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Unit 1
Linear Programming Problem

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Formulation of Linear Programming Problem

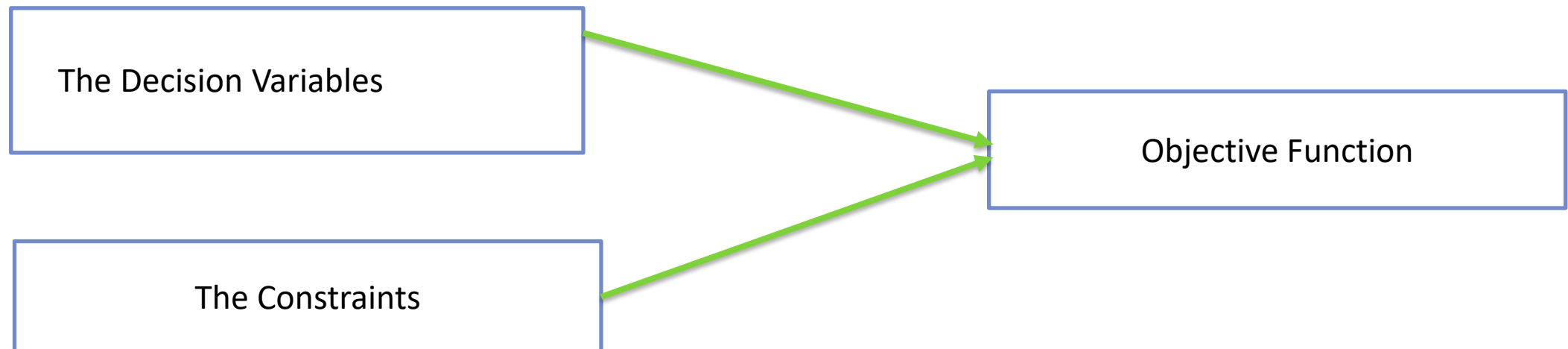


FORMULATION OF LINEAR PROGRAMMING PROBLEM

INTRODUCTION TO LINEAR PROGRAMMING

- Linear Programming is a problem solving approach that has been developed to help managers to make decisions.
- Linear Programming is a mathematical technique for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the resources, money, manpower, material, machine and other facilities.

FORMULATION OF THE LINEAR PROGRAMMING MODEL





TERMINOLOGY USED IN LINEAR PROGRAMMING PROBLEM

1. Components of LP Problem:

Every LPP is composed of a. Decision Variable, b. Objective Function, c. Constraints.

2. **Optimization:** Linear Programming attempts to either maximise or minimize the values of the objective function.

$$\text{Max } z \text{ or min } z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

3. **Profit of Cost Coefficient:** The coefficient of the variable in the objective function express the rate at which the value of the objective function increases or decreases by including in the solution by chnaging of one unit of each of the decision variable.

4. **Constraints:** The maximization (or minimization) is performed subject to a set of constraints. Therefore LP can be defined as a constrained optimisation problem. They reflect the limitations of the resources.

5. **Input-Output coefficients:** The coefficient of constraint variables are called the Input-Output Coefficients. They indicate the rate at which a given resource is unitized or depleted. They appear on the left side of the constraints.

6. **Capacities:** The capacities or availability of the various resources are given on the right hand side of the constraints.



BASIC DEFINITIONS

1. Decision Variables

These are the variables, whose quantitative values are to be found from the solution of the model so as to maximize or minimize the objective function. The decision variables are usually denoted by $x_1, x_2, x_3, \dots, x_n$. It may be controllable or uncontrollable.

Controllable variables are those, whose values are under control of the decision makers. Uncontrollable variables are those, whose values are not under control.

2. Objective Function

It is the determinants of quantity either to be maximized or to be minimized. An objective function must include all the possibilities with profit or cost coefficient per unit of output. It is denoted by z . The objective function can be stated as $\text{Max } z$ or $\text{min } z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

3. Constraints (Inequalities)

These are the restrictions imposed on decision variables. It may be in terms of availability of raw materials, machine hours, man-hours, etc.

$$\begin{aligned}
 &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, =, \geq) b_1 \\
 &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, =, \geq) b_2 \\
 &\dots \\
 &\dots \\
 &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, =, \geq) b_m \\
 &x_1, x_2, x_3 \dots x_n \geq 0.
 \end{aligned}$$



THE MATHEMATICAL EXPRESSION OF THE LP MODEL

The general LP Model can be expressed in mathematical terms as shown below:

Let →

a_{ij} → Input-Output Coefficient

c_j → Cost (Profit) Coefficient

b_i → Capacities (Right Hand Side)

x_j Decision Variables

Objective Function:

Find a vector $(x_1, x_2, x_3, \dots, x_n)$ that minimize or maximize a linear objective function z where

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to linear **constraints**

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, =, \geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, =, \geq) b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, =, \geq) b_m$$

and non-negativity constraints

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0.$$

Here x_1, x_2, \dots, x_n are the **decision variables**.



THE MATHEMATICAL EXPRESSION OF THE LP MODEL IN MATRIX FORM

The LPP is usually expressed in matrix form and then it becomes:

$$\text{Maximize or Minimize } z = c^T x$$

Subjected to $Ax(\leq, =, \geq)b, x \geq 0$. Here

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ is a matrix of order } m \times n, c = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ are the matrix of order } n \times 1,$$

$$\text{and } b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} \text{ is the matrix of order } m \times 1.$$



MATHEMATICAL FORMULATION

Introduction

In fact, the most difficult problem in the application of management science is the formulation of a model. Therefore, it is important to consider model formulation before launching into the details of linear programming solution. Model formulation is the process of transforming a real word decision problem into an operations research model.

The procedure for mathematical formulation of linear programming problem consists of the following major steps:

1. Write down the decision variables of the problem.
2. Formulate the objective function to be optimized (maximized or minimized) as a linear function of the decision variables.



MATHEMATICAL FORMULATION

3. Formulate the other conditions of the problem such as resource limitations, market constraints, inter-relation between variables etc. as linear equations or in equations in terms of the decision variables.
4. Add the ‘Non-negativity’ constraint from the consideration that negative values of the decision variables do not have any valid physical interpretation

Now the objective function, the set of constraints and the non-negative constraint together form a Linear Programming Problem.



MATHEMATICAL FORMULATION

Example:

A farmer has an area of A square kilometres to be planted with a combination of wheat and barley. A limited amount F of fertilizer and P of insecticide can be used, each of which is required in different amounts per unit area for wheat (F_1, P_1) and barley (F_2, P_2). Let S_1 be the selling price of wheat per square Kilometre, and S_2 the price of barley, and denote the area planted with wheat and barley as x_1 and x_2 respectively. Formulate this for the optimal number of square kilometres to plant with wheat vs. barley can be expressed as a linear programming problem.

Solution:

Here We need to maximise something, but what? $z = S_1x_1 + S_2x_2$

What stops us making z as big as we want?

constraints on x_1 and x_2

- Must fit in the field $x_1 + x_2 \leq A$
- Need enough fertilizer $F_1x_1 + F_2x_2 \leq F$
- Need enough pesticide $P_1x_1 + P_2x_2 \leq P$
- cannot plant a negative area $x_1 \geq 0, x_2 \geq 0$.



CONTD...

Hence Mathematical formulation of given problem as LPP is given by

$$\text{Maximize } z = S_1x_1 + S_2x_2$$

subjected to constraints

$$\begin{aligned} x_1 + x_2 &\leq A \\ F_1x_1 + F_2x_2 &\leq F \\ P_1x_1 + P_2x_2 &\leq P \\ x_1 &\geq 0, x_2 > 0. \end{aligned}$$

Example 2: A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (Minutes per Day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430



CONTD...

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

Formulation of Linear Programming Model

Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be x_1 , x_2 and x_3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1 , x_2 and x_3 units respectively.

Since negative production has no meaning and is not feasible. So x_1, x_2 and $x_3 \geq 0$.



CONTD...

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit. i.e. Maximize $z = 4x_1 + 3x_2 + 6x_3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables. Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \leq 440, \quad 4x_1 + 0x_2 + 3x_3 \leq 470, \quad 2x_1 + 5x_2 + 0x_3 \leq 430.$$

Hence L.P.P. formulation of given problem is

$$\text{Maximize } z = 4x_1 + 3x_2 + 6x_3$$

Subjected to constraint

$$2x_1 + 3x_2 + 2x_3 \leq 440,$$

$$4x_1 + 3x_3 \leq 470,$$

$$2x_1 + 5x_2 \leq 430,$$

$$x_1, x_2 \text{ and } x_3 \geq 0.$$



CONTD...

Example 3: An advertising agency is planning to launch an ad campaign. Media under consideration are T.V., Radio & Newspaper. Each medium has different reach potential and different cost. Minimum 10, 000, 000 households are to be reached through T.V. Expenditure on newspapers should not be more than Rs. 10, 00, 000. Total advertising budget is Rs. 20 million. Following data is available:

Medium	Cost per Unit (Rs.)	Reach per unit (No. of households)
Television	2, 00, 000	20, 00, 000
Radio	80, 000	10, 00, 000
Newspaper	40, 000	2, 00, 000

Formulate the LPP for best media selection that maximizes the total number of household reach.

Solution:

Decision Variables:

x_1 = Number of units of T.V. ads,

x_2 = Number of units of Radio ads,

x_3 = Number of units of Newspaper ads.

Objective function: (Maximize reach)

$$\text{Max. } z = 20,00,000 x_1 + 10,00,000 x_2 + 2,00,000 x_3$$



CONTD..

Subject to constraints:

$$20,00,000 x_1 \geq 10,000,000 \text{ ie } x_1 \geq 5 \dots\dots\dots (\text{for T.V.})$$

$$40,000 x_3 \leq 10,00,000 \text{ ie } x_3 \leq 25 \dots\dots\dots (\text{for Newspaper})$$

$$2,00,000x_1 + 80,000x_2 + 40,000x_3 \leq 20,000,000 \text{ ie } 5x_1 + 2x_2 + x_3 \leq 500 \dots\dots\dots (\text{Ad. budget})$$
$$x_1, x_2, x_3 \geq 0.$$

Hence mathematical formulation of given L.P.P is

$$\text{Max. } z = 20,00,000 x_1 + 10,00,000 x_2 + 2,00,000x_3$$

Subjected to constraint

$$x_1 \geq 5,$$
$$x_3 \leq 25,$$
$$5x_1 + 2x_2 + x_3 \leq 500,$$
$$x_1, x_2, x_3 \geq 0.$$



CONTD...

Example 4:

A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes one hours to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for A and Rs. 7 per bottle for B. Formulate this problem as a linear programming problem.



CONTD...

Solution:

Let production of each of A and B are x_1 unit bottles and x_2 unit bottles respectively. Since profits on each bottle of A and B are Rs. 8 and Rs. 7 per bottle respectively. So profit on x_1 bottles of A and x_2 bottles of B are $8x_1$ and $7x_2$ respectively.

Let z be total profit on bottles so, $z = 8x_1 + 7x_2$.

Since, it takes 3 hours and 1 hour to prepare enough material to fill 1000 bottles of Type A and Type B respectively. so x bottles of A and y bottles of B are preparing is $\frac{3x_1}{1000}$ hours and $\frac{x_2}{1000}$ hours respectively, but only 66 hours are available, so

$$\frac{3x_1}{1000} + \frac{x_2}{1000} \leq 66 \dots\dots\dots(\text{Time available constraint})$$

$$\text{or, } 3x_1 + x_2 \leq 66000.$$



CONTD...

Since raw material available to make 20000 bottles of A and 40000 bottles of B but there are 45000 bottles in which either of these medicines can be put so,

$$x_1 \leq 20000, \dots\dots\dots(\text{Bottle A constraint})$$

$$x_2 \leq 40000, \dots\dots\dots(\text{Bottle B constraint})$$

$$x_1 + x_2 \leq 45000 \dots\dots(\text{combined production constraint})$$

$$\text{with } x_1, x_2 \geq 0.$$

So mathematical formulation of the given LPP is

$$\text{Max } z = 8x_1 + 7x_2$$

Subject to constraints

$$3x_1 + x_2 \leq 66000,$$

$$x_1 \leq 2000,$$

$$x_2 \leq 40000,$$

$$x_1 + x_2 \leq 45000$$

$$x_1, x_2 \geq 0.$$



CONTD...

Example 5:

A firm can produce three types of cloth A , B and C . Three kinds of wool is required for it, say red, green and blue wools. One unit length of type A cloth needs 2 yards of red wool, 5 yards of blue wools, one unit length of type B cloth needs 3 yards of red wool, 4 yards of green wool, and 2 yards of blue wool, and one unit length of type C cloth needs 6 yards of green and 5 yards of blue wools. The firm has only a stock of 10 yards of red wool, 12 yards of green wool, and 17 yards of blue wool. It is assumed that the income obtained from one unit length of type A , B and C are Rs 4.00, 5.00 and 6.00 respectively. Determine how the firm should use the available material, so as to maximize the income from the finished cloths.



CONTD...

Solution:

The above problem can be tabulated as:

Kinds of wool	Types of cloth			Stock of wool (yards)
	A	B	C	
Red	2	3	0	10
Green	0	4	6	12
Blue	5	2	5	17
Income from one unit of clothes (Rs)	4.00	5.00	6.00	



CONTD...

Let the firm produce x_1 , x_2 and x_3 yards of three types of cloth A , B and C respectively. Therefore, x_1 , x_2 and x_3 can be treated as decision variables.

Since the profit per unit length of type A , B and C are given and we have to maximize the profit, therefore, we have

$$\text{Max } Z = 4x_1 + 5x_2 + 6x_3$$

As per the statement of given problem, we have

$$2x_1 + 3x_2 + 0x_3 \leq 10$$

$$0x_1 + 4x_2 + 6x_2 \leq 12$$

$$5x_1 + 2x_2 + 5x_3 \leq 17,$$

$$x_1, x_2, x_3 \geq 0.$$



CONTD...

So mathematical formulation of the given LPP is

$$\text{Max } z = 4x_1 + 5x_2 + 6x_3$$

Subjected to constraints

$$2x_1 + 3x_2 \leq 10$$

$$4x_2 + 6x_3 \leq 12,$$

$$5x_1 + 2x_2 + 5x_3 \leq 17$$

$$x_1, x_2, x_3 \geq 0.$$



CONTD...

Example 6 A research laboratory has two melts *A* and *B* of copper (Cu), Nickel (Ni) and Zinc (Zn) alloy to make up a new alloy. The composition of metals are as follows.

Melt	Composition (Parts)		
	Cu	Ni	Zn
<i>A</i>	3	2	1
<i>B</i>	2	2	1

To make up a new alloy at least 15 kg of copper, 10 kg of nickel, and 6 kg of zinc are needed. Melt *A* cost Rs 45 per kg and melt *B* cost Rs 50 per kg. Formulate the L.P.P. for the quantities of each melt to be used to minimized cost.



Solution:

The above data can be tabulated as follows.

Composition	Melt		Requirement of elements(Rs.)
	A	B	
Cu	3	2	15
Ni	2	2	10
Zn	1	1	6
Cost per kg of melt (Rs)	45	50	

Decision Variables: Let x_1 and x_2 be the quantity of melt A and B respectively. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Function: Since cost per kg melt of product A and B are given and we have to minimize the cost. Therefore,

$$\text{Min } z = 45x_1 + 50x_2$$



Contd

Constraints: As per the statement of problem, we have

$$\frac{3}{6}x_1 + \frac{2}{5}x_2 \geq 15 \quad \text{or} \quad 5x_1 + 4x_2 \geq 150,$$

$$\frac{2}{6}x_1 + \frac{2}{5}x_2 \geq 10 \quad \text{or} \quad 5x_1 + 6x_2 \geq 150,$$

$$\frac{1}{6}x_1 + \frac{1}{5}x_2 \geq 6 \quad \text{or} \quad 5x_1 + 6x_2 \geq 180,$$

Non-negative Restrictions: $x_1, x_2 \geq 0$.

So mathematical formulation of the given LPP is

$$\text{Min } Z = 45x_1 + 50x_2$$

subjected to constraints

$$5x_1 + 4x_2 \geq 150$$

$$5x_1 + 6x_2 \geq 150$$

$$5x_1 + 6x_2 \geq 180$$

$$x_1, x_2 \geq 0$$



SOLUTION OF LINEAR PROGRAMMING PROBLEM- GRAPHICAL METHOD



SOLUTION OF LINEAR PROGRAMMING PROBLEM

Solution of L.P.P.

The set of values of decision variables that satisfy the constraints of an LPP is said to be solution of L.P.P.

Feasible Solution L.P.P.

The solution of L.P.P. which satisfy nonnegative restriction is called feasible solution.

Infeasible solution of L.P.P.

The set of values of decision variables x_j ($j = 1 2 \dots n$) which do not satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the infeasible solution to that linear programming problem.



METHOD

Unbounded solution of L.P.P. :

A solution which increases (in case of maximization) or decreases (in case of minimization) the value of objective function of the LP problem infinitely is called unbounded solution.

Optimal Solution of L.P.P.

The feasible solution which optimize objective function is called optimal solution.

Method for Solving L.P.P

- Graphical Method
- Simplex Method
- Dual Simplex Method
- Two-phase Simplex Method
- Big M Method



GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEMS

Graphical method is applicable to find the simple linear programming problem with two decision variables. Various steps for solving the problems are given below:

1. Draw the nonnegative restrictions as axes.
2. Consider each inequality constraint as equation and draw each equations on the graph such that each will geometrically represent a straight line.
3. Take region of the constraint as an inequality form. If the inequality constraint corresponding to that line is \leq , then the region below the line in the first quadrant is to be shaded. For the inequality constraint \geq , then the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region.



CONTD....

4. Locate the corner points (Vertices) of the feasible region. These corner points are also called extreme point.
5. Test the objective function at each corner point of the feasible region and choose the point, where objective function obtains optimal value.



GRAPHICAL METHOD

EXAMPLE 1:

Solve the following L.P.P. by graphical method

$$\text{Min } z = 20x_1 + 10x_2$$

Subject to constraint

$$x_1 + 2x_2 \leq 40,$$

$$3x_1 + x_2 \geq 30,$$

$$4x_1 + 3x_2 \geq 60,$$

$$x_1, x_2 \geq 0.$$



GRAPHICAL METHOD

Solution

Convert all the inequalities of the constraints into equations, we have

$$x_1 + 2x_2 = 40$$

$$3x_1 + x_2 = 30$$

$$4x_1 + 3x_2 = 60$$

$$x_1 + 2x_2 = 40 \text{ passes through } (0, 20) \text{ \& } (40, 0)$$

$$3x_1 + x_2 = 30 \text{ passes through } (0, 30) \text{ \& } (10, 0)$$

$$4x_1 + 3x_2 = 60 \text{ passes through } (0, 20) \text{ \& } (15, 0)$$

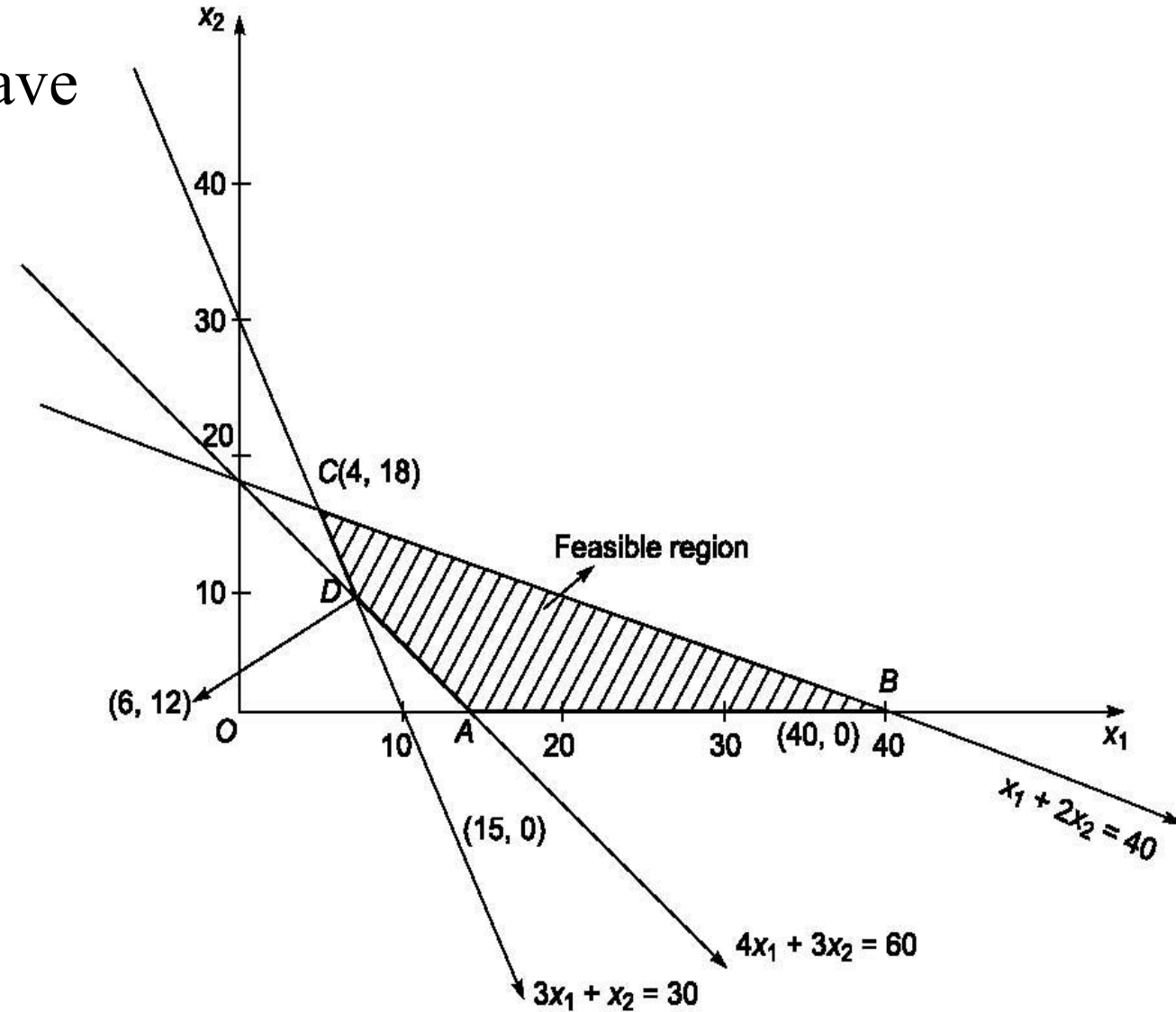


CONTD.....

Plot above equations on graph, we have

Here feasible region is $ABCD$.
The coordinates of $ABCD$ are
 $A(15, 0)$, $B(40, 0)$, $C(4, 18)$,
and $D(6, 12)$

$$\begin{aligned}x_1 + 2x_2 &= 40 \\3x_1 + x_2 &= 30 \\4x_1 + 3x_2 &= 60\end{aligned}$$





CONTD...

Here feasible region is ABCD.

The coordinates of extreme points of region $ABCD$ are $A(15, 0)$ $B(40, 0)$, $C(4, 18)$ $D(6, 12)$

Corner Points	Coordinate	Value of objective function z
A	$(15, 0)$	300
B	$(40, 0)$	800
C	$(4, 18)$	260
D	$(6, 12)$	240

Therefore, minimum value of Z occurs at $D(6, 12)$. Hence, optimal solution is $x_1 = 6, x_2 = 12$.



CONTD...

Example 2:

Solve the following L.P.P. using graphical methods

$$\text{Max } z = 6x_1 + 8x_2$$

Subject to

$$5x_1 + 10x_2 \leq 60,$$

$$4x_1 + 4x_2 \leq 40,$$

$$x_1, x_2 \geq 0.$$



CONTD...

Solution: Convert all the equalities of the constraint into equations, we have

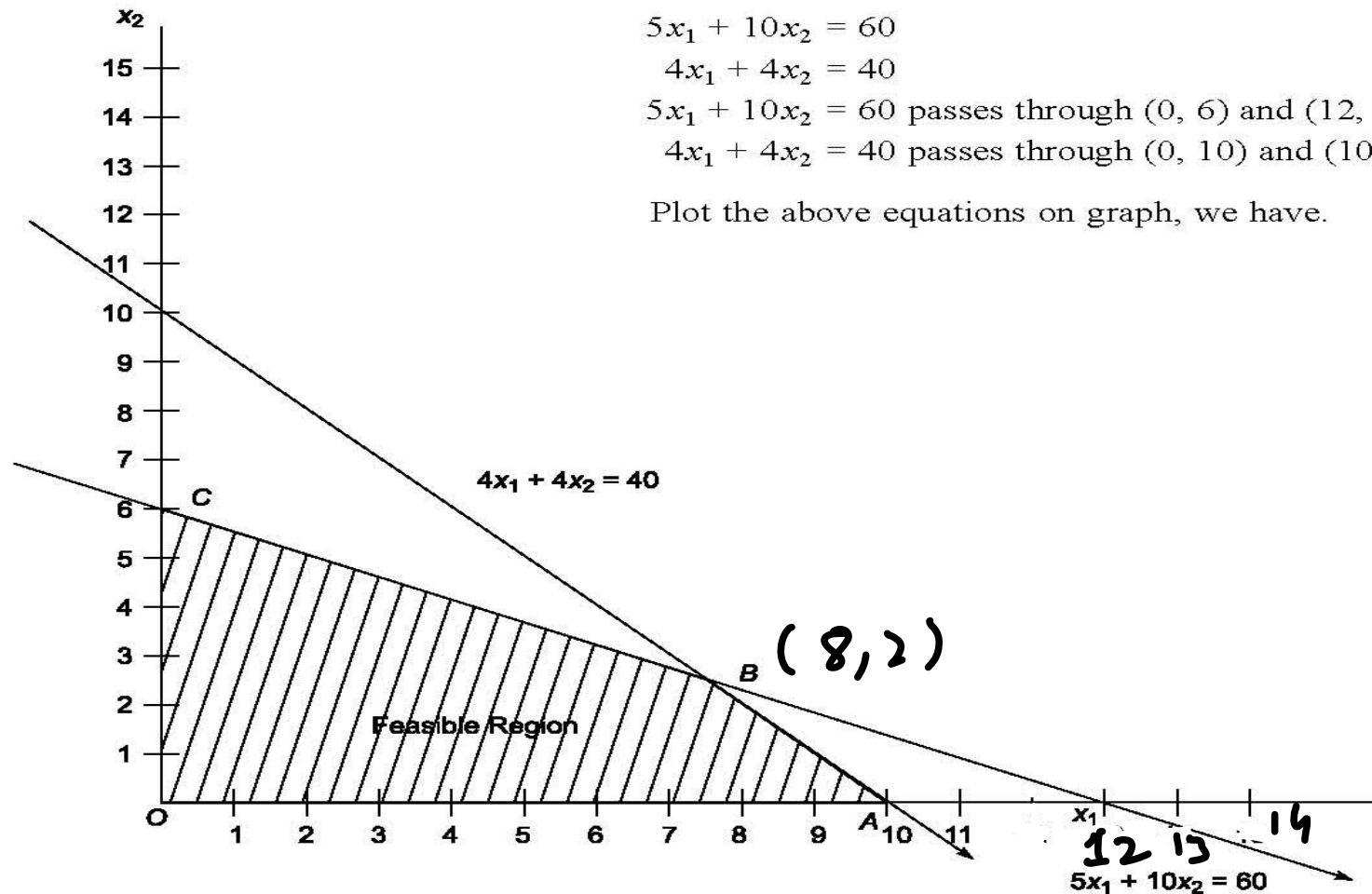
$$5x_1 + 10x_2 = 60$$

$$4x_1 + 4x_2 = 40$$

$5x_1 + 10x_2 = 60$ passes through (0, 6) and (12, 0)

$4x_1 + 4x_2 = 40$ passes through (0, 10) and (10, 0)

Plot the above equations on graph, we have.





CONTD...

Here feasible region is $OABC$.

The coordinates of $OABC$ are $O(0, 0)$, $A(10, 0)$, $B(8, 2)$, $C(0, 6)$

Now

Corner Points	Coordinate	Value of z
O	$(0, 0)$	0
A	$(10, 0)$,	60
B	$(8, 2)$	64
C	$(0, 6)$	48

Therefore, Maximum value of Z occurs at $B(8, 2)$. Hence, optimal solution is $x_1 = 8, x_2 = 2$.



Special Cases in Linear Programming Problem

Unbounded Solution

When values of decision variable may be increased infinitely without violating any of the constraints, the solution space (feasible region) is said to be unbounded. If value of objective function increases (for maximization case) or decreases (for minimization case) infinitely, solution space and objective function both are said to be unbounded.



CONTD...

Example 3:

Solve the following L.P.P. by graphical method

$$\text{Min } z = 3x_1 + 2x_2$$

Subject to constraint

$$x_1 + 4x_2 \geq 12,$$

$$x_1 + x_2 \geq 6,$$

$$5x_1 + x_2 \geq 10,$$

$$x_1, x_2 \geq 0.$$

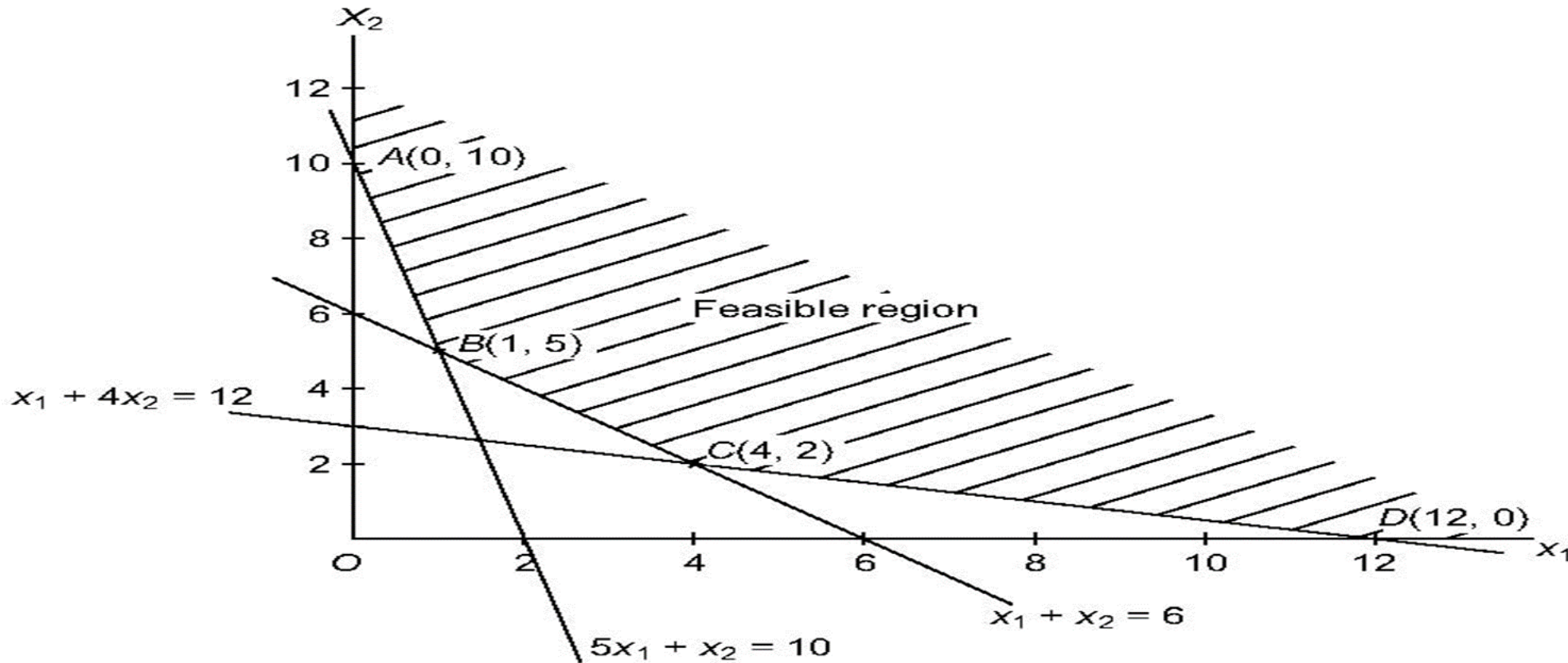


CONTD...

Convert all the equalities of the constraint into equations, we have

$$x_1 + 4x_2 = 12, x_1 + x_2 = 6, 5x_1 + x_2 = 10,$$

Plot the above equations on graph, we have





CONTD...

Now the coordinates of points $ABCD$ are $A(0, 10)$, $B(1, 5)$, $C(4, 2)$, $D(12, 0)$

Corner Points	Coordinate	Value of Z
A	$(0, 10)$	20
B	$(1, 5)$	13
C	$(4, 2)$	16
D	$(12, 0)$	36

Hence, minimum value occurs at point $B(1, 5)$. Therefore, optimum solution is given by

$$x_1 = 1, x_2 = 5 \text{ and } \min Z = 13.$$



CONTD...

Example 4

Solve the following L.P.P. by graphical method

$$\text{Max } z = 3x_1 + 2x_2$$

Subject to constraint

$$x_1 - x_2 \leq 1,$$

$$x_1 + x_2 \geq 3,$$

$$x_1, x_2 \geq 0.$$



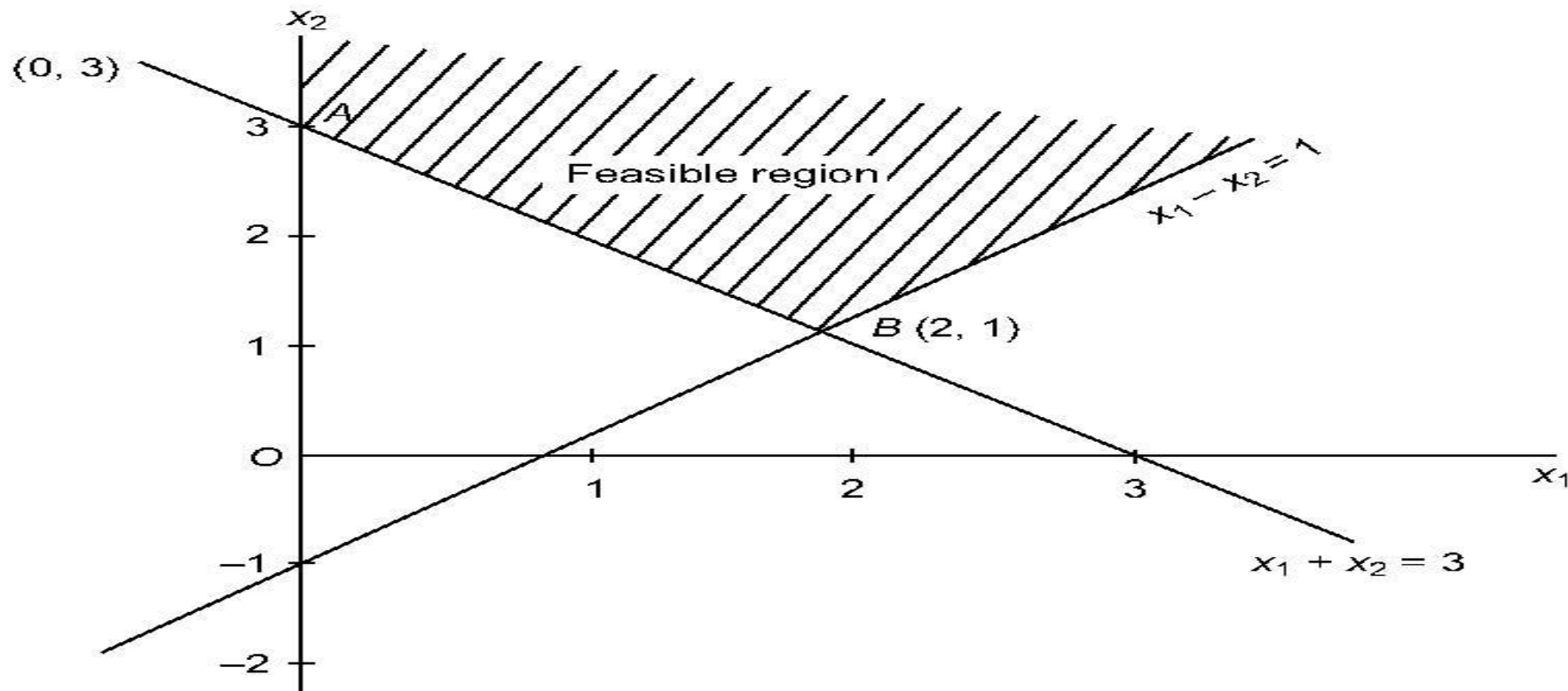
CONTD...

Convert the inequality constraints into equations. We have

$$x_1 - x_2 = 1, \quad x_1 + x_2 = 3,$$

$x_1 - x_2 = 1$ passes through $(0, -1)$ and $(1, 0)$, $x_1 + x_2 = 3$ passes through $(0, 3)$ and $(3, 0)$

Plot above equations on graph, we have





CONTD...

- Here the solution space is unbounded. The value of objective function at the vertices A and B are $Z(A) = 6$, $Z(B) = 8$. But there exists points in the feasible region for which the value of the objective function is more than 8. In fact, the maximum value of z occurs at infinity. Hence, the problem has an unbounded solution.



Special Cases in Linear Programming Problem

Infeasible Solution

Infeasible solution arises when there is no solution to linear programming problem that satisfies all the constraint simultaneously. This means there would be no unique feasible region. Such a problem arises when wrong model is formulated that has conflicting constraints. Every possible solution **violates** at least one constraint.

Example 5: Use Graphical method to solve the following LP problem

$$\text{Maximize } z = 6x_1 - 4x_2$$

subjected to constraints

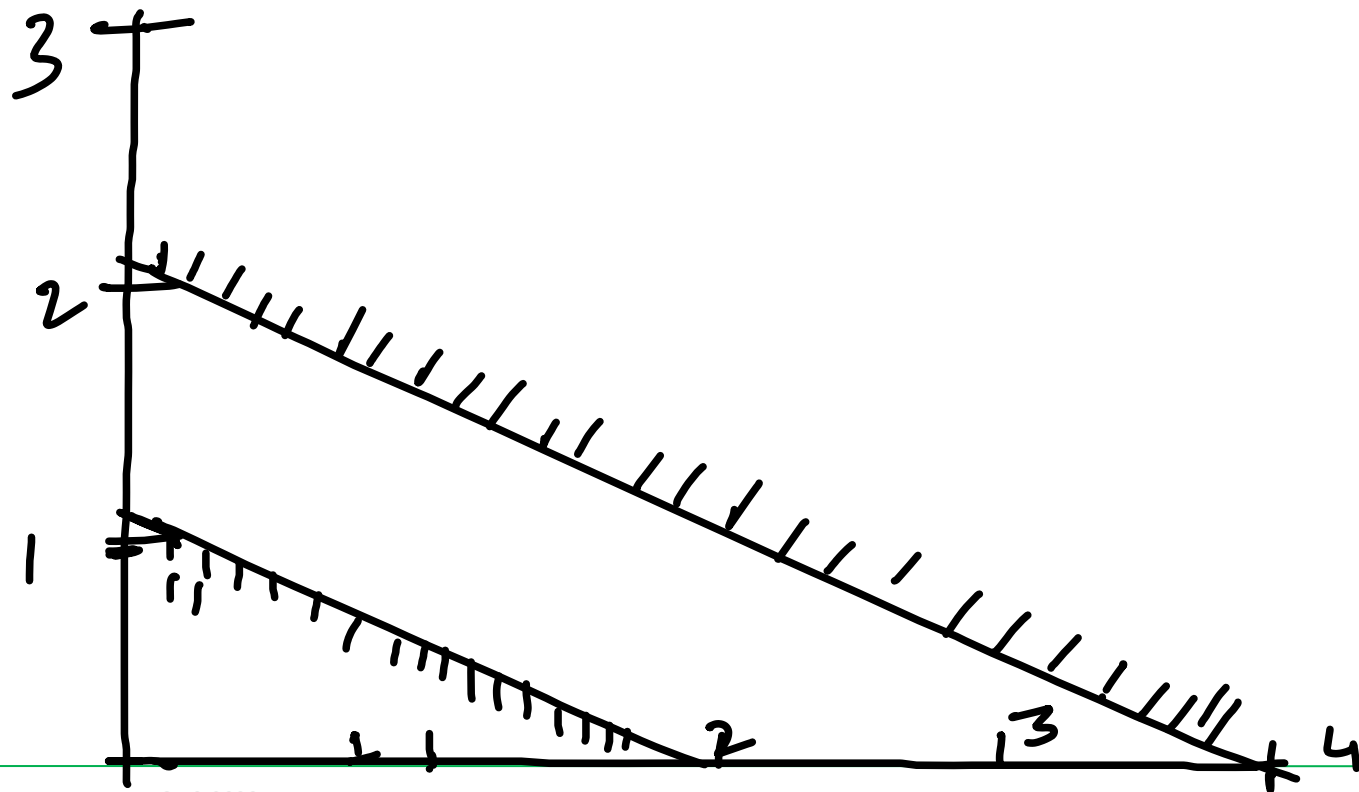
$$2x_1 + 4x_2 \leq 4;$$

$$4x_1 + 8x_2 \geq 16;$$

$$x_1, x_2 \geq 0$$

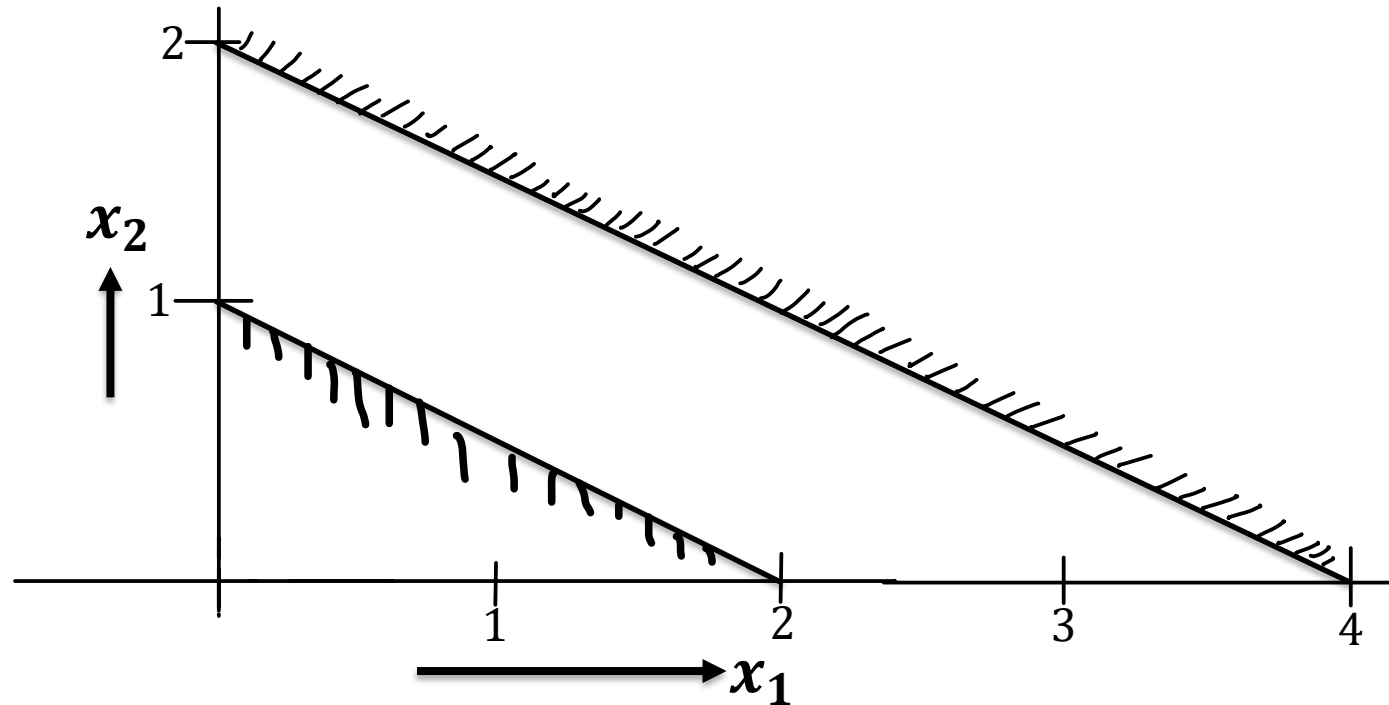


$$\begin{aligned} 2x_1 + 4x_2 &\leq 4; \\ 4x_1 + 8x_2 &\geq 16; \\ x_1, x_2 &\geq 0 \end{aligned}$$





Solution:



The constraints are plotted on graph shown in the figure. We can see that there is no unique feasible solution space allows us to obtain unique set of values of x_1 and x_2 that satisfy all the constraints simultaneously. Hence there is no feasible solution to the given problem.



An Infeasible Solution

Example 6

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

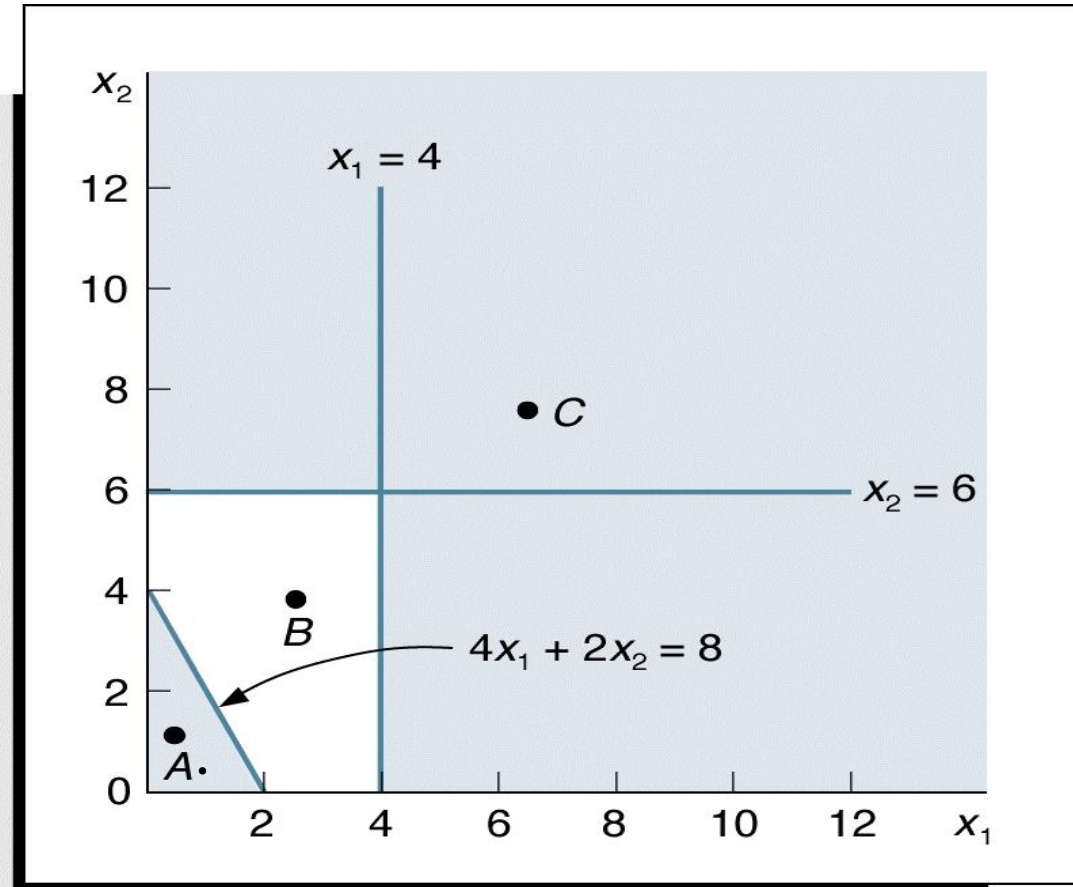
$x_2 \geq 6$

$x_1, x_2 \geq 0.$

Solution:

There is no common feasible region generated by two constraints together i.e. we cannot identify even a single point satisfying all the constraints simultaneously. Hence there is no feasible solution to the given problem.

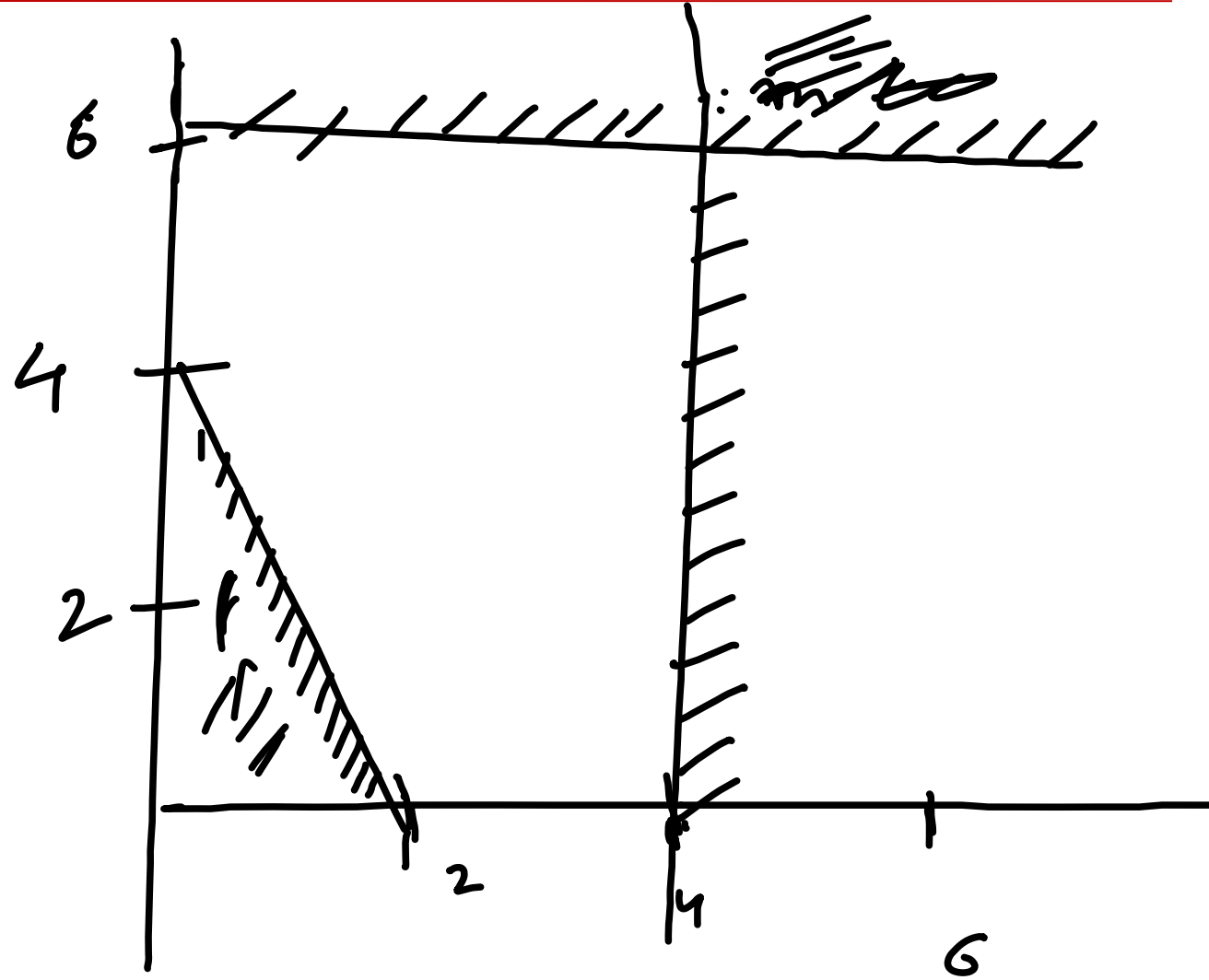
Plot given inequalities on graph, we have





$$4x_1 + 2x_2 \leq 8$$

- $x_1 \geq 4$
- $x_2 \geq 6$
- $x_1, x_2 \geq 0.$





Special Cases in Linear Programming Problem Alternative or Multiple Optimal Solutions

There are two conditions that should be satisfied for the alternative solution

1. The given objective function should be parallel to the constraint that forms the boundary of the feasible region.
2. The constraints should form a boundary of the feasible region in the direction of the optimal movement of the objective function.

Note: If we get an optimal solution of the objective function at two extreme points $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then L.P. problem has infinite numbers of solution at the point $\lambda x + (1 - \lambda)y = (\lambda x_1 + (1 - \lambda)y_1, \lambda x_2 + (1 - \lambda)y_2)$ $0 \leq \lambda \leq 1$.

$$\begin{array}{c} \lambda x + (1 - \lambda)y \\ \lambda = (\lambda_1, \lambda_2) \\ y = (y_1, y_2) \end{array}$$



Alternative or Multiple Optimal Solutions

Example 7:

Maximize $z = x_1 + 2x_2$

subject to

$$x_1 \leq 80,$$

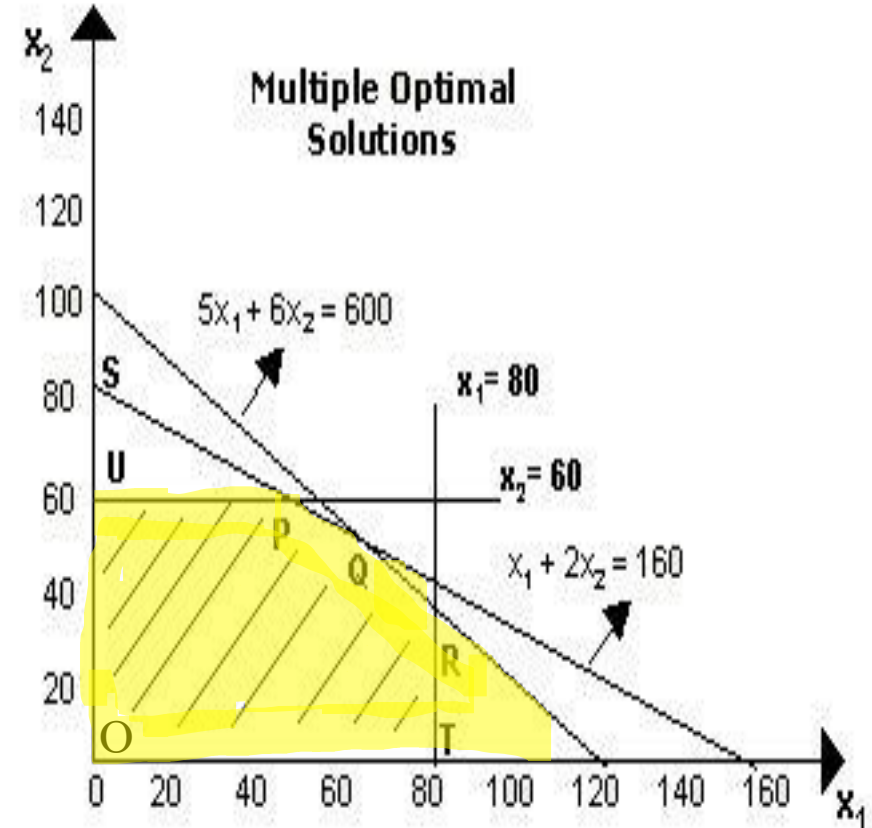
$$x_2 \leq 60,$$

$$5x_1 + 6x_2 \leq 600,$$

$$x_1 + 2x_2 \leq 160,$$

$$x_1, x_2 \geq 0.$$

$$60 - 20\lambda, 10\lambda + 50$$





CONTD...

Here feasible region is OTRQPU. The coordinates of extreme points of region OTRQPU are $O(0, 0)$, $T(80, 0)$, $R(80, 100/3)$, $Q(60, 50)$, $P(40, 60)$, $U(0, 60)$.

Since at the two different points P and Q, the optimum value of objective function is same. Hence all points from P to Q lying on line PQ represent optimal solutions and all these will give the same optimal value (maximum profit) of 160. This is indicated by the fact that both the points P with co-ordinates (40, 60) and Q with co-ordinates (60, 50) are on the line $x_1 + 2x_2 = 160$. Thus, every point on the

line PQ maximizes the value of the objective function and the problem has infinite number of optimal solutions. Hence given LPP has infinitely many solution at the point $\bar{x} = \lambda x + (1 - \lambda)y = \lambda(40, 60) + (1 - \lambda)(60, 50) = (60 - 20\lambda, 10\lambda + 50) \ 0 \leq \lambda \leq 1$.

Corner point or Extreme point	Coordinate	Value of the objective function
O	(0,0)	0
T	(80,0)	80
R	(80, 100/3),	440/3
Q	(60, 50),	160
P	(40,60),	160
U	(0,60).	120



Alternative or Multiple Optimal Solutions

Example 8:

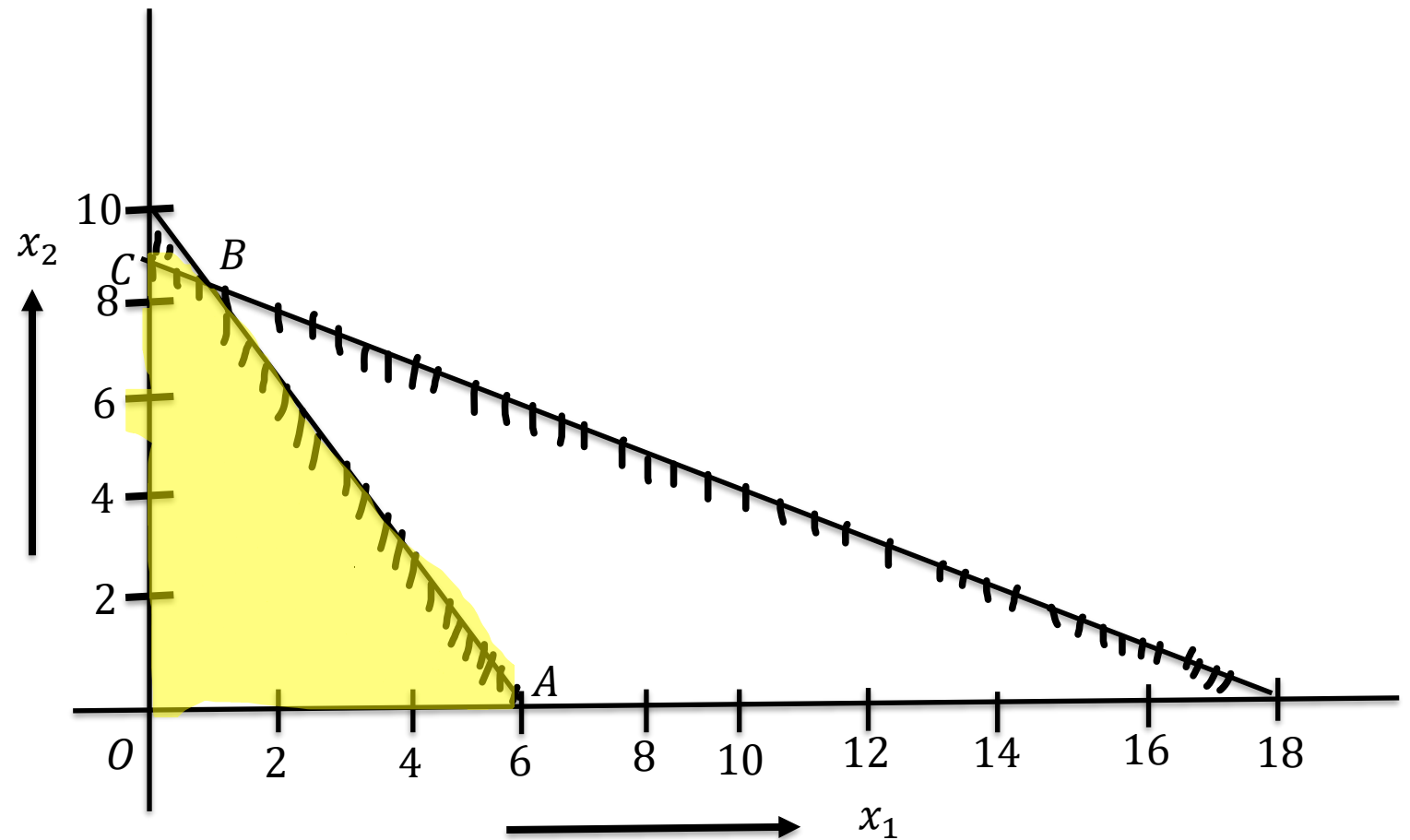
$$\text{Max } z = 10x_1 + 6x_2$$

subjected to
constraint

$$5x_1 + 3x_2 \leq 30;$$

$$x_1 + 2x_2 \leq 18;$$

$$x_1, x_2 \geq 0.$$





CONTD...

Here feasible region is $OABC$. The coordinates of extreme points region $OABC$ are $O(0, 0)$, $A(6, 0)$, $B(6/7, 60/7)$, $C(0, 9)$.

Since at the two different points A and B , the optimum value of objective function is same. Hence all points from A to B lying on line AB represent optimal solutions and all these will give the same optimal value (maximum value of z) of 60. This is indicated by the fact that

Corner point or Extreme point	Coordinate	Value of the objective function
O	$(0,0)$	0
A	$(6,0)$	60
B	$(6/7, 60/7)$,	60
C	$(0, 9)$,	54

both the points A with co-ordinates $(6, 0)$ and B with co-ordinates $(6/7, 60/7)$ are on the line $5x_1 + 3x_2 = 30$. Thus, every point on the line AB maximizes the value of the objective function and the problem has infinite number of optimal solutions. Hence given LPP has infinitely many solution at the point $\bar{x} = \lambda x + (1 - \lambda)y = \lambda(6, 0) + (1 - \lambda)(6/7, 60/7) = \left(\frac{36}{7}\lambda + \frac{6}{7}, (1 - \lambda)\frac{60}{7}\right) 0 \leq \lambda \leq 1$.



Limitations or Assumptions of Linear Programming Problem

Linearity of relations: A primary requirement of linear programming is that the objective function and every constraint must be linear. However, in real life situations, several business and industrial problems are nonlinear in nature.

Single objective: Linear programming takes into account a single objective only, i.e., profit maximization or cost minimization. However, in today's dynamic business environment, there is no single universal objective for all organizations.

Certainty or Deterministic: Linear Programming assumes that the values of co-efficient of decision variables are known with certainty. Due to this restrictive assumption, linear programming cannot be applied to a wide variety of problems where values of the coefficients are probabilistic.

Constant parameters: Parameters appearing in LP are assumed to be constant, but in practical situations it is not so.

Divisibility: In linear programming, the decision variables are allowed to take non-negative integer as well as fractional values. However, we quite often face situations where the planning models contain integer valued variables. For instance, trucks in a fleet, generators in a powerhouse, pieces of equipment, investment alternatives and there are a myriad of other examples. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases, linear programming techniques cannot be used.



CONTD...

Example 4:

A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes one hours to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is Rs. 8 per bottle for A and Rs. 7 per bottle for B. Solve the problem to maximize the profit.



CONTD...

Solution:

Let production of each of A and B are x_1 unit bottles and x_2 unit bottles respectively. Since profits on each bottle of A and B are Rs. 8 and Rs. 7 per bottle respectively. So profit on x_1 bottles of A and x_2 bottles of B are $8x_1$ and $7x_2$ respectively.

Let z be total profit on bottles so, $z = 8x_1 + 7x_2$.

Since, it takes 3 hours and 1 hour to prepare enough material to fill 1000 bottles of Type A and Type B respectively. so x bottles of A and y bottles of B are preparing is $\frac{3x_1}{1000}$ hours and $\frac{x_2}{1000}$ hours respectively, but only 66 hours are available, so

$$\frac{3x_1}{1000} + \frac{x_2}{1000} \leq 66 \dots\dots\dots(\text{Time available constraint})$$

$$\text{or, } 3x_1 + x_2 \leq 66000.$$



CONTD...

Since raw material available to make 20000 bottles of A and 40000 bottles of B but there are 45000 bottles in which either of these medicines can be put so,

$$x_1 \leq 20000, \dots\dots\dots(\text{Bottle A constraint})$$

$$x_2 \leq 40000, \dots\dots\dots(\text{Bottle B constraint})$$

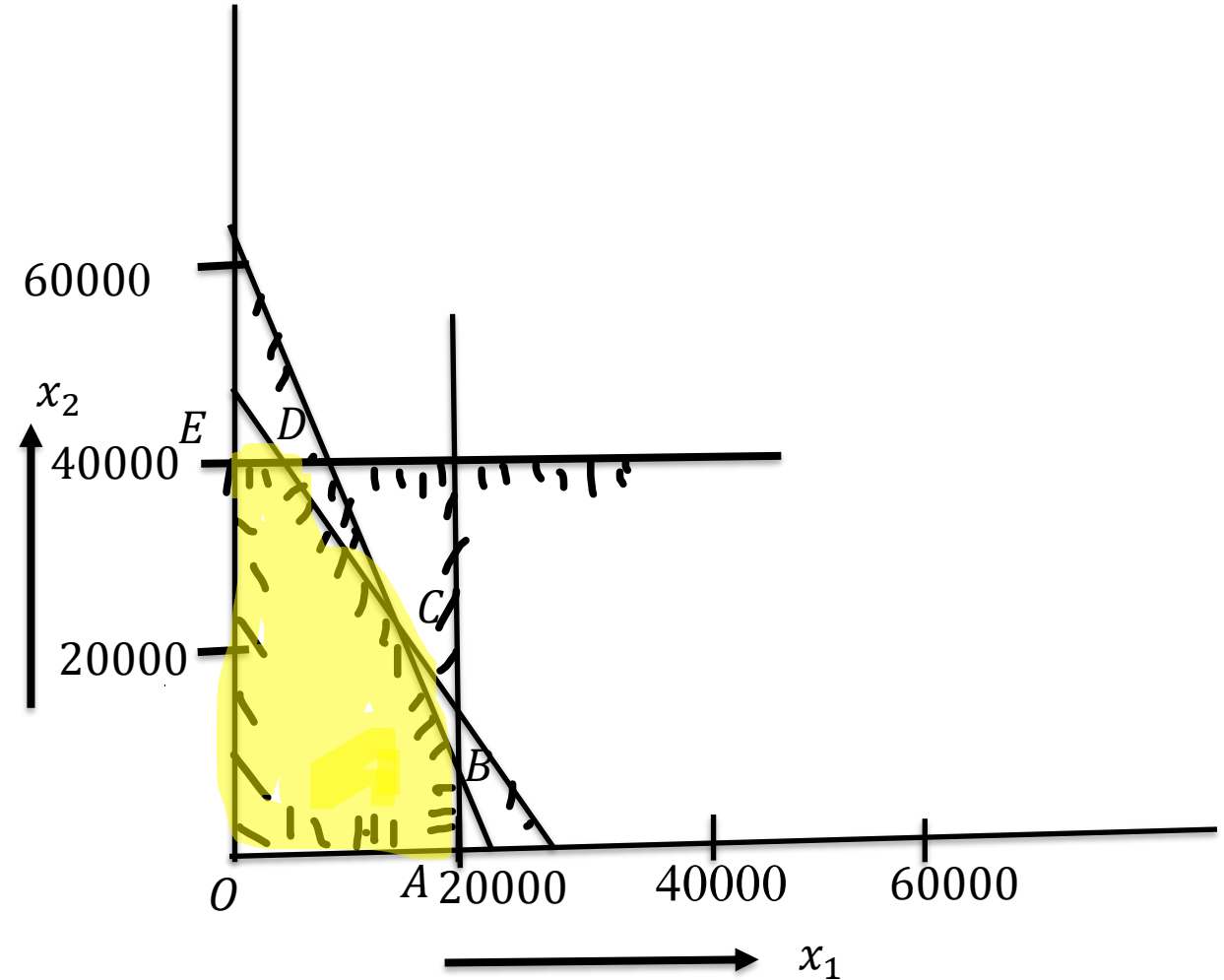
$$x_1 + x_2 \leq 45000 \dots\dots(\text{combined production constraint})$$

with $x_1, x_2 \geq 0$.

So mathematical formulation of the given LPP is $\text{Max } z = 8x_1 + 7x_2$

Subject to constraints

$$\begin{aligned} 3x_1 + x_2 &\leq 66000, \\ x_1 &\leq 20000, \quad x_2 \leq 40000, \\ x_1 + x_2 &\leq 45000 \\ x_1, x_2 &\geq 0. \end{aligned}$$





CONTD...

Here feasible region is $OABCDE$. The coordinates of $OABC$ are $O(0, 0)$, $A(20000, 0)$, $B(20000, 6000)$, $C(10500, 34500)$, $D(5000, 40000)$, $E(0, 40000)$.

Corner Points	Coordinate	Value of z
O	$(0, 0)$	0
A	$(20000, 0)$,	160000
B	$(20000, 6000)$	202000
C	$(10500, 34500)$	325500
D	$(5000, 40000)$	320000
E	$(0, 40000)$	280000

Therefore, Maximum value of z occurs at $C(10500, 34500)$. Hence, optimal solution is $x_1 = 10500$, $x_2 = 10500$ and max profit =Rs. 325500



Simplex Method for Solving Linear Programming Problem



Canonical form of L.P.P.

1. Max $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1, x_2, x_3 \dots x_n \geq 0.$$

2. Min $z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$

$$x_1, x_2, x_3 \dots x_n \geq 0.$$



Standard form of L.P.P.

$$\text{Max } z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

$$x_1, x_2, x_3 \dots x_n \geq 0.$$

The standard form of L.P.P. Should have following characteristics

1. Objective function should be of maximization type.
2. All the constraint should be expressed in equation form.
3. The right hand side of each constraints should be non-negative.



Procedure for changing into Standard form

1. The minimization of function z is equivalent to the maximization of negative expression of this function ie $-z$. Hence objective function $\text{Min } z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ is equivalent to $\text{Max } z^* = -z = -(c_1 x_1 + c_2 x_2 + \dots + c_n x_n)$
2. The inequality constraints are changed into equality constraints by adding or subtracting a nonnegative variable from left side of the constraint . These new variables are known as slack variable. These variable are added if the constraints are \leq and subtracted if the constraint are \geq . Since in case of \geq constraints, the subtracted variable represent the surplus of left hand side over right hand side, it is commonalty known as surplus variable in fact negative slack.
3. If R.H.S. of any constraints is negative, then it should be made non-negative by multiplying -1 on both side of the constraints.
4. Replace each unrestricted decision variable with difference of two non-negative variable.



Standard Form

Example: Express the following LPP in standard form

$$\text{Min } z = 2x_1 + x_2 + 4x_3$$

Subjected to constraints

$$\begin{aligned} -2x_1 + 4x_2 &\leq 4 \\ x_1 + 2x_2 + x_3 &\geq 5, \\ 2x_1 + 3x_2 &\leq 2, \end{aligned}$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Solution: Since x_3 is unrestricted. So suppose $x_3 = x_3' - x_3''$

Hence Standard form Is given by

$$\text{Max } z^* = -z = -(2x_1 + x_2 + 4x_3' - 4x_3'')$$

Subjected to constraints

$$\begin{aligned} -2x_1 + 4x_2 + s_1 &= 4 \\ x_1 + 2x_2 + x_3' - x_3'' - s_2 &= 5, \\ 2x_1 + 3x_2 + s_3 &= 2, \\ x_1, x_2, x_3', x_3'', s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Here s_1 and s_3 are the slacks variable and s_2 is surplus variable.



Standard Form

Example: Express the following LPP in standard form

$$\text{Max } z = 3x_1 + 2x_2 + 5x_3$$

Subjected to constraints

$$2x_1 + 3x_2 - 2x_3 \leq 40$$

$$4x_1 - 2x_2 + x_3 \leq 24,$$

$$x_1 - 5x_2 - 6x_3 \geq 2,$$

$$x_1 \geq 0.$$

Sol. Since x_2 and x_3 are unrestricted. So suppose $x_2 = x_2' - x_2''$ and $x_3 = x_3' - x_3''$.

Hence Standard form Is given by

$$\text{Max } z = 3x_1 + 2(x_2' - x_2'') + 5(x_3' - x_3'')$$

Subjected to constraints

$$2x_1 + 3(x_2' - x_2'') - 2(x_3' - x_3'') + s_1 = 40$$

$$4x_1 - 2(x_2' - x_2'') + (x_3' - x_3'') + s_2 = 24,$$

$$x_1 - 5(x_2' - x_2'') - 6(x_3' - x_3'') - s_3 = 2,$$

$$x_1, x_2', x_2'', x_3', x_3'', s_1, s_2, s_3 \geq 0.$$

Here s_1 and s_2 are the slack variable and s_3 is surplus variable.



BASIC AND BASIC FEASIBLE SOLUTIONS

Matrix form of L.P.P. (Standard Form)

$$\text{Maximize } z = c^T x$$

subjected to constraints

$$Ax = b, \quad x \geq 0.$$

Here

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ is a matrix of order } m \times n, c = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

$$\text{are the matrix of order } n \times 1, \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} \text{ is the matrix of order } m \times 1.$$



BASIC AND BASIC FEASIBLE SOLUTIONS

Here given system of equations $Ax = b$ has n unknowns and m equations ($m < n$).

Suppose A be $m \times n$ matrix of rank m . Let B be any $m \times m$ non-singular submatrix after deleting $(n - m)$ columns of A . Now equate zero to $(n - m)$ variables which are not associated with columns of B .

A basic solution is one in which $(n - m)$ variables not associated with columns of B equal to zero and solution obtained for the remaining m number variables.

The m variable which may be different from zero, are called basic variables and remaining variables are called non-basic variables.

$$\begin{aligned} Bx_B &= b, \\ x_B &= B^{-1}b \end{aligned}$$

Basic feasible Solution:

This is the basic solution that satisfies the non-negativity conditions.



NONDEGENERATE BASIC FEASIBLE SOLUTION AND OPTIMAL SOLUTION

Nondegenerate basic feasible solution:

This is a basic feasible solution that has got exactly m positive decision variables x_i .

Optimal solution:

A feasible solution that optimized the objective function is called an optimal solution



Example

Example: Find all basic solution of the following system of equation

$$\begin{aligned}x_1 + 2x_2 + x_3 &= 4 \\2x_1 + x_2 + 5x_3 &= 5\end{aligned}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$m = 2, n = 3$$

(i) $x_1 = 0$

$$\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$



Basic Solutions

(ii) $x_2 = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

(iii) $x_3 = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence

- All basic solutions are $\left[0, \frac{5}{3}, \frac{2}{3}\right]$, $[5, 0, -1]$ and $[2, 1, 0]$,
- All basic feasible solution are $\left[0, \frac{5}{3}, \frac{2}{3}\right]$, and $[2, 1, 0]$.
- All basic feasible solution are nondegenerate basic feasible solution.



Example:

Example: Find the basic solutions to the following problem

$$\text{Maximize } z = x_1 + 3x_2 + 3x_3$$

Subjected to constraint

$$x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 3x_2 + 5x_3 = 7.$$

Also find which of the basic solution are

- (a) Basic feasible solution
- (b) Non-degenerate basic feasible solution
- (c) Optimal basic feasible solution

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\text{Here } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$m = 2, n = 3$$



Basic Solutions

(i) $x_1 = 0$

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(ii) $x_2 = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(iii) $x_3 = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Basic Solutions

Hence basic solutions are $[0, -1, 2]$, $[1, 0, 1]$, $[2, 1, 0]$

(a) Basic feasible solutions are $[1, 0, 1]$, $[2, 1, 0]$

(b) Nondegenerate basic feasible solutions are $[1, 0, 1]$, $[2, 1, 0]$

(c) Values of objective function at basic feasible solutions are

(i) $x_1 = 1, x_2 = 0, x_3 = 1, \quad z = 4$

(ii) $x_1 = 2, x_2 = 1, x_3 = 0, \quad z = 5$

Hence optimal basic feasible solution is

$$x_1 = 2, x_2 = 1, x_3 = 0, \quad z = 5$$



How many numbers of basic solutions?

How many numbers of basic solutions?

$$n_{C_m} = \frac{n!}{n-m!m!}$$

The problem we have just solved has 3 basic solutions.

Now consider a problem of 10 variables and 8 equations, we will have 45 different basic solutions.

If a problem of 15 variables and 10 equations, we will have 3003 different basic solutions.

As such, it is not possible to find solutions for all the basic solutions. Moreover, many basic solutions, we may get infeasible solutions. As such we need some set of rules to switch from one feasible solution another feasible solution.

This rule may be known as Simplex method.



Simplex Method For solving LPP

We use this method for solving LPP where we use only slack variable in its standard form.

Example: Solve the following LP problem by simplex method

$$\text{Min } z = x_1 - 3x_2 + 2x_3$$

Subjected to constraints

$$\begin{aligned} 3x_1 - x_2 + 3x_3 &\leq 7, & -2x_1 + 4x_2 &\leq 12, \\ -4x_1 + 3x_2 + 8x_3 &\leq 10, & x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Solution:

Step 1: Write the given LPP in to standard form.

$$\text{Max } z^* = -z = -x_1 + 3x_2 - 2x_3$$

Subjected to constraints

$$\begin{aligned} 3x_1 - x_2 + 3x_3 + s_1 &= 7, \\ -2x_1 + 4x_2 + s_2 &= 12, \\ -4x_1 + 3x_2 + 8x_3 + s_3 &= 10, \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Here s_1, s_2 and s_3 are the slack variable.



Simplex Method

Step 2: Write down the coefficient of all the variables in the standard form of L.P.P. in tabular form, as shown in the table below. In order to find the initial basic feasible solution $x_B = B^{-1}b$.

Here B is the identity matrix. Since we have taken $x_1 = 0, x_2 = 0, x_3 = 0$ as non basic variable and s_1, s_2 and s_3 are the basic variable.

Hence $s_1 = 7, s_2 = 12$ and $s_3 = 10$ $x_B = \{s_1, s_2, s_3\}$



Write the given LPP in to standard form.

$$\text{Max } z^* = -z = -x_1 + 3x_2 - 2x_3$$

Subjected to constraints

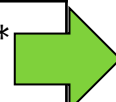
$$\begin{aligned} 3x_1 - x_2 + 3x_3 + s_1 &= 7, \\ -2x_1 + 4x_2 + s_2 &= 12, \\ -4x_1 + 3x_2 + 8x_3 + s_3 &= 10, \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Here s_1, s_2 and s_3 are the slack variable.



Simplex Method

C_j			-1	3	-2	0	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
s_1	0	7	3	-1	3	1	0	0	<u> </u>
s_2	0	12	-2	4	0	0	1	0	3*
s_3	0	10	-4	3	8	0	0	1	10/3
$Z_j - C_j$			1	-3*	2	0	0	0	



Key Row



Key Column

- $Z_j - C_j = \sum C_B x_i - C_j$. Here C_B denotes coefficient of basic variable in the objective function.



Simplex Method

Step 3: (Condition for optimality) If all the $z_j - c_j \geq 0$, then this basic feasible will be optimal. We shall always get an identity matrix corresponding to basic variable and $z_j - c_j$ will be zero corresponding to basic variable.

Step 4: (Select the variable to enter the basis) If $z_j - c_j \leq 0$ for any non basic variable, this indicate an improvement in the value of the objective is possible. Now non basic variable for which $z_j - c_j$ is most negative will enter in to new solution and column corresponding to this variable is called key column or pivot column.



Simplex Method

Step 5: Test the feasibility (Variable leave the basis)

After identifying variable to become basic variable, one variable to be removed from existing set of basic variable is to be determined.

For this each number in x_B column is divided by the corresponding (positive) number in key column and a row is selected for which this ratio is nonnegative and minimum.

$$\frac{x_{B_r}}{a_{rj}} = \min \left\{ \frac{x_{B_i}}{a_{ij}}, a_{ij} > 0 \right\}$$

It may be noted that division by zero or negative number is not permitted.

Note: If all element of key column is zero or negative solution will be unbounded.

The row selected in this manner is called key row or pivot row and it will represent the variable which will leave the solution.

The element that lies at the intersection key row and key column is called key element or pivot element.



Simplex Method

Step 6: (Find new solution)

- (i) If the key element is 1, then row remain same in the new simplex table.
- (ii) If key element is other than 1, then divide each element in the key row (including the element in x_B –column) by the key element to new values in that row.
- (iii) The new values of the elements in remaining rows of new simplex table can be obtained by performing elementary rows operations on all rows so that all element except key element in key column are zero.

The new entries in C_B (coefficient of basic variable) and x_B (value of basic variable) columns are updated in new simplex table of current solution.

Step 7: (Repeat the procedure) Go to step 3 and repeat the procedure until all entries in $z_j - c_j$ row is either positive or zero.



Simplex Method

C_j			-1	3	-2	0	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
s_1	0	7	3	-1	3	1	0	0	
s_2	0	<u>12</u>	-2	4	0	0	1	0	<u>3*</u>
s_3	0	10	-4	3	8	0	0	1	10/3
$Z_j - C_j$			1	<u>-3*</u>	2	0	0	0	

Key Row

Key Column

Key element

$$R_2 \rightarrow \frac{R_2}{4}, R_1 \rightarrow R_1 + \frac{1}{4}R_2, R_3 \rightarrow R_3 - \frac{3}{4}R_2$$



Simplex Method

C_j			-1	3	-2	0	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
s_1	0	10	5/2	0	3	1	1/4	0	20/5 =4
x_2	3	3	-1/2	1	0	0	1/4	0	-
s_3	0	1	-5/2	0	8	0	-3/4	1	-
$Z_j - C_j$			-1/2	0	2	0	3/4	0	

Key
Row

Key Column

Key element

$$R_1 \rightarrow \frac{2}{5}R_1, R_2 \rightarrow R_2 + \frac{1}{5}R_1, R_3 \rightarrow R_3 + R_1$$



FINAL TABLE OF SIMPLEX METHOD

C_j			-1	3	-2	0	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
x_1	-1	4	1	0	6/5	2/5	1/10	0	-
x_2	3	5	0	1	3/5	1/5	3/10	0	-
s_3	0	11	0	0	11	1	-1/2	1	-
$Z_j - C_j$			0	0	13/5	1/5	4/5	0	

Since $z_j - c_j \geq 0$ for all variables.

Therefore optimal solution reached with

$$x_1 = 4, x_2 = 5, \max z = -4 + 15 = 11$$

Hence $\text{Min } z = -z^* = -11$



Simplex Method For solving LPP

Example: Solve the following LP problem by simplex method

$$\text{Max } z = 4x_1 + 5x_2$$

Subjected to constraints

$$\begin{aligned} 2x_1 + 3x_2 &\leq 12, \\ 2x_1 + x_2 &\leq 8, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: Write the given LPP in to standard form.

$$\text{Max } z = 4x_1 + 5x_2 + 0.s_1 + 0.s_2$$

Subjected to constraints

$$\begin{aligned} 2x_1 + 3x_2 + s_1 &= 12, \\ 2x_1 + x_2 + s_2 &= 8, \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

Here s_1 and s_2 are the slack variable.



Simplex Method

C_j			4	5	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2	Min Ratio
s_1	0	12	2	3	1	0	$12/3 = 4$
s_2	0	8	2	1	0	1	$8/1 = 8$
$Z_j - C_j$			-4	-5*	0	0	

Key Row

Key Column

Key element

$$R_1 \rightarrow \frac{R_1}{3}, R_2 \rightarrow R_2 - \frac{1}{3}R_1$$



Simplex Method

C_j			4	5	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2	Min Ratio
x_2	5	4	2/3	1	1/3	0	12/2 = 6
s_2	0	4	4/3	0	-1/3	1	12/4 = 3
$Z_j - C_j$			-2/3	0	5/3	0	

Key Row

Key Column

Key element

$$R_2 \rightarrow \frac{3}{4} R_2, R_1 \rightarrow R_1 - \frac{1}{2} R_2$$



Simplex Method

C_j			4	5	0	0	
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2	Min Ratio
x_2	5	2	0	1	1/2	-1/2	-
x_1	4	3	1	0	-1/4	3/4	-
$Z_j - C_j$			0	0	3/2	1/2	

Since $z_j - c_j \geq 0$ for all variables.

Therefore optimal solution reached with

$$x_1 = 2, x_2 = 3$$

Hence Max $z = 5 \times 2 + 4 \times 3 = 22$.



Dual Simplex Method for Solving Linear Programming Problem



Dual Simplex Method for Solving LPP

We use this method for solving LPP where initial solution is optimal but infeasible ie this method is used when value of basic variables are negative and optimality condition is satisfied. Hence

$z_j - c_j \geq 0$ (for optimal) and value of basic variable ≤ 0 , for using this method.

Computational Procedure of Dual Simplex Algorithm :

Step 1: (1) Write all the constraint in the form of inequality of \leq sign

(2) Now write the given LPP in to standard form.

To write the given LPP in to standard form,

(1) If problem is minimization type, convert it in to maximization type.

(2) Introduce slack variable in the constraint to reduce them into equality form.

Step 2: Find initial basic solution

To find the initial basic solution suppose all the given variable equal to zero and calculate the slack variable which may not be feasible.



Dual Simplex Method

Step 3: Construct the simplex table as usual in the simplex method.

Step 4: To test the initial solution optimality compute $z_j - c_j = C_B x_i - c_j$ for each x_i column.

- (1) If all the $z_j - c_j \geq 0$ and x_{B_i} are non-negative, the solution is optimal basic feasible solution.
- (2) If all the $z_j - c_j \geq 0$ and at least one x_{B_i} is negative, then proceed for the next step.
- (3) If any $z_j - c_j < 0$, the method is not possible.

Step 5: (1) Determine the outgoing vector: Row corresponding to most negative basic variable is known as key row.



Dual Simplex Method

(2) Determine the incoming vector:

$$\text{minimum ratio} = \left\{ -\frac{z_j - c_j}{-ve \text{ value of key row}} \right\}$$

If all the element of key row is $+ve$ or zero then we shall get infeasible solution. Column corresponding to minimum ratio is key column and corresponding variable will be incoming variable.

Mark the element which is intersection of key row and key column. This element is known as key element.

(3) Make key element unity, carry out the row operation as Regular simplex method.

Step 6: If all the basic variable all positive then this solution is optimal basic feasible solution but at least one basic variable is $-ve$ then solution is not optimal F.S. In this case repeat the step (4) and (5) till an optimal solution achieved.



Dual Simplex Method

Example: Solve the following LP problem by Dual Simplex method

$$\text{Min } z = 3x_1 + x_2$$

Subjected to constraints

$$\begin{aligned} x_1 + x_2 &\geq \underline{1}, \\ 2x_1 + 3x_2 &\geq \underline{2}, \quad x_1, x_2 \geq 0. \end{aligned}$$

Solution:

Step 1: Write the given LPP in to standard form.

$$\text{Max } z^* = -z = -3x_1 - x_2$$

Subjected to constraints

$$\begin{aligned} -x_1 - x_2 &\leq -1, \\ -2x_1 - 3x_2 &\leq -2, \\ x_1, x_2, &\geq 0. \end{aligned}$$

Introduce slack variable s_1, s_2 to reduce the constraint in to equality form

$$\begin{aligned} -x_1 - x_2 + s_1 &= -1, \\ -2x_1 - 3x_2 + s_2 &= -2, \end{aligned}$$



Dual Simplex Method

Take $x_1 = x_2 = 0$, we get $s_1 = -1, s_2 = -2$, which is the initial basic infeasible solution. Now simplex table is given by

$$\begin{aligned} \text{Max } z^* &= -z = -3x_1 - x_2 \\ -x_1 - x_2 + s_1 &= -1, \\ -2x_1 - 3x_2 + s_2 &= -2, \end{aligned}$$



Dual Simplex Method

C_j			-3	-1	0	0
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2
s_1	0	-1	-1	-1	1	0
s_2	0	-2	-2	-3	0	1
$Z_j - C_j$			3	1	0	0
<i>Minimum Ratio</i>			$-\frac{3}{-2} = \frac{3}{2}$	$-\frac{1}{-3} = \frac{1}{3}$	0	0

Key Row

Key Column

Key element

Here $z_j - c_j \geq 0$ and $X_B \leq 0$. Hence initial basic feasible is infeasible optimal.



Dual Simplex Method

Now since $s_2 = -2$ is most $-ve$ value. So it will be ongoing variable.

Now *minimum ratio* = $\left\{ -\frac{z_j - c_j}{-ve \text{ value of key row}} \right\} = \left\{ -\frac{3}{-2}, -\frac{1}{-3} \right\} = \left\{ \frac{3}{2}, \frac{1}{3} \right\} = \frac{1}{3}$,
which is correspond to x_2 column.

Hence now x_2 will work as basic variable.

$$R_2 \rightarrow -\frac{1}{3}R_2, \quad R_1 \rightarrow R_1 - \frac{1}{3}R_3$$



Dual Simplex Method

C_j			-3	-1	0	0
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2
s_1^*	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1	$-\frac{1}{3}$
x_2	-1	$\frac{2}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{3}$
$Z_j - C_j$			$\frac{7}{3}$	0	0	$\frac{1}{3}$
Min ratio			7	-	0	1

Key Row

Key element

Key Column

$$R_1 \rightarrow -3R_1, \quad R_2 \rightarrow R_2 - R_1$$



Dual Simplex Method

c_j			-3	-1	0	0
Basic variable in the basis	C_B	X_B	x_1	x_2	s_1	s_2
s_2	0	1	1	0	-3	1
x_2	-1	1	1	1	-1	0
$z_j - c_j$			2	0	1	0

Since $z_j - c_j \geq 0$ and $X_B \geq 0$. Hence this solution is optimal solution. Hence $x_1 = 0, x_2 = 1$, $\text{Max } z = -3 \times 0 - 1 \times 1 = -1$. So $\text{Min } z = -(-1) = 1$



Assignment Work

Use dual simplex method to solve following problem:

1. Min $z = x_1 + 2x_2$ subject to $2x_1 + x_2 \geq 4$, $x_1 + 7x_2 \geq 0$, $x_1, x_2 \geq 0$.
2. Min $z = 2x_1 + x_2$ subjected to $2x_1 + x_2 \geq 3$, $4x_1 + 3x_2 \geq 3$, $x_1, x_2 \geq 0$.

Use simplex method to solve following problem.