec{p}} = u(p) = \begin{bmatrix} u_1(p) \\ u_2(p) \\ u_3(p) \\ u_4(p) \end{bmatrix} = \begin{bmatrix} u_1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\, u_1 \end{bmatrix}$$

$$= \begin{bmatrix} \phi \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\, \phi \end{bmatrix}$$

arbitrary two-spinor

$\vec{\sigma}$ are the Pauli matrices

E_p is the +ve square root.

$$E_p = +\sqrt{m^2 + \vec{p}^2}$$

& or $px = p_\mu x^\mu \equiv p^0 x^0 - \vec{p}\cdot\vec{x}$
$= Et - \vec{p}\cdot\vec{x}$

(4-position, 4-momentum)

put (2) in (1)

$$i\gamma^\mu \left\{ \frac{\partial}{\partial x^\mu} e^{-ip_\mu x^\mu} \right\} u(p) - m\, e^{-ip_\mu x^\mu} u(p) = 0$$

$\Rightarrow i\gamma^\mu(-ip_\mu)\, e^{-ipx}\, u(p) - m(\;) = 0$

$\Rightarrow (\gamma^\mu p_\mu - m)\, u(p) = 0$

or, $\boxed{(\not p - m)\, u(p) = 0}$ → (4) (in natural unit here $c = 1$)

So, we have to find out $u(p)$ which satisfy eqn (4) so that $\psi(x)$ can be find out.

As, $u(p)$ has 4-component, hence (4) is a system of 4-eqn linear homogeneous eqns for having components u_α, $\alpha = 1,2,3,4$ for which

The non-trivial soln of eqn (4) can be find out only if

$$\det(\not p - m) = 0$$

(10)

KG-equation $(\partial_\mu \partial^\mu + m^2)\phi = 0$, problem?

'Surprise' of the plane-wave $\phi = Ne^{-ip\cdot x} = Ne^{-iEt+i\vec{p}\cdot\vec{x}}$ ***solutions, if you plug them in the KG equation you find:***

$$\begin{cases} E^2 = \vec{p}^{\,2} + m^2 \;\Rightarrow\; E = \pm\sqrt{\vec{p}^{\,2} + m^2} \\ \rho = 2|N|^2 E \begin{cases} \geq 0 & E \geq 0 \\ \leq 0 & E \leq 0 \end{cases} \end{cases}$$

solutions with E<0

solutions with ρ ***<0***

One way out: drop KG equation! That is what Dirac successfully did!

Other way out: re-interpret in terms of charge density & charge flow:

$$j^\mu \rightarrow q \times j^\mu = 2|N|^2(q \times E, q \times \vec{p}) \begin{cases} j^0 \geq 0 & q>0 \text{ particles} \\ j^0 < 0 & q<0 \text{ particles} \end{cases}$$

In reality (electrons negatively charged) just the opposite way …:

$$\begin{cases} E > 0 \text{ particles with } q = -|e| \\ E < 0 \text{ anti-particles with } q = +|e| \end{cases}$$

Dirac equation:
free particles

Schrödinger – Klein-Gordon – Dirac

Quantum mechanical E & p operators:

$$\begin{cases} E = i\dfrac{\partial}{\partial t} \\ \vec{p} = -i\vec{\nabla} \end{cases}$$

$$p^{\mu} = (E, \vec{p}) \rightarrow i\partial^{\mu} = i\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

You simply 'derive' the Schrödinger equation from classical mechanics:

$$E = \frac{\boldsymbol{p}^2}{2m} \quad \rightarrow \quad i\frac{\partial}{\partial t}\phi = -\frac{1}{2m}\nabla^2\phi$$

Schrödinger equation

With the relativistic relation between E, p & m you get:

$$E^2 = \boldsymbol{p}^2 + m^2 \quad \rightarrow \quad \frac{\partial^2}{\partial t^2}\phi = \nabla^2\phi - m^2\phi$$

Klein-Gordon equation

The negative energy solutions led Dirac to try an equation with first order derivatives in time (like Schrödinger) as well as in space

$$i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$$

Dirac equation

Does it make sense?

Also Dirac equation should reflect: $E^2 = \vec{p}^2 + m^2$

Basically squaring: $i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi = \vec{\alpha}\cdot\vec{p}\phi + \beta m\phi$

Tells you:

$$
\begin{aligned}
(\vec{\alpha}\cdot\vec{p} + \beta mc)^2 &= (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc) \\
&= \beta^2 m^2 c^2 \\
&\quad + \sum_i \left[\alpha_i^2 p_i^2 + (\alpha_i\beta + \beta\alpha_i) p_i mc\right] \\
&\quad + \sum_{i>j} \left[(\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j\right]
\end{aligned}
$$

$(\vec{\alpha}\cdot\vec{p} + \beta mc)^2$ → E^2

$\beta^2 m^2 c^2$ → m^2; $\beta^2 = 1$

$\sum_i \left[\ldots\right]$ → $\vec{p}^2$; $\alpha_i^2 = 1$, $\alpha_i\beta + \beta\alpha_i = 0$

$\sum_{i>j}\left[\ldots\right]$ → $i \neq j$: $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$

Properties of α_i and β

β and α can not be simple commuting numbers, but must be matrices

Because $\beta^2 = \alpha_i^2 = 1$, both β and α must have eigenvalues ± 1

Since the eigenvalues are real (± 1), both β and α must be Hermitean

$$\alpha_i^\dagger = \alpha_i \quad \text{en} \quad \beta^\dagger = \beta$$

$$A_{ij}B_{jk}C_{ki} = C_{ki}\,A_{ij}B_{jk} = B_{jk}C_{ki}\,A_{ij}$$

Both β and α must be traceless matrices: Tr(ABC) = Tr(CAB) = Tr(BCA)

$\beta^2=1$ — cyclic — anti commutation — $\beta^2=1$

$\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\alpha_i\beta\beta) = \mathrm{Tr}(\beta\alpha_i\beta) = -\mathrm{Tr}(\alpha_i\beta\beta) = -\mathrm{Tr}(\alpha_i)$ and hence $\mathrm{Tr}(\alpha_i)=0$

You can easily show the dimension d of the matrices β,α to be even:

either: $$i \neq j:\ |\alpha_i\alpha_j| = |-\alpha_j\alpha_i| = (-1)^d|\alpha_j\alpha_i| = \begin{cases} -|\alpha_i\alpha_j|, & \text{d odd} \\ +|\alpha_i\alpha_j|, & \text{d even} \end{cases}$$

or: with eigenvalues ± 1, matrices are only traceless in even dimensions

Explicit expressions for α_i and β

In 2 dimensions, you find at most 3 anti-commuting matrices,
Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In 4 dimensions, you can find 4 anti-commuting matrices,
numerous possibilities, Dirac-Pauli representation:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Any other set of 4 anti-commutating matrices will give same physics
(if the Dirac equation is to make any sense at all of course .....
and ... if it would not: we would not be discussing it here!)

Co-variant form: Dirac γ-matrices

$$i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$$ ***does not look that Lorentz invariant***

Multiplying on the left with β and collecting all the derivatives gives:

$$m\phi = i\beta\frac{\partial}{\partial t}\phi + i\,\beta\vec{\alpha}\cdot\vec{\nabla}\phi \equiv i\gamma^{\mu}\partial_{\mu}\phi$$

note: $\partial_{\mu} = (\partial_t, +\vec{\nabla})$

Hereby, the Dirac γ-matrices are defined as:

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k \equiv \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

And you can verify that: $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

As well as: $(\gamma^0)^2 = +1$, $(\gamma^k)^2 = -1$

and: $\gamma^{0\dagger} = +\gamma^0$, $\gamma^{k\dagger} = -\gamma^k$ $\rightarrow \gamma^{\mu+} = \gamma^0\gamma^{\mu}\gamma^0$

Co-variant form: Dirac γ-matrices

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$m\phi = i\gamma^{\mu}\partial_{\mu}\phi$ **with the Dirac γ-matrices defined as:**

$$\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k \equiv \beta\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Warning!

This $m\phi = i\gamma^{\mu}\partial_{\mu}\phi$ ***notation is misleading,*** γ^{μ} ***is not a 4-vector!***

The γ^{μ} ***are just a set of four 4×4 matrices, which do no not transform at all i.e. in every frame they are the same, despite the*** μ***-index.***

The Dirac wave-functions (ϕ ***or*** ψ***), so-called ´spinors´ have <u>interesting</u> Lorentz transformation properties which we will discuss shortly.***

After that it will become clear why the notation with γ^{μ} ***is useful!***

& beautiful!

To make things even worse, we define:

$$\begin{cases} \gamma_0 = +\gamma^0 \\ \gamma_k = -\gamma^k \end{cases}$$

Spinors & (Dirac) matrices

$$\phi = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \qquad \phi^{+} = \begin{pmatrix} * & * & * & * \end{pmatrix} \qquad \gamma^{\mu} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\gamma^{\mu}\phi = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix}$$

$$\gamma^{\mu}\phi^{+} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \end{pmatrix} = \text{↯}$$

$$\phi^{+}\gamma^{\mu} = \begin{pmatrix} * & * & * & * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \end{pmatrix}$$

$$\phi\,\gamma^{\mu} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \text{↯}$$

$$\phi^{+}\phi = \begin{pmatrix} * & * & * & * \end{pmatrix} \times \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} = \begin{pmatrix} * \end{pmatrix}$$

$$\phi\,\phi^{+} = \begin{pmatrix} * \\ * \\ * \\ * \end{pmatrix} \times \begin{pmatrix} * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

this one we will encounter later …

Dirac current & probability densities

Proceed analogously to Schrödinger & Klein-Gordon equations, but with Hermitean instead of complex conjugate wave-functions:

$$
\begin{aligned}
0 &= -i\hbar\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - mc\psi^\dagger \\
&= -i\hbar\partial_0\psi^\dagger\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k - mc\psi^\dagger \qquad \longleftarrow \times\gamma^0 \\
&\longrightarrow -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 + i\hbar\partial_k\psi^\dagger\gamma^k\gamma^0 - mc\psi^\dagger\gamma^0 \\
&= -i\hbar\partial_0\psi^\dagger\gamma^0\gamma^0 - i\hbar\partial_k\psi^\dagger\gamma^0\gamma^k - mc\psi^\dagger\gamma^0 \\
&= -i\hbar\partial_\mu\psi^\dagger\gamma^0\gamma^\mu - mc\psi^\dagger\gamma^0 \\
(\bar\psi \equiv \psi^\dagger\gamma^0) &\longrightarrow -i\hbar\partial_\mu\bar\psi\gamma^\mu - mc\bar\psi
\end{aligned}
$$

Dirac equations for $\bar\psi$ ***&*** ψ ***:***

$$
\begin{cases}
i\hbar(\partial_\mu\bar\psi)\gamma^\mu + mc\bar\psi = 0 & \longleftarrow \times\,\psi \\
\times\,\bar\psi \longrightarrow \quad i\hbar\gamma^\mu(\partial_\mu\psi) - mc\psi = 0
\end{cases}
$$

Add these two equations to get:

Conserved 4-current: $0 = i\hbar(\partial_\mu\bar\psi)\gamma^\mu\psi + i\hbar\bar\psi\gamma^\mu(\partial_\mu\psi) = i\hbar\partial_\mu\left[\bar\psi\gamma^\mu\psi\right]$

$$
j^\mu = \bar\psi\gamma^\mu\psi
\begin{cases}
j^0 = \bar\psi\gamma^0\psi = |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \geq 0 \\
j^k = \bar\psi\gamma^k\psi
\end{cases}
$$

(exactly what Dirac aimed to achieve …)

Solutions: particles @ rest $\vec{p} = \vec{0}$

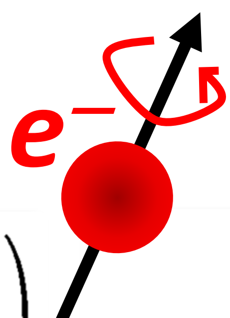
Dirac equation for $\vec{p} = \vec{0}$ is simple: $i\hbar\gamma^0\partial_0\psi - mc\psi = 0$

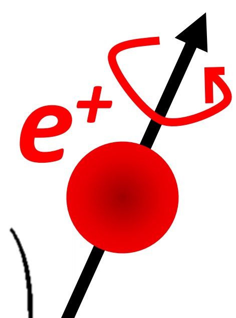
Solve by splitting 4-component in two 2-components: $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$

with $\partial_0 \equiv (1/c)\partial_t$ follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A/\partial_t \\ \partial\psi_B/\partial_t \end{pmatrix} = -\frac{imc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

solutions:





$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solutions: moving particles $\vec{\mathbf{p}} \neq \vec{\mathbf{0}}$

Dirac equation for $\vec{p} \neq \vec{0}$ less simple:

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0$$

Anticipate plane-waves:

$$\psi = u(p)e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{x})} = u(p)e^{-\frac{i}{\hbar}p\cdot x}$$

And again anticipate two 2-components:

$$u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix}$$

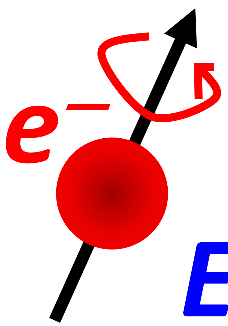
Plugging this in gives:

$$\begin{aligned} 0 &= (\gamma^\mu p_\mu - mc)\,u(p) = \left(\gamma^0 p_0 - \gamma^k p_k - mc\right)u(p) \\ &= \begin{pmatrix} E/c - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c - mc \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} \\ &= \begin{pmatrix} (E/c - mc)u_A(p) - \vec{p}\cdot\vec{\sigma}u_B(p) \\ \vec{p}\cdot\vec{\sigma}u_A(p) - (E/c + mc)u_B(p) \end{pmatrix} \end{aligned}$$

$$\Rightarrow \begin{cases} u_A(p) = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p) \\ u_B(p) = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p) \end{cases}$$

Solutions: moving particles $\vec{p} \neq \vec{0}$

Solutions: pick $u_A(p)$ & calculate $u_B(p)$: $$u_B(p) = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p)$$



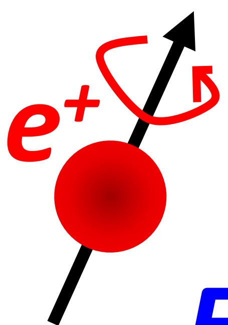
E>0

$u_A = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow$ $$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}1\\0\\\frac{cp_z}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$$

$u_A = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow$ $$\psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}0\\1\\\frac{c(p_x-ip_y)}{E+mc^2}\\\frac{-cp_z}{E+mc^2}\end{pmatrix}$$

In limit $\vec{p} \rightarrow \vec{0}$ you retrieve the E>0 solutions, hence these are $\vec{p} \neq \vec{0}$ electron solutions

Similarly: pick $u_B(p)$ & calculate $u_A(p)$: $$u_A(p) = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p)$$

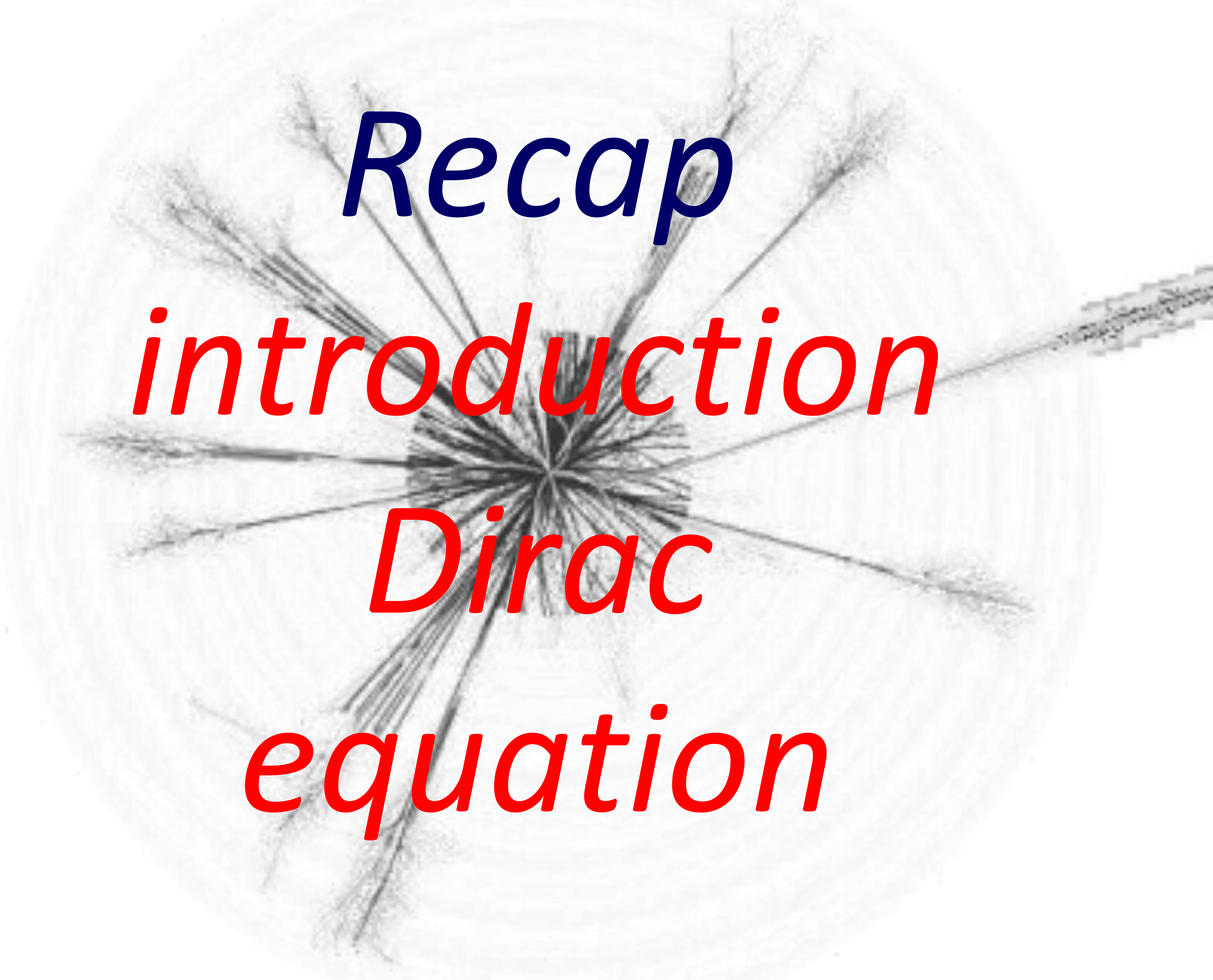


E<0

$u_B = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow$ $$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}\frac{cp_z}{E-mc^2}\\\frac{c(p_x+ip_y)}{E-mc^2}\\1\\0\end{pmatrix}$$

$u_B = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow$ $$\psi^{(4)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix}\frac{c(p_x-ip_y)}{E-mc^2}\\\frac{-cp_z}{E-mc^2}\\0\\1\end{pmatrix}$$

In limit $\vec{p} \rightarrow \vec{0}$ you retrieve the E<0 solutions, hence these are $\vec{p} \neq \vec{0}$ positron solutions



Recap

introduction Dirac equation

Dirac equation

$\partial^\mu = (\partial_t, -\vec{\nabla})$

From: $E^2 = \vec{p}^2 + m^2$ **& classical → QM 'transcription':** $\begin{cases} E = i\dfrac{\partial}{\partial t} \\ \vec{p} = -i\vec{\nabla} \end{cases}$

We found: $i\dfrac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi = \vec{\alpha}\cdot\vec{p}\phi + \beta m\phi$

$E^2 \stackrel{?}{=} \vec{p}^2 + m^2$

With β, α_1, α_2 **&** α_3 **(4×4) matrices, satisfying:**

$$(\vec{\alpha}\cdot\vec{p} + \beta mc)^2 = (\alpha_i p_i + \beta mc)(\alpha_j p_j + \beta mc)$$

$$= \beta^2 m^2 c^2 \longrightarrow \beta^2 = 1$$

$$+ \sum_i \left[\alpha_i^2 p_i^2 + (\alpha_i\beta + \beta\alpha_i)p_i mc\right] \longrightarrow \alpha_i^2 = 1, \quad \alpha\beta + \beta\alpha = 0$$

$$+ \sum_{i>j} \left[(\alpha_i\alpha_j + \alpha_j\alpha_i)p_i p_j\right] \longrightarrow i \neq j: \alpha_i\alpha_j + \alpha_j\alpha_i = 0$$

$(\vec{\alpha}\cdot\vec{p} + \beta mc)^2 \rightarrow E^2$; $\beta^2 m^2 c^2 \rightarrow m^2$; $\sum_i \alpha_i^2 p_i^2 \rightarrow \vec{p}^2$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Co-variant form: Dirac γ-matrices

Dirac's original form does not look covariant: $i\frac{\partial}{\partial t}\phi = -i\,\vec{\alpha}\cdot\vec{\nabla}\phi + \beta m\phi$

Multiplying on the left with β and collecting all the derivatives gives covariant form:

$$m\phi = i\beta\frac{\partial}{\partial t}\phi + i\,\beta\vec{\alpha}\cdot\vec{\nabla}\phi \equiv i\gamma^{\mu}\partial_{\mu}\phi \quad \textbf{note: } \boldsymbol{\partial_{\mu}} = \left(\boldsymbol{\partial_t}, +\vec{\boldsymbol{\nabla}}\right)$$

With Dirac γ-matrices defined as: $\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^k = \beta\alpha^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From the properties of β, α_1, α_2 & α_3 follows: $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

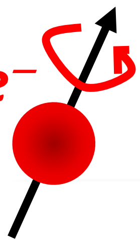
$$(\gamma^0)^2 = +1 \qquad \gamma^{0\dagger} = +\gamma^0$$
$$(\gamma^k)^2 = -1 \qquad \gamma^{k\dagger} = -\gamma^k \;\rightarrow\; \boldsymbol{\gamma^{\mu+} = \gamma^0\gamma^{\mu}\gamma^0}$$

Dirac particle solutions: spinors

Ansatz solution: $\psi = \begin{bmatrix} u_A(p) \\ u_B(p) \end{bmatrix} e^{-ip\cdot x}$ ⟶ **Dirac eqn.:**

$$\begin{cases} u_A(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E-m}\, u_B(p) \\ u_B(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\, u_A(p) \end{cases}$$

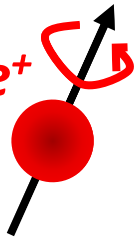
$\vec{p} = \vec{0}$ solutions:



$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

spin ½ electrons — **E>0**



$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

spin ½ positrons — **E<0**

$\vec{p} \neq \vec{0}$ solutions:

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

Dirac equation:

more on free particles

normalisation

4-vector current

anti-particles

sorry for the c's

One more look at $\vec{p}\cdot\vec{\sigma}$

The conditions:
$$\begin{cases} u_A(p) &= \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B(p) \\ u_B(p) &= \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A(p) \end{cases}$$

Imply:
$$u_A(p) = \frac{c^2}{E^2 - m^2c^4}(\vec{p}\cdot\vec{\sigma})^2 u_A(p)$$

$$\Rightarrow 1 = \frac{c^2}{E^2 - m^2c^4}(\vec{p}\cdot\vec{\sigma})^2$$

$$\Rightarrow \; p^2c^2 = E^2 - m^2c^4$$

i.e. energy-momentum relation, as expected

Check this:

$$(\vec{p}\cdot\vec{\sigma}) = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} p_z & (p_x - ip_y) \\ (p_x + ip_y) & -p_z \end{pmatrix} \Rightarrow (\vec{p}\cdot\vec{\sigma})^2 = \begin{pmatrix} p_z^2 + (p_x - ip_y)(p_x + ip_y) & \cdots \\ \cdots & \cdots \end{pmatrix} = \vec{p}^2$$

Normalisation of the Dirac spinors

Just calculate it!:

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}$$

Spinors 1 & 2, E>0:

$$\begin{aligned}\psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E+mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E+mc^2)^2} \\ &= 1+\frac{E-mc^2}{E+mc^2} = \frac{2E}{E+mc^2} = \frac{2|E|}{|E|+mc^2} \rightarrow N = \sqrt{|E|+mc^2}\end{aligned}$$

To normalize @ 2E particles/unit volume

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x}\begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix}$$

Spinors 3 & 4, E<0:

$$\begin{aligned}\psi^\dagger\psi &= 1+\frac{p_x^2c^2+p_y^2c^2+p_z^2c^2}{(E-mc^2)^2} \\ &= 1+\frac{E^2-m^2c^4}{(E-mc^2)^2} \\ &= 1+\frac{E+mc^2}{E-mc^2} = \frac{2E}{E-mc^2} = \frac{2|E|}{|E|+mc^2} \rightarrow N = \sqrt{|E|+mc^2}\end{aligned}$$

To normalize @ 2E particles/unit volume

Current & probability densities

Again, just plug it in!

$$j^\mu = \bar\psi\gamma^\mu\psi \begin{cases} j^0 = \bar\psi\gamma^0\psi \\ j^k = \bar\psi\gamma^k\psi \end{cases}$$

always using

N= $\sqrt{|E| + mc^2}$

particle @ rest

$N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$\begin{cases} j^0 = \bar\psi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi = \psi^\dagger\psi = 2mc^2 & \rightarrow 2|E| \geq 0 \\ j^k = \bar\psi\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix}\psi = \psi^\dagger\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}\psi = \vec{0} \end{cases}$$

moving particle

$N\begin{pmatrix}1\\0\\ \frac{cp_z}{E+mc^2}\\ \frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}$

$$\begin{cases} j^0 = \bar\psi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi = \psi^\dagger\psi = \begin{cases}+2E & E>0\\ -2E & E<0\end{cases} & \rightarrow 2|E| \geq 0 \\ j^k = \bar\psi\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix}\psi = \psi^\dagger\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}\psi = \begin{cases}+2\vec{p} & E>0\\ -2\vec{p} & E<0\end{cases} \end{cases}$$

not that easy, next slide!

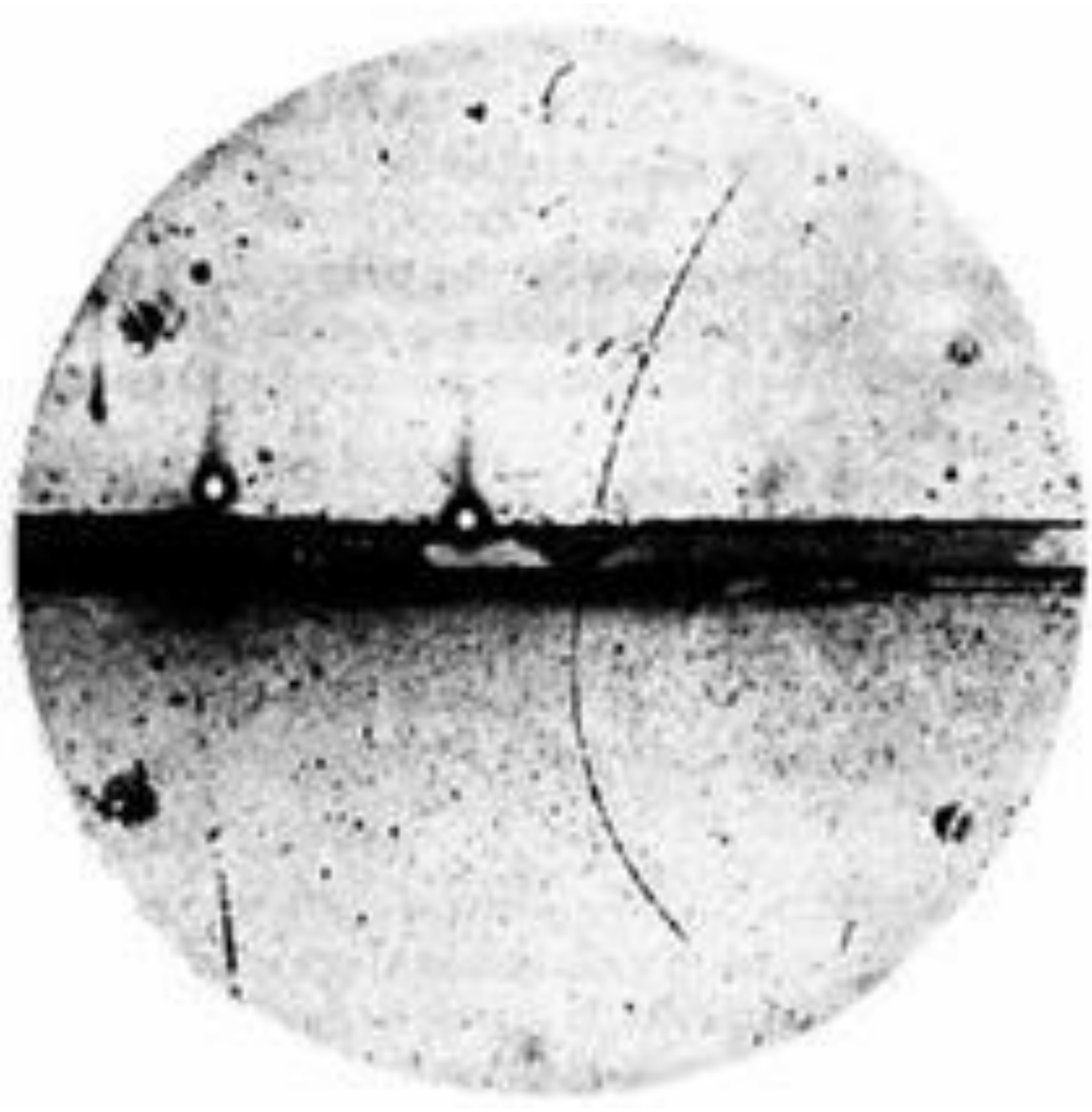
Current & probability densities

Explicit verification of j_x for moving particle solution $\psi^{(1)}$:

$$
\begin{aligned}
j_x &= |N|^2\left(1,0,\tfrac{cp_z}{E+mc^2},\tfrac{c(p_x-ip_y)}{E+mc^2}\right)\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp_z}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}\\
&= |N|^2\left(1,0,\tfrac{cp_z}{E+mc^2},\tfrac{c(p_x-ip_y)}{E+mc^2}\right)\begin{pmatrix}\frac{c(p_x+ip_y)}{E+mc^2}\\\frac{cp_z}{E+mc^2}\\0\\1\end{pmatrix}\\
&= |N|^2\left(\frac{c(p_x+ip_y)}{E+mc^2}+\frac{c(p_x-ip_y)}{E+mc^2}\right)\\
&= |N|^2\frac{2cp_x}{E+mc^2} \quad\rightarrow\quad 2p_x
\end{aligned}
$$

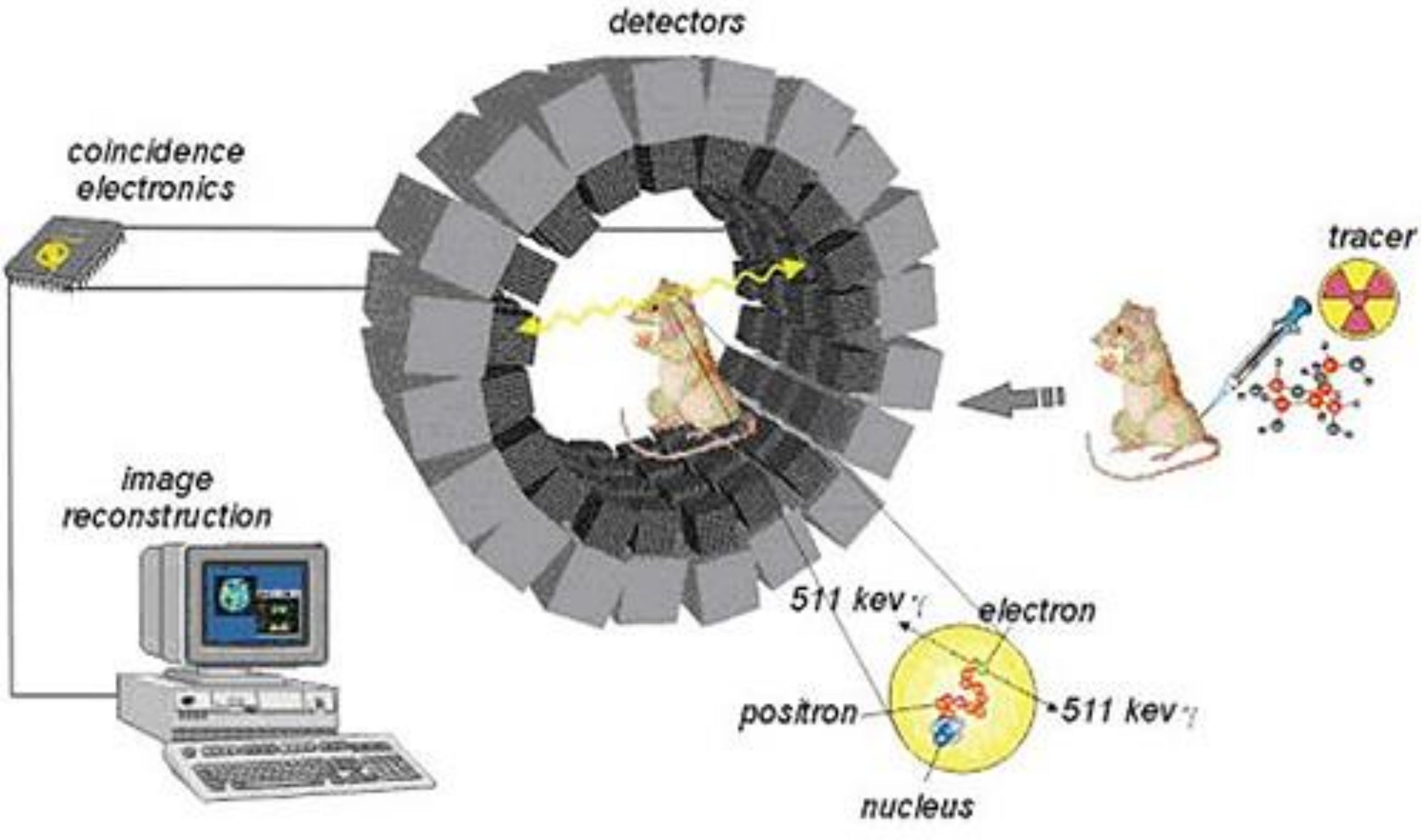
And j_x for moving anti-particle solution $\psi^{(3)}$: $j_x = |N|^2\dfrac{2cp_x}{E-mc^2} \rightarrow -2p_x$

Antiparticles



Surprising applications

PET – Positron Emission Tomography



Particles & Anti-particles

4-component Dirac spinors $\rightarrow$ 4-solutions.

These represent: 2 spin states of the electron

2 spin states of the anti-electron i.e. the positron

Different ways how to proceed:

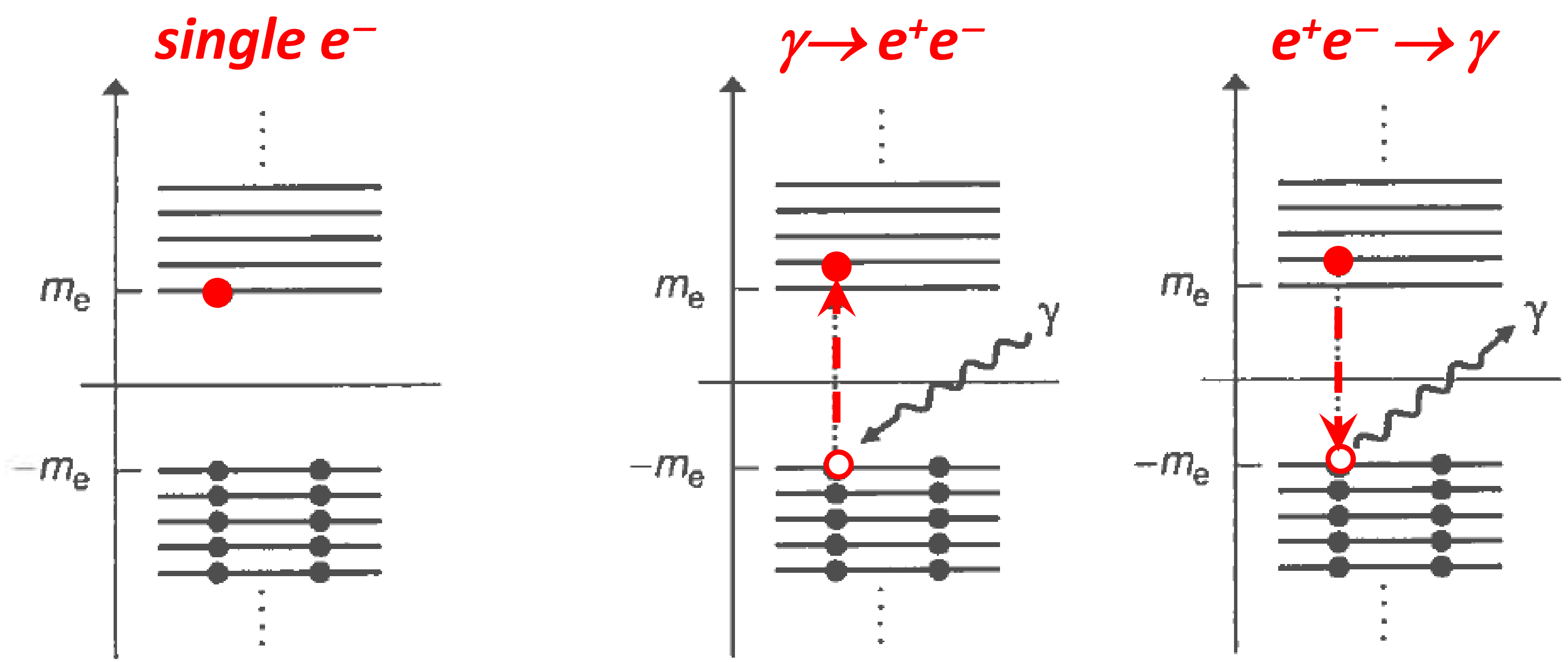
- ***Use E>0 & E<0 solutions of the electron Dirac eqn.***
- ***Use E>0 & E<0 solutions of the positron Dirac eqn.***
- ***Use E>0 solutions for the 'particle' i.e. electron &***
  Use E<0 solutions for the 'anti-particle' i.e. positron

Will opt for the last option:

i.e. using the physical E & $\vec{p}$ to characterize states

And now: E<0 → antiparticles

*‘Dirac sea’: **fill all E<0 states (thanks to Pauli exclusion principle)***

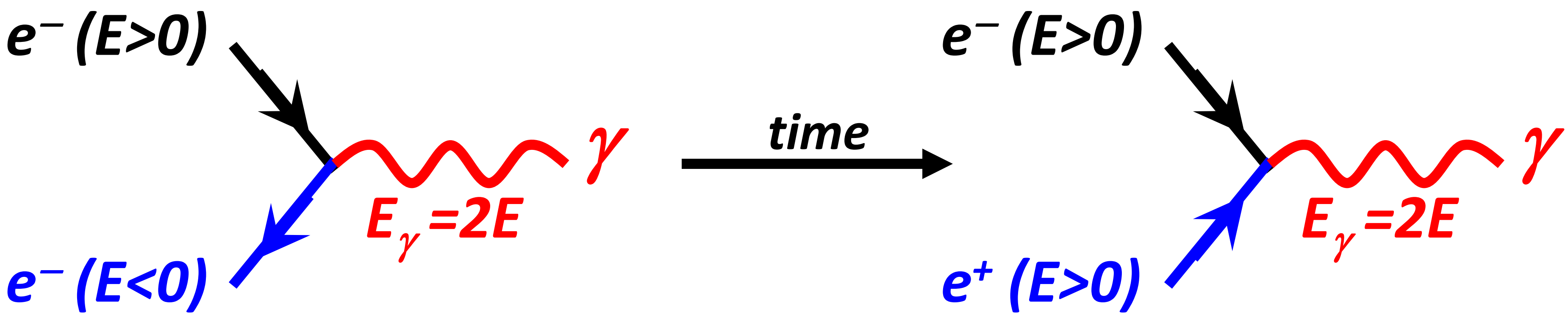


But:
- ***does not work for bosons***
- ***an infinite energy sea not a nice concept …***

And now: E<0 $\rightarrow$ antiparticles

'Feynman-Stückelberg': E<0 particle solutions propagating backwards in time
E>0 anti-particle solutions propagating forwards in time

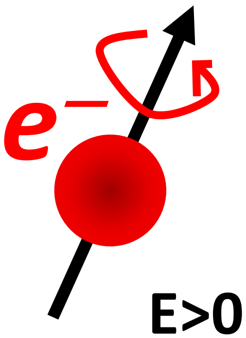
$$e^{-i(-E)(-t)} = e^{-iEt}$$



'Up-shot': Dirac equation accommodates both particle & antiparticles!

Sequel: will use particle & anti-particle spinors labelled with their physical, E>0 & real $\vec{p}$, kinematics. (exponents remain opposite)

We had: Dirac 'u'-spinors

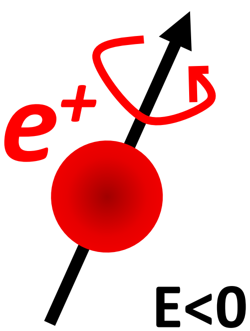


$$\psi^{(1)} = \sqrt{E+m}\; e^{-ip\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

$$\equiv\; u_1(E,\vec{p})\; e^{-ip\cdot x}$$

$$\psi^{(2)} = \sqrt{E+m}\; e^{-ip\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$\equiv\; u_2(E,\vec{p})\; e^{-ip\cdot x}$$



$$\psi^{(3)} = \sqrt{|E|+m}\; e^{-ip\cdot x} \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}$$

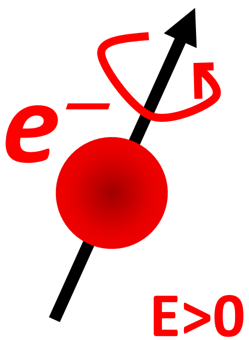
$$\equiv\; u_3(E,\vec{p})\; e^{-ip\cdot x}$$

$$\psi^{(4)} = \sqrt{|E|+m}\; e^{-ip\cdot x} \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$$\equiv\; u_4(E,\vec{p})\; e^{-ip\cdot x}$$

From now on use: Dirac ‘u’- & ‘v’-spinors

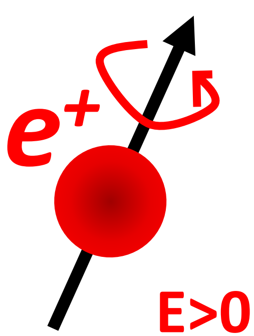
***u-spinors:* for electrons, labeled with physical E>0 & $\vec{p}$**



$$\psi^{(1)} = \sqrt{E+m}\, e^{-ip\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \equiv \boldsymbol{u_1(E,\vec{p})\, e^{-ip\cdot x}}$$

$$\psi^{(2)} = \sqrt{E+m}\, e^{-ip\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \equiv \boldsymbol{u_2(E,\vec{p})\, e^{-ip\cdot x}}$$

***v-spinors:* for positrons, labeled with physical E>0 & $\vec{p}$**



$$u_4(-E,-\vec{p})\, e^{+ip\cdot x} \equiv \boldsymbol{v_1(E,\vec{p})\, e^{+ip\cdot x}}$$

$$u_3(-E,-\vec{p})\, e^{+ip\cdot x} \equiv \boldsymbol{v_2(E,\vec{p})\, e^{+ip\cdot x}}$$

Dirac equation

Dirac equation in original form with matrices $\vec{\alpha}$ & β:

$$i\frac{\partial}{\partial t}\psi = -i\,\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m\psi$$

With plane-wave solutions: $\psi = u(p)e^{-ip\cdot x} = \begin{bmatrix} u_A(p) \\ u_B(p) \end{bmatrix} e^{-ip\cdot x}$ *you find for spinor u(p):*

$$Eu(p) = \vec{\alpha}\cdot\vec{p}\,u(p) + \beta m u(p)$$

This algabraic equation for u(p) you can solve for particles with $p^{\mu} = (E, \vec{p})$

Co-variant form of Dirac equation with matrices γ^{μ}:

$m\psi = i\gamma^{\mu}\partial_{\mu}\psi$ *with* $\psi = u(p)e^{-ip\cdot x} = \begin{bmatrix} u_A(p) \\ u_B(p) \end{bmatrix} e^{-ip\cdot x}$ *you get* $(\gamma^{\mu}p_{\mu} - m)u(p) = 0$

Explicit expressions for the γ^{μ} matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

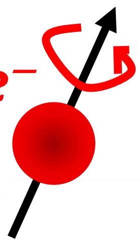
And the algebra for the γ^{μ} matrices: $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

Spinors

Ansatz solution: $\psi = \begin{bmatrix} u_A(p) \\ u_B(p) \end{bmatrix} e^{-ip\cdot x}$ ⟶ **Dirac eqn.:** $\begin{cases} u_A(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E-m} u_B(p) \\ u_B(p) = \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A(p) \end{cases}$

$\vec{p} = \vec{0}$ solutions:

e^-

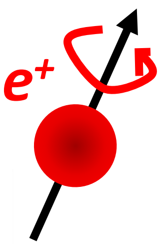


$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(1)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

spin ½ electrons — **E>0**

e^+



$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\psi^{(3)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{+\frac{imc^2}{\hbar}t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

spin ½ positrons — **E<0**

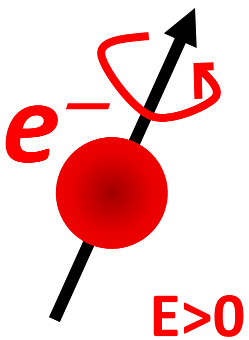
$\vec{p} \neq 0$ solutions:

$$\psi^{(1)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad \psi^{(2)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}$$

$$\psi^{(3)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{cp_z}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} \propto e^{-\frac{i}{\hbar}p\cdot x} \begin{pmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{-cp_z}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

Particles & anti-particles

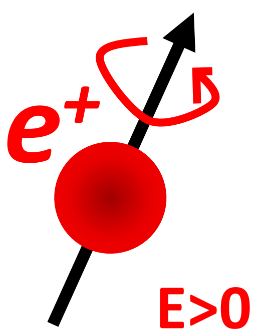
***u-spinors:** for electrons, labeled with physical E>0 & $\vec{p}$*



$$\psi^{(1)} = \sqrt{E+m}\; e^{-ip\cdot x} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} = \boldsymbol{u_1(E,\vec{p})\, e^{-ip\cdot x}}$$

$$\psi^{(2)} = \sqrt{E+m}\; e^{-ip\cdot x} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = \boldsymbol{u_2(E,\vec{p})\, e^{-ip\cdot x}}$$

***v-spinors:** for positrons, labeled with physical E>0 & $\vec{p}$*



$$\psi^{(1)} = u_4(-E,-\vec{p})\, e^{+ip\cdot x} = \boldsymbol{v_1(E,\vec{p})\, e^{+ip\cdot x}}$$

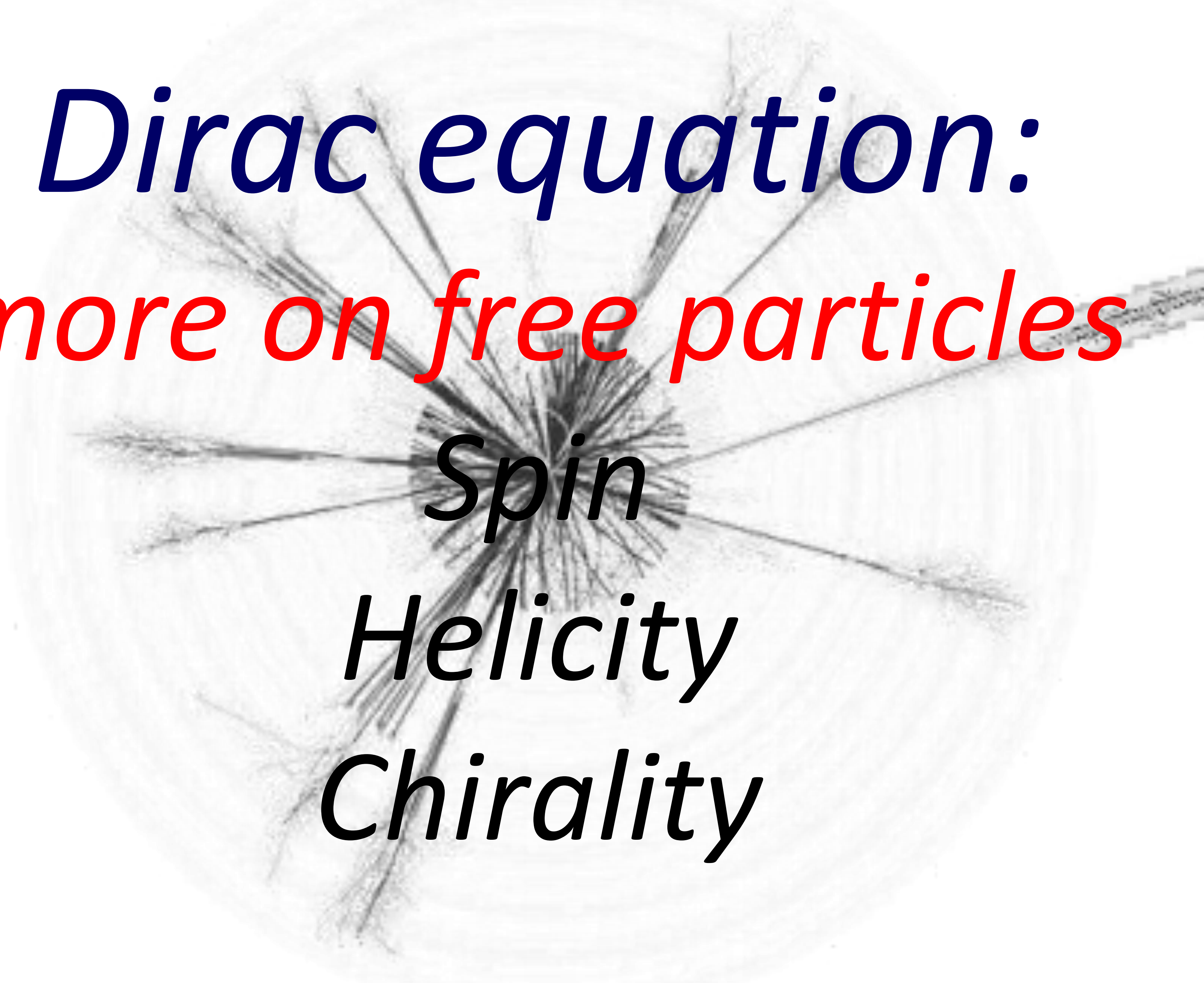
$$\psi^{(2)} = u_3(-E,-\vec{p})\, e^{+ip\cdot x} = \boldsymbol{v_2(E,\vec{p})\, e^{+ip\cdot x}}$$

Dirac equation: more on free particles

Spin

Helicity

Chirality



Dirac particles & spin

As you might guess, the two-fold degeneracy is because of the spin=½ nature of the particles the Dirac equation describes!

How do you see this?

Use commutator with Hamiltonian $H=\vec{\alpha}\cdot\vec{p}+\beta mc$ ***to find conserved quantities***

First attempt: orbital angular momentum $\vec{L}\equiv\vec{r}\times\vec{p}$ ***tells you:***

$$[H,\vec{L}]=[\vec{\alpha}\cdot\vec{p}+\beta mc,\vec{r}\times\vec{p}]=\alpha_l[p_l,\vec{r}\times\vec{p}]=\alpha_l p_l(\vec{r}\times\vec{p})-\alpha_l(\vec{r}\times\vec{p})p_l$$

$$=\alpha_l p_l \varepsilon_{ijk} r_j p_k-\alpha_l \varepsilon_{ijk} r_j p_k p_l$$

Used: $p_l r_j = r_j p_l + \frac{\hbar}{i}\delta_{lj}$

$$=\alpha_l \frac{\hbar}{i}\delta_{lj}\varepsilon_{ijk} p_k=\alpha_l \frac{\hbar}{i}\varepsilon_{ilk} p_k=\frac{\hbar}{i}\vec{\alpha}\times\vec{p}=-i\hbar\vec{\alpha}\times\vec{p}\neq\vec{0}$$ ☹

Second attempt: internal angular momentum $\vec{\Sigma}\equiv\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$ ***tells you:***

$$[H,\vec{\Sigma}]=[\vec{\alpha}\cdot\vec{p}+\beta mc,\vec{\Sigma}]=p_k[\alpha_k,\Sigma_l]=p_k\begin{pmatrix}0 & +[\sigma_k,\sigma_l]\\ +[\sigma_k,\sigma_l] & 0\end{pmatrix}$$

$$=p_k\, 2i\varepsilon_{klm}\begin{pmatrix}0 & +\sigma_m\\ +\sigma_m & 0\end{pmatrix}=2i\, p_k\, \varepsilon_{klm}\alpha_m\equiv 2i\vec{\alpha}\times\vec{p}\neq\vec{0}$$ ☹

☺ ***total spin*** $\vec{J}\equiv\vec{L}+1/2\hbar\vec{\Sigma}$ ***is conserved!***

Dirac particles & spin

Do we indeed describe particles with spin =½?

$$\left(\frac{1}{2}\vec{\Sigma}\right)^2 \psi = \frac{3}{4}\psi \sim s(s+1)\psi \longrightarrow s = \frac{1}{2}$$ ***Yes!***

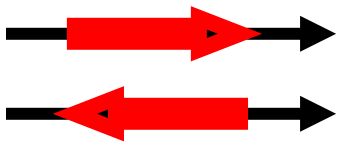
For particles with p=0: $$\frac{1}{2}\Sigma_3\psi = \begin{cases} \psi^{(1)}: & +\frac{1}{2} \times \psi^{(1)} \\ \psi^{(2)}: & -\frac{1}{2} \times \psi^{(2)} \\ \psi^{(3)}: & +\frac{1}{2} \times \psi^{(3)} \\ \psi^{(4)}: & -\frac{1}{2} \times \psi^{(4)} \end{cases}$$ ***can use (Σ^2, Σ_3) to classify states***

For particles with p≠0 we can not use Σ_3, but we can use spin // p: $1/2\vec{\Sigma}\cdot\hat{p}$

$$\left[H, \vec{\Sigma}\cdot\hat{\vec{p}}\right] = \hat{\vec{p}}\cdot\left[H, \vec{\Sigma}\right] = \hat{\vec{p}}\cdot 2i\vec{\alpha}\times\vec{p} = 0$$

Are you sure? Check it yourself!

$1/2\vec{\Sigma}\cdot\hat{p}$ ***is called helicity with eigenvalues: ±½***

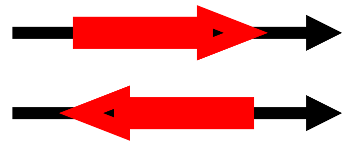


helicity + ½

helicity – ½

Helicity states

$1/2\vec{\Sigma}\cdot\hat{p}$ ***is called helicity with eigenvalues:*** $\pm$***½***



right-handed
helicity + ½ RH
helicity – ½ LH
left-handed

Instead of u_1 & u_2 spinors, we could use helicity ±½ : $u_\uparrow$ & $u_\downarrow$ spinors (& similarly for v-spinors)

You 'simply' solve the eigenvalue equation:

$$\frac{1}{2\text{p}}\begin{pmatrix}\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}\begin{pmatrix}u_A \\ u_B\end{pmatrix} = \lambda\begin{pmatrix}u_A \\ u_B\end{pmatrix} \Rightarrow \begin{array}{l}(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 2\text{p}\,\lambda u_A \\ (\boldsymbol{\sigma}\cdot\mathbf{p})u_B = 2\text{p}\,\lambda u_B\end{array}$$

Eigenvalues, use $(\boldsymbol{\sigma}\cdot\mathbf{p})^2 = \text{p}^2$**:** $\quad \text{p}^2 u_A = 2\text{p}\lambda(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 4\text{p}^2\lambda^2 u_A \Rightarrow$ **$\lambda = \pm$ ½ as it should**

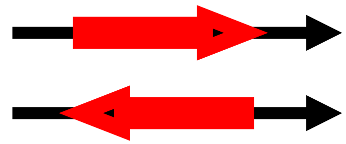
With u_A, you get u_B using the Dirac eqn. as we did before (easier now $\boldsymbol{\sigma}\cdot\vec{\boldsymbol{p}}\boldsymbol{u}_A = 2\boldsymbol{p}\lambda\boldsymbol{u}_A$):

$$(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = (E+m)u_B \Rightarrow u_B = 2\lambda\left(\frac{\text{p}}{E+m}\right)u_A$$

*

Helicity states

$1/2\vec{\Sigma}\cdot\hat{p}$ ***is called helicity with eigenvalues: ±½***



right-handed

helicity + ½ RH

helicity – ½ LH

left-handed

solving $(\boldsymbol{\sigma}\cdot\mathbf{p})u_A = 2\mathrm{p}\,\lambda u_A$

easiest using spherical coordinates: $\mathbf{p} = (\mathrm{p}\sin\theta\cos\phi,\ \mathrm{p}\sin\theta\sin\phi,\ \mathrm{p}\cos\theta)$

yields: $$\frac{1}{2\mathrm{p}}(\boldsymbol{\sigma}\cdot\mathbf{p}) = \frac{1}{2\mathrm{p}}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

with $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ ***follows:*** $\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\lambda\begin{pmatrix} a \\ b \end{pmatrix}$ ***or:*** $\begin{cases} a\cos\theta + b\sin\theta\, e^{-i\varphi} = 2\lambda a \\ a\ \sin\theta\, e^{+i\varphi} - b\cos\theta = 2\lambda b \end{cases}$

$$\Rightarrow \quad \frac{b}{a} = \frac{2\lambda - \cos\theta}{\sin\theta}e^{i\phi}$$

For λ = +½: $$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta}e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \Rightarrow$$

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{\mathrm{p}}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{\mathrm{p}}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

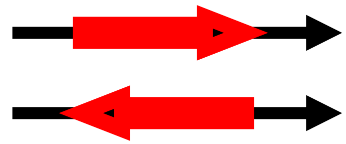
*

Helicity states

$c \equiv \cos(\theta/2)$
$s \equiv \sin(\theta/2)$

$1/2\vec{\Sigma}\cdot\hat{p}$ ***is called helicity with eigenvalues: ±½***

right-handed

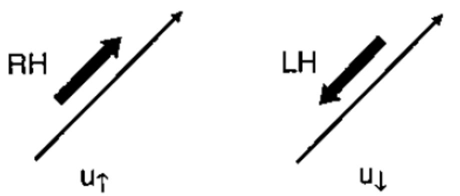


helicity + ½ RH

helicity – ½ LH

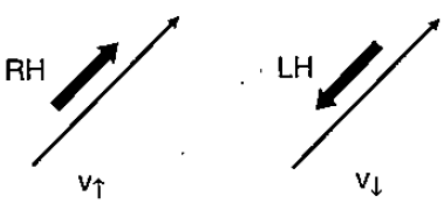
left-handed

Particles



$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \qquad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix}$$

Anti-particles



$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \qquad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

Remark:

we have used physical E & p for the v-spinors. Nevertheless: exponents still reflect negative energy (& momentum)! This means that the physical E, p and even helicity of v-spinors are obtained using the opposite of the operators used for u-spinors!
Afteral: we are re-interpreting the unwanted negative energy solutions of the Dirac eqn.!

Chirality

For massless & extremely relativistic particles, helicity states become simple:

Particles

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\varphi} \\ c \\ se^{i\varphi} \end{pmatrix} \equiv \boldsymbol{u_R} \qquad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\varphi} \\ s \\ -ce^{i\varphi} \end{pmatrix} \equiv \boldsymbol{u_L}$$

Anti-particles

$$v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\varphi} \\ -s \\ ce^{i\varphi} \end{pmatrix} \equiv \boldsymbol{v_R} \qquad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\varphi} \\ c \\ se^{i\varphi} \end{pmatrix} \equiv \boldsymbol{v_L}$$

These four states are also eigenstates of:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \equiv \gamma^5$$

$$\boldsymbol{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

Simple check: $\gamma^5 u_\uparrow = +u_\uparrow$, $\gamma^5 u_\downarrow = -u_\downarrow$ ***and:*** $\gamma^5 v_\uparrow = -v_\uparrow$, $\gamma^5 v_\downarrow = +v_\downarrow$

Eigenstates of γ^5 called:
Left-handed (L)
Right handed (R)
chiral states.

Weak interactions!

Dirac equation:

more on free particles

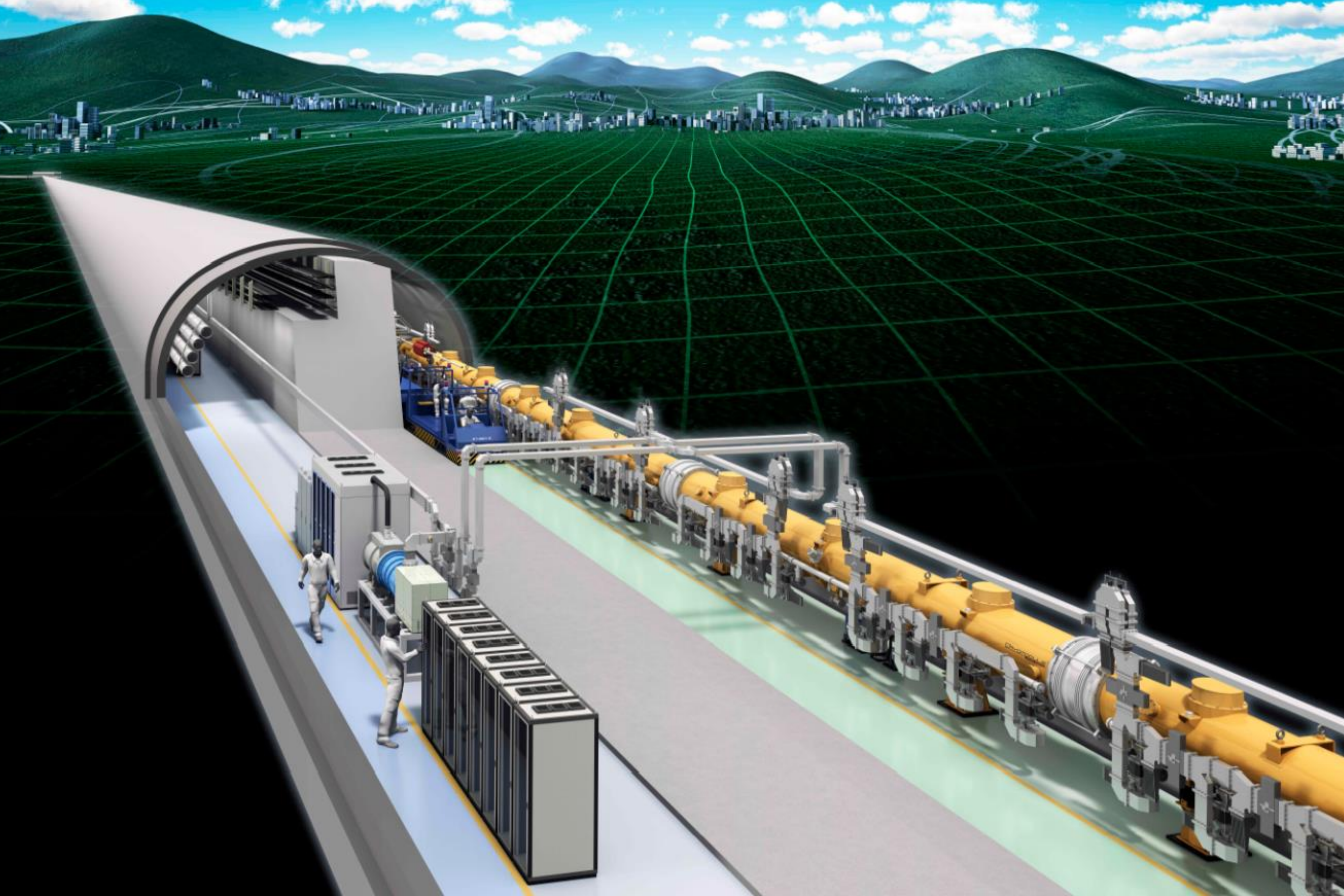
* *transformation properties*

! *normalisation*

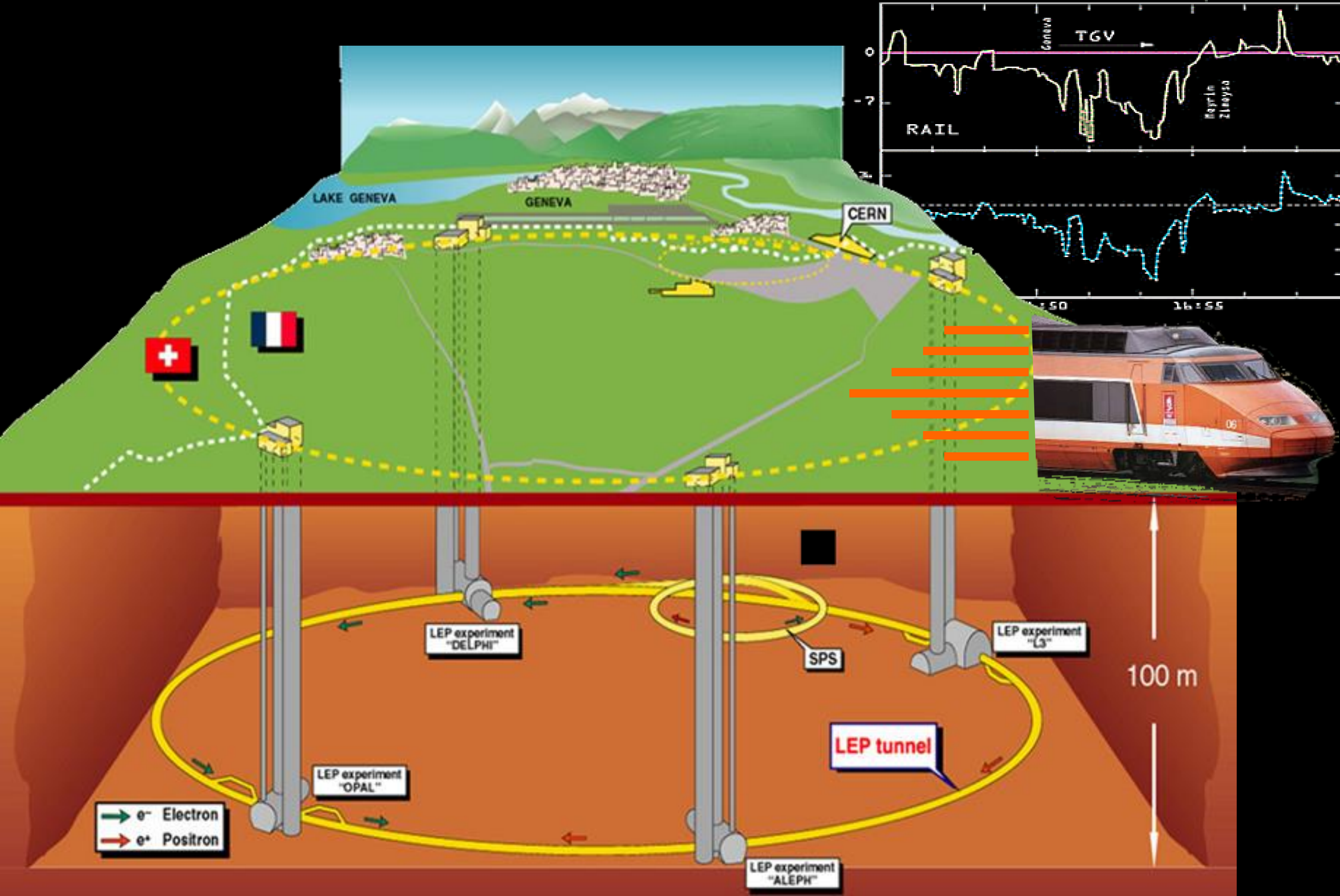
! *orthogonality*

! *completeness*

International Linear Collider (Japan?)

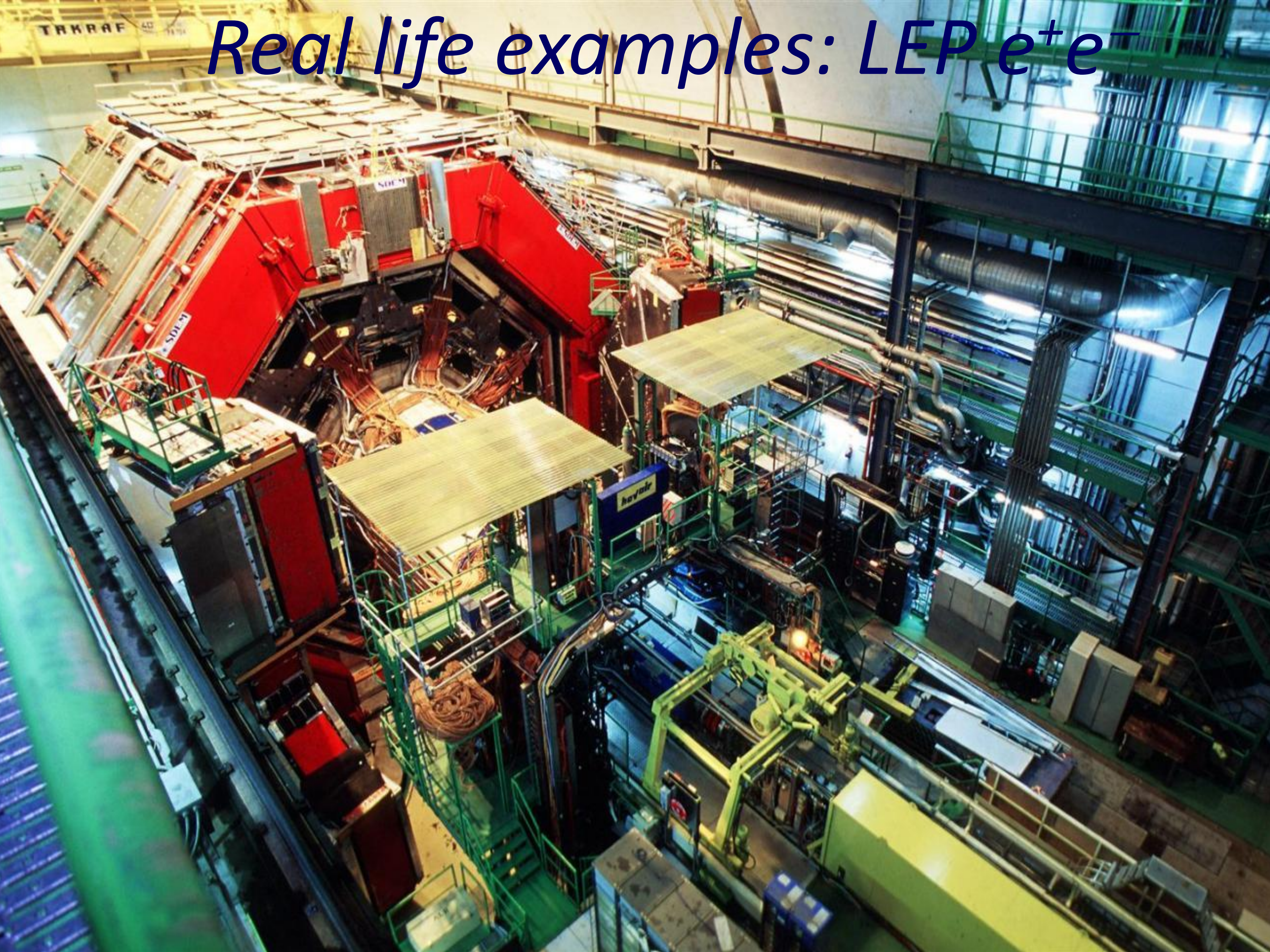


Real life examples: LEP e^+e^-

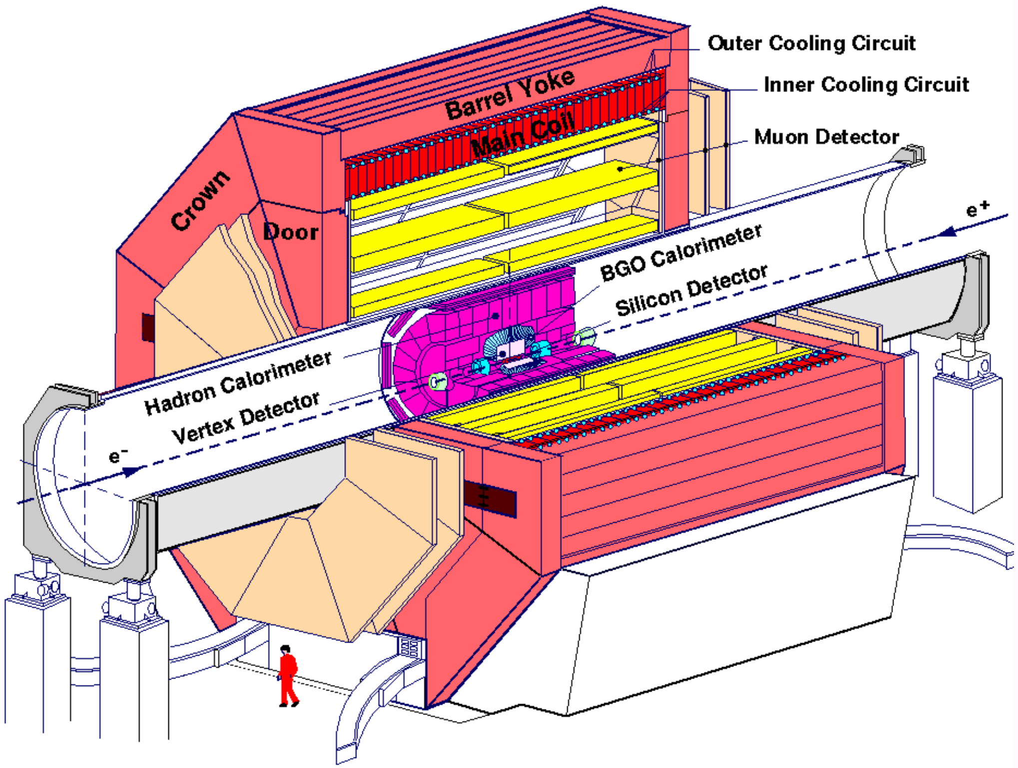


Real life examples: LEP e^+e^-

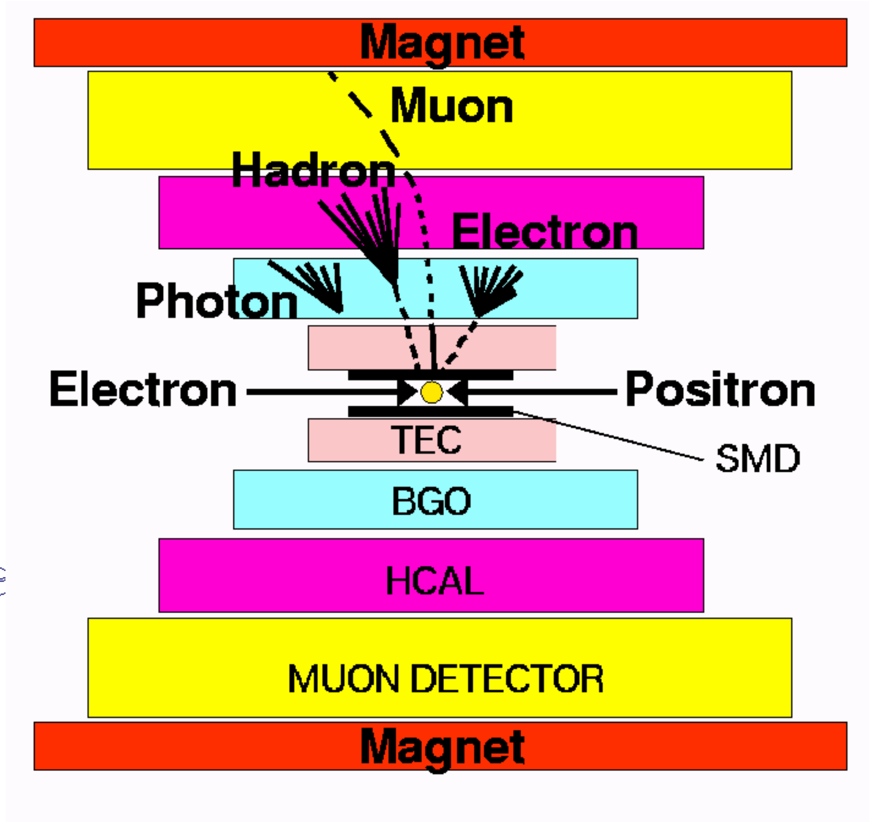
Real life examples: LEP e^+e^-



Real life examples: LEP e^+e^-

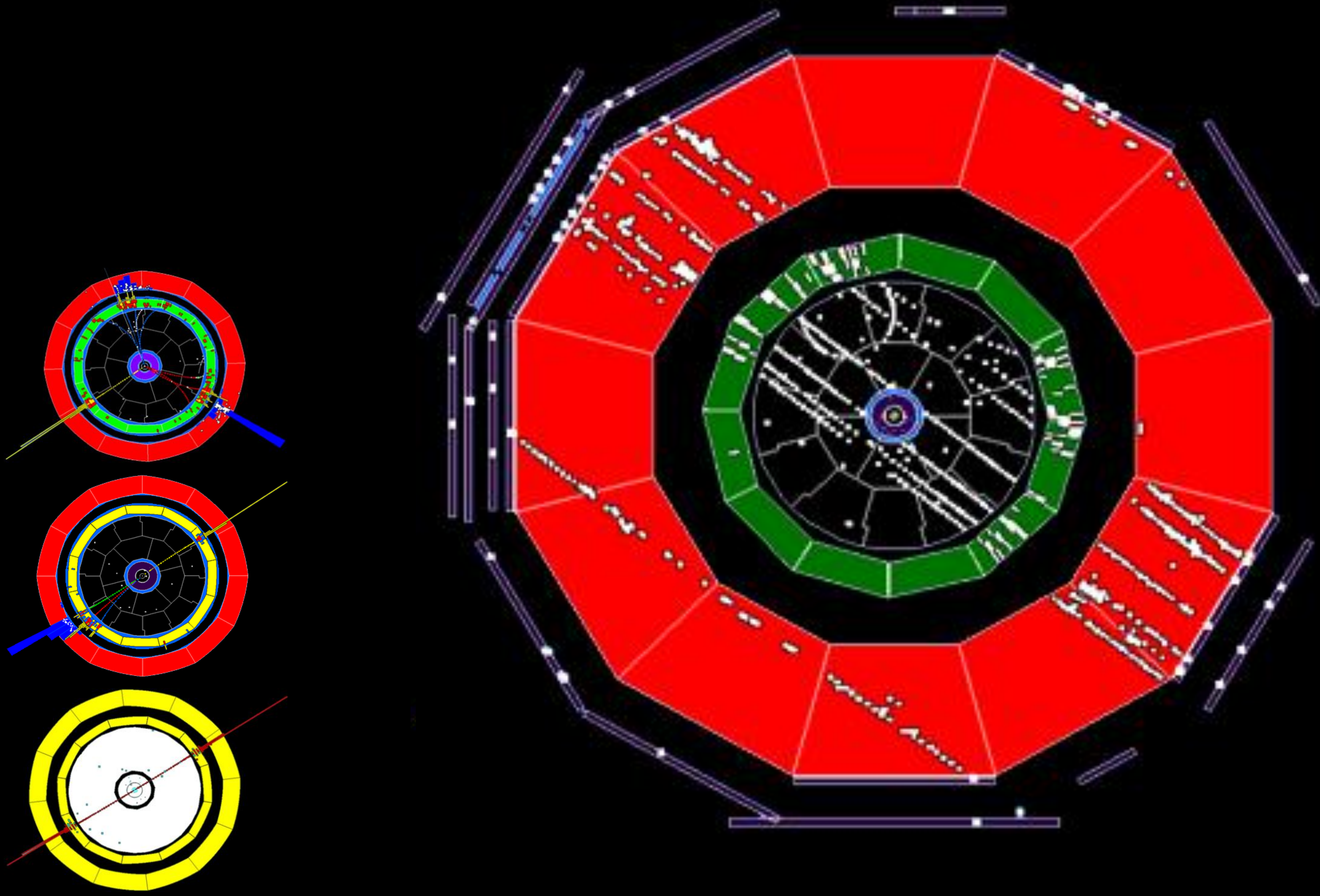


detector



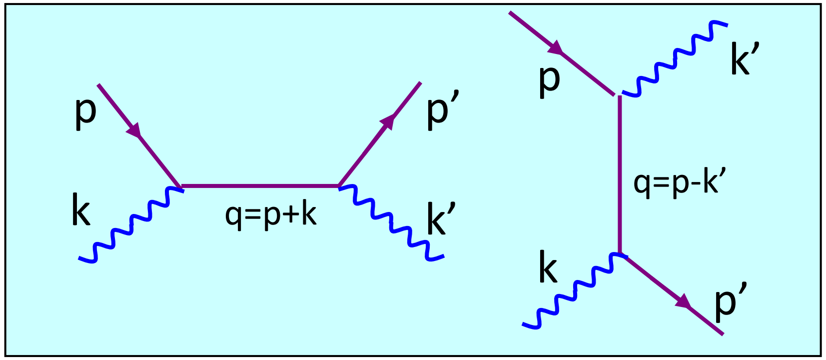
particle identification

Real life examples: LEP e^+e^-

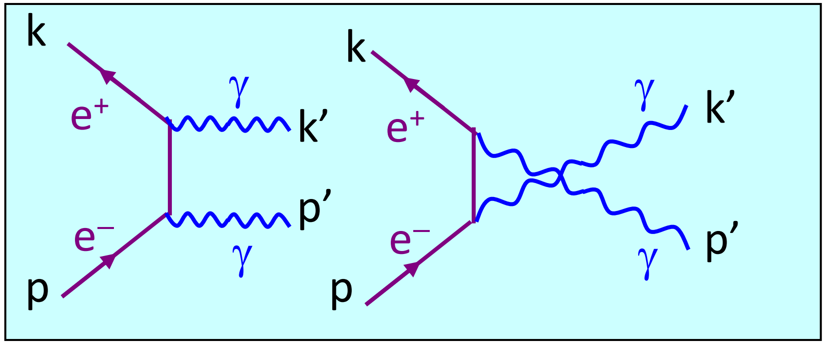


Other processes .....

Compton scattering: $e^-\gamma \rightarrow e^-\gamma$



Pair creation: $e^+e^- \rightarrow \gamma\gamma$



Lecture 8
Study Thomson 6