Diffraction

UNIT III

Optics

Lecture-4

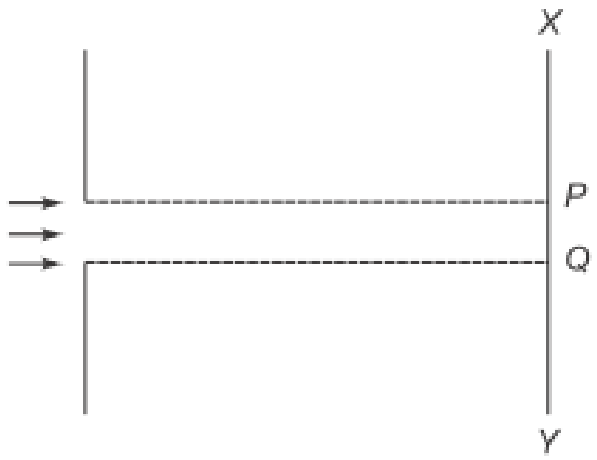
Content of Lecture

- DIFFRACTION OF LIGHT
- RESULTANT OF n SIMPLE HARMONIC WAVES
- FRAUNHOFER DIFFRACTION AT A SINGLE SLIT
- DIRECTIONS OF MAXIMA AND MINIMA

DIFFRACTION

- According to geometrical optics when a plane wave is incident on a long narrow slit the region PQ of the screen XY is illuminated and the remaining portion (known as the geometrical shadow) is found absolutely dark.

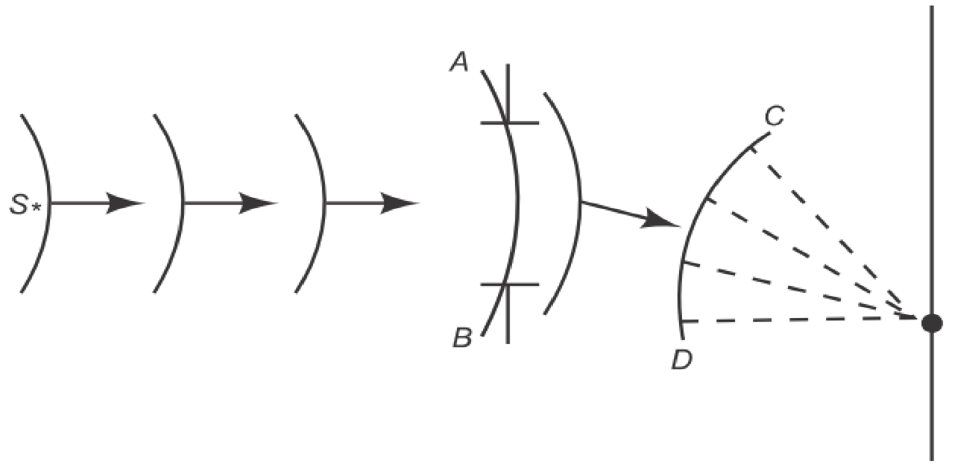
- However, if the width of the slit is not very large compared to the wavelength, then the light intensity in the region PQ is not uniform and there is also some intensity inside the geometrical shadow



Class of Diffraction

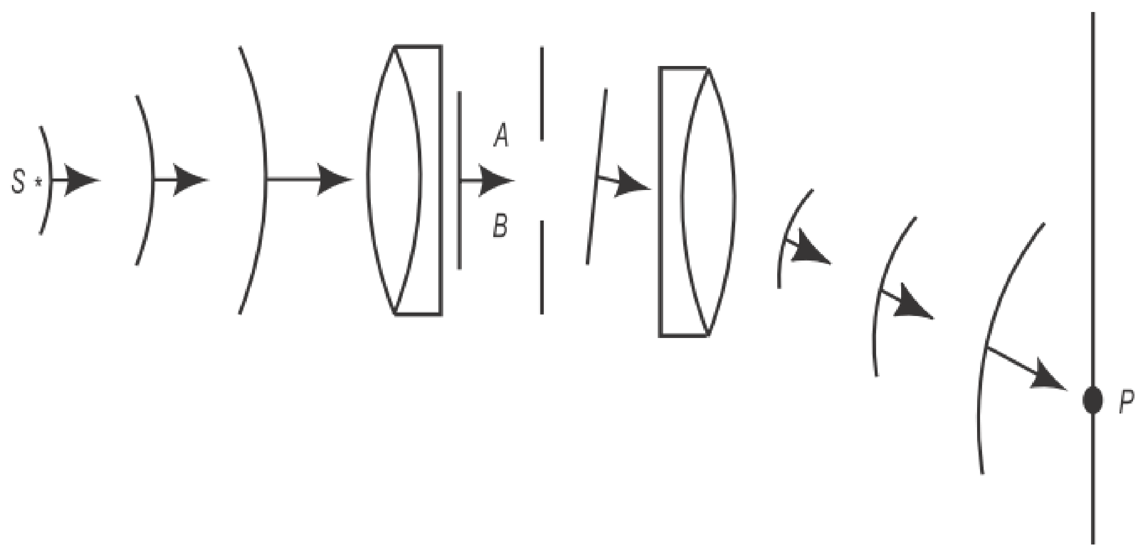
- The diffraction phenomena are usually divided into two categories:
- (i) Fresnel diffraction and
- (ii) Fraunhofer diffraction.

➢ **In the Fresnel class of diffraction, the source of light or screen on which diffraction pattern is observed, or usually both, is at finite distance from the diffraction obstacle or aperture.**



Fraunhofer Class of Diffraction

- In the Fraunhofer class of diffraction, the source of light and the screen are effectively at infinite distances from the diffracting obstacle. This is achieved by placing the source and the screen in the focal planes of the two lenses.



RESULTANT OF n SIMPLE HARMONIC WAVES

- Consider n simple harmonic vibrations having equal amplitude a and common phase difference δ between successive vibrations.

- Component of resultant amplitude can be given as

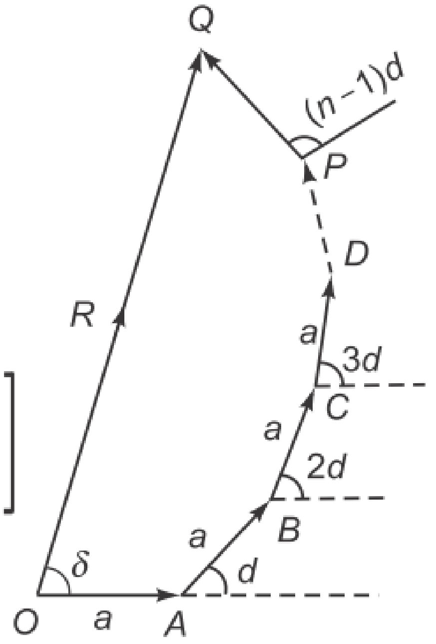
$$R \cos \delta = a[1 + \cos d + \cos 2d + \ldots + \cos (n-1)d]$$

$$R \sin \delta = a[0 + \sin d + \sin 2d + \ldots + \sin (n-1)d]$$

Multiplying Eq. (13.1) by $2 \sin (d/2)$, we get

$$2R \cos \delta \sin \frac{d}{2} = a\left[2 \sin \frac{d}{2} + 2 \cos d \sin \frac{d}{2} + 2 \cos 2d \sin \frac{d}{2} + \ldots + 2 \cos (n-1) d \sin \frac{d}{2}\right]$$

Using formula $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$, we have



RESULTANT OF n SIMPLE HARMONIC WAVES CONTD…

$$2R\cos\delta\sin\frac{d}{2} = a\left[2\sin\frac{d}{2}+\left(\sin\frac{3d}{2}-\sin\frac{d}{2}\right)+\left(\sin\frac{5d}{2}-\sin\frac{3d}{2}\right)\right.$$

$$\left.+\cdots+\left(\sin\left\{n-\frac{1}{2}\right\}d-\sin\left\{n-\frac{3}{2}\right\}d\right)\right]$$

$$= a\left[\sin\frac{d}{2}+\sin\left(n-\frac{1}{2}\right)d\right]$$

$$= 2\,a\sin\frac{nd}{2}\cos\frac{(1-n)d}{2}$$

$$= 2\,a\sin\frac{nd}{2}\cos\frac{(n-1)d}{2} \qquad \{\because \cos(-\theta)=\cos\theta\}$$

$$\left(\text{using formula } \sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right)$$

RESULTANT OF n SIMPLE HARMONIC WAVES CONTD…

or $$R\cos\delta = \frac{a\sin(nd/2)}{\sin(d/2)}\cos\frac{(n-1)d}{2}$$

Similarly, multiplying Eq. (13.2) by 2 sin $(d/2)$ and simplifying, we get

$$R\sin\delta = \frac{a\sin(nd/2)}{\sin(d/2)}\sin\frac{(n-1)d}{2}$$

Squaring and adding Eqs. (13.3) and (13.4), we get

$$R^2 = \frac{a^2\sin^2(nd/2)}{\sin^2(d/2)}$$

or $$R = \frac{a\sin(nd/2)}{\sin(d/2)}$$

Dividing Eq. (13.4) by Eq. (13.3), we get

$$\tan\delta = \tan\frac{(n-1)d}{2}$$

or $$\delta = \frac{(n-1)d}{2}$$

RESULTANT OF n SIMPLE HARMONIC WAVES CONTD…

n is the number of rays and hence infinitely large, while a and d are infinitesimally small.

Let $\quad \dfrac{nd}{2} = \alpha$ (finite quantity)

Then from Eq. (13.5),

$$R = \frac{a\sin\alpha}{\sin(\alpha/n)}$$

$$= \frac{a\sin\alpha}{\alpha/n} \qquad \left(\because \frac{\alpha}{n} \text{ is very small, } \therefore \sin\left(\frac{\alpha}{n}\right) = \frac{\alpha}{n}\right)$$

$$= \frac{na\sin\alpha}{\alpha}$$

na is still finite. Let $na = A$, which gives

$$R = \frac{A\sin\alpha}{\alpha}$$

Also, Eq. (13.6) will give

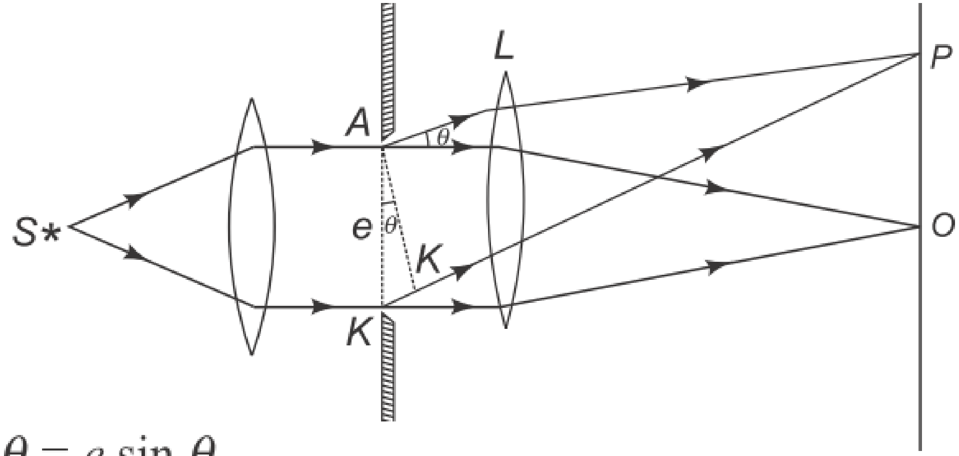
$$\delta = \frac{(n-1)d}{2} = \frac{nd}{2} \qquad \text{(as } n \text{ is very large)}$$

$$\delta = \alpha$$

FRAUNHOFER DIFFRACTION AT A SINGLE SLIT

- Let a parallel beam of monochromatic light of wavelength λ be incident normally upon a slit having width e. The diffraction pattern obtained on the screen consists of a central bright band, having alternate dark and bright bands of decreasing intensity on both sides.

The path difference between the wavelets from A and B in the direction θ is



$$BK = AB \sin\theta = e \sin\theta$$

The corresponding phase difference $= \dfrac{2\pi}{\lambda} e \sin\theta.$

Analysis

- Let the width of the slit AB be divided into n equal parts, and the amplitude of each wave coming out from these n parts be a. Then the phase difference between the waves from any two consecutive parts is

$$d = \frac{1}{n}\left[\frac{2\pi}{\lambda} e \sin\theta\right]$$

Hence, the resultant amplitude at P will be

$$R = \frac{a \sin(nd/2)}{\sin(d/2)} = \frac{a \sin(\pi e \sin\theta/\lambda)}{\sin(\pi e \sin\theta/n\lambda)}$$

Let $\pi e \sin\theta/\lambda = \alpha$, then

Analysis

$$R = \frac{a \sin \alpha}{\sin(\alpha/n)} = \frac{a \sin \alpha}{\alpha / n} \qquad [\because \alpha/n \text{ is very small}]$$

$$= \frac{n a \sin \alpha}{\alpha}$$

As $n \to \infty$, $a \to 0$, but the product na remains finite.

Let $na = A$, then

$$R = \frac{A \sin \alpha}{\alpha}$$

Therefore, resultant intensity at P is

$$I = R^2 = \frac{A^2 \sin^2 \alpha}{\alpha^2}$$

Directions of Maxima and Minima

For intensity to be maximum, the value of (sin α/α) should also be maximum

Here, $\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$ (using L′ Hospital's rule)

$\therefore$ $\alpha = 0$ corresponds to maxima

or $$\frac{\pi e \sin \theta}{\lambda} = 0$$

or $$\sin \theta = 0$$

or $$\theta = 0$$

The intensity will be minimum when

$$\sin \alpha = 0 \qquad (\text{but } \alpha \neq 0)$$

or $$\alpha = \pm m\pi \begin{pmatrix} m \neq 0 \\ m = 1, 2, 3, \ldots \end{pmatrix}$$

or $$\frac{\pi e \sin \theta}{\lambda} = \pm m\pi$$

$$e \sin \theta = \pm m\lambda$$

Analysis

Now it is obvious that due to these minima some other maxima called secondary or subsidiary maxima will also be present in the pattern, whose intensity distribution can be analysed by putting

$$\frac{dI}{d\alpha} = 0$$

or $$\frac{d}{d\alpha}\left[A^2 \frac{\sin^2\alpha}{\alpha^2}\right] = 0$$

or $$A^2\left[\frac{\alpha^2 2\sin\alpha\cos\alpha - 2\alpha\sin^2\alpha}{\alpha^4}\right] = 0$$

or $$\alpha\cos\alpha - \sin\alpha = 0$$

or $$\alpha - \tan\alpha = 0$$

Analysis

- This equation is solved graphically by plotting the curves y = α and y = tan α.

These values of α are approximately given out as α = 3π/2, 5π/2, 7π/2…..

$$I_0 = A^2\left(\frac{\sin 0}{0}\right)^2 \approx A^2$$

$$I_1 \approx A^2\left\{\frac{\sin(3\pi/2)}{3\pi/2}\right\} = \frac{4A^2}{9\pi^2} \approx \frac{A^2}{22}$$

$$I_2 \approx A^2\left\{\frac{\sin(5\pi/2)}{5\pi/2}\right\} = \frac{4A^2}{25\pi^2} \cong \frac{A^2}{61}$$

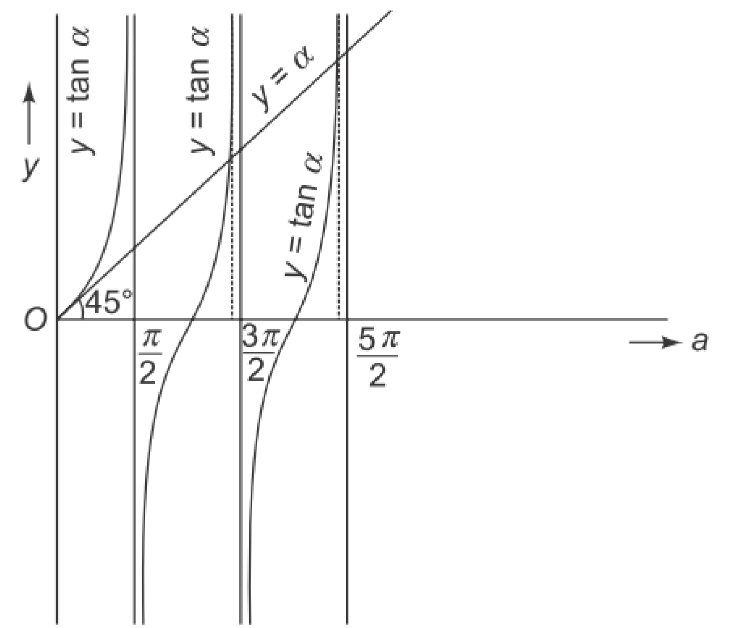


Fig. 13.5 The plot of y = α and y = tan α

Analysis

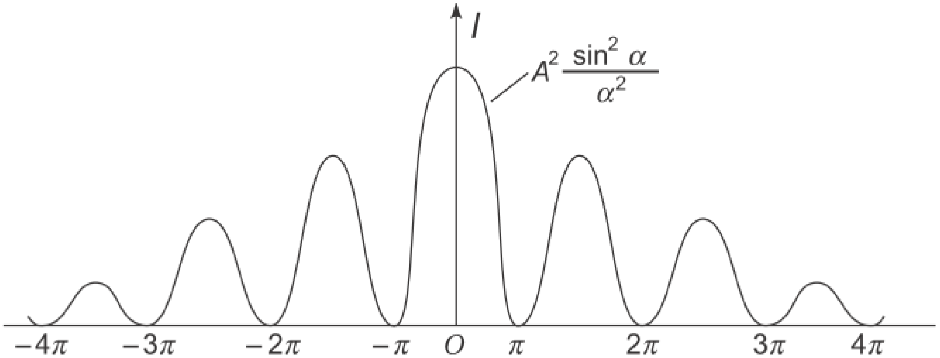
Thus, the intensities of the successive maxima are in the ratio

$$1:\frac{4}{9\pi^2}:\frac{4}{25\pi^2}:\frac{4}{49\pi^2}:\cdots$$

or

$$1:\frac{1}{22}:\frac{1}{61}:\frac{1}{121}:\cdots$$

Thus, the diffraction pattern consists of a bright principal maximum in the direction of incident light having alternate minima and weak subsidiary maxima of rapidly decreasing intensity on either sides of it. The intensity pattern is shown in Fig.



Example-1 Light of wavelength 5000 Å is incident normally on a single slit. The central maximum spreads out at 30° on both sides of the direction of incident light. Calculate the width of the slit.

Solution

The direction of minima in Fraunhofer diffraction due to single slit is given by

$$\alpha = \pm m\pi$$

or $$\frac{\pi}{\lambda} e \sin\theta = \pm m\pi \qquad \text{where } m = 1, 2, 3, \ldots$$

Therefore, the angular spread of the central maximum on either side of the incident light is

$$\sin\theta = \frac{\lambda}{e} \qquad \text{(for } m = 1\text{, the position of first minimum)}$$

Given $\theta = 30°$ and $\lambda = 5000$ Å $= 5 \times 10^{-5}$ cm.

$$\sin\theta = \frac{\lambda}{e}$$

On putting the given values, we have

$$\sin 30° = \frac{5 \times 10^{-5}}{e}$$

$$\Rightarrow \qquad e = \frac{5 \times 10^{-5}}{\sin 30°} = \frac{5 \times 10^{-5}}{(1/2)}$$

$$= 2 \times 5 \times 10^{-5}$$

$$= 10 \times 10^{-5}$$

$$= 10^{-4} \text{ cm}$$

Therefore, the width of the slit is 10^{-4} cm.

Example-2 A single slit is illuminated by light composed of two wavelengths λ_1 and λ_2. One observes that due to Fraunhofer diffraction, the first minimum obtained for λ_1 coincides with the second diffraction minimum of λ_2. What will be the relation between λ_1 and λ_2?

Solution

In Fraunhofer diffraction due to single slit, the directions of minima are given by

$$e \sin\theta = \pm m\lambda$$

For wavelength λ_1 the position of first minimum ($m = 1$) is

$$e \sin\theta_1 = \lambda_1 \qquad (1)$$

For wavelength λ_2, the position of second minima ($m = 2$) is

$$e \sin\theta_2 = 2\lambda_2$$

It is given that the direction of first minimum (θ_1) due to λ_1 coincides with the second minimum due to λ_2, i.e.,

$$\theta_1 = \theta_2 = \theta$$

Now from Eqs. (1) and (2), we have

$$e \sin\theta = \lambda_1 = 2\lambda_2$$

$$\lambda_1 = 2\lambda_2$$

Assignment Based on this Lecture

- Discuss the phenomena Diffraction of Light.
- Obtain the expression of Resultant of n-Harmonic waves.
- Discuss the Fraunhofer Diffraction at a Single Slit. Also obtain the expression of Subsidiary Maxima and Minima.
- Discuss the phenomenon of Fraunhofer diffraction at a single slit and show that the relative intensities of the successive maxima are nearly

$$1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2} : \cdots$$

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