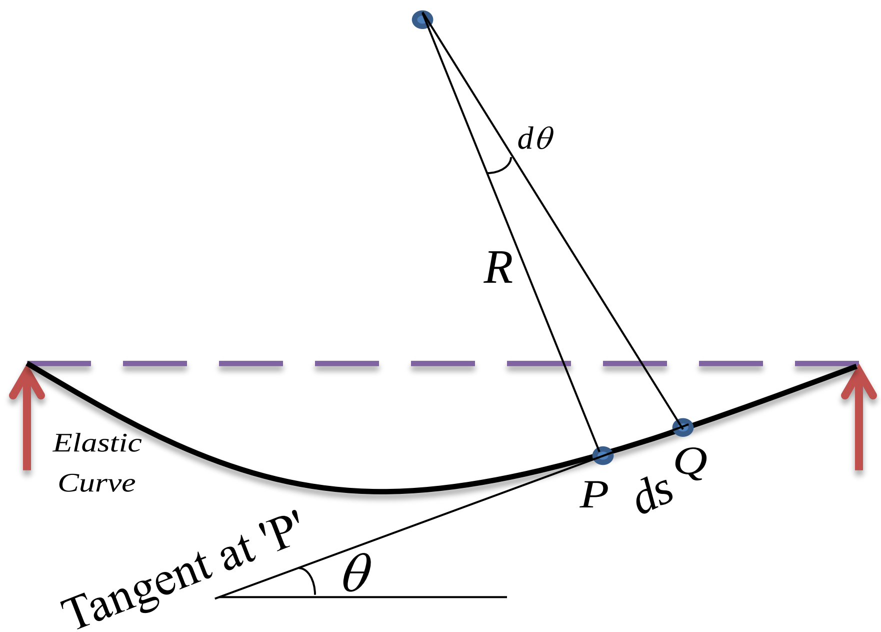
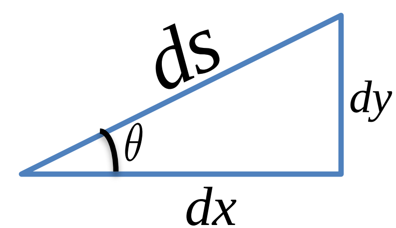
DEFLECTION OF BEAM

Beam Differential Equation





Where dx and dy represent the projected lengths of the segment ds along X and Y axes.

$$Angle = \frac{Arc}{Radius}$$

$$d\theta = \frac{ds}{R} \text{.................................}(1)$$

$$\tan\theta = \frac{dy}{dx} \text{.................................}(2)$$

$$differenting\left(w.r.t - x\right)$$

$$\sec^2\theta.\frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\sec^2\theta.\frac{ds}{R}.\frac{1}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{\sec^3\theta}{R} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(1+\tan^2\theta\right)^{3/2}}{R}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{R} = \frac{M}{EI}$$

$$EI\,\frac{d^2 y}{dx^2} = M$$

Note :-

1) The above equation is the governing differential equation of the beam.

2) we only take the effect of bending moment. The effect of shear on the deflection is extremely small and usually neglected.

3) EI is an index which is known as flexural strength of an element.

Boundary Condition for Beams

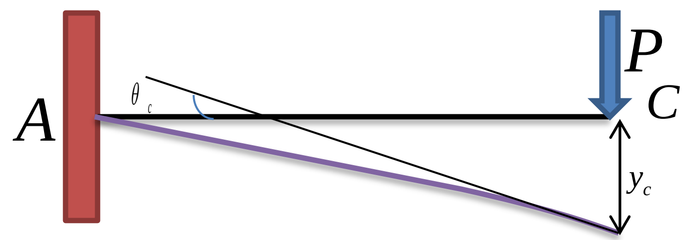
Case (1) Cantilever beam

Fixed

End

$$y = y_A = 0$$

$$\theta = \theta_A = 0$$



Transverse

Deflection

Free

End

$$y = y_C = Maximum$$

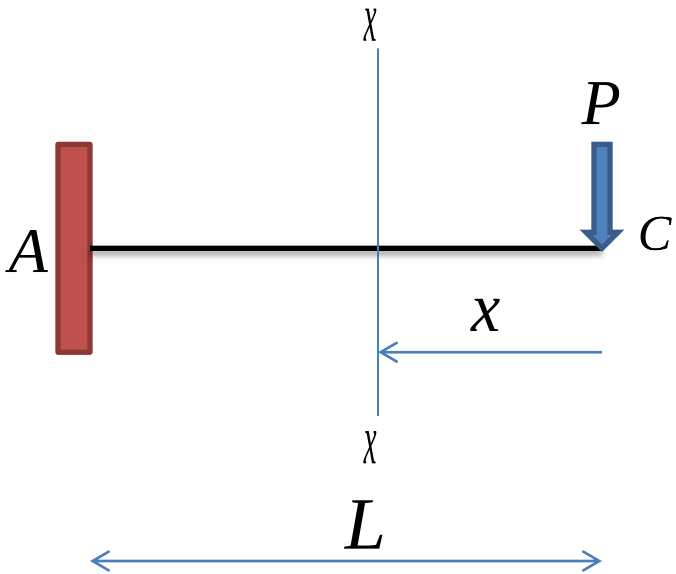
$$\theta = \theta_c = Maximum$$

y_c - Transverse deflection

θ_c - Angular deflection of beam (or) slope of beam

Ankit Saxena (saxena01ankit@gmail.com)

Cantilever beam concentrated load at free end



$$EI\frac{d^2y}{dx^2} = M_{x-x}$$

$$M_{x-x} = -Px$$

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating w.r.t 'x'

$$EI\frac{dy}{dx} = \frac{-Px^2}{2} + C_1$$

$$at, x = L$$

$$\frac{dy}{dx} = 0$$

$$0 = \frac{-PL^2}{2} + C_1$$

$$C = \frac{PL^2}{2}$$

$$EI\frac{dy}{dx} = \frac{-Px^2}{2} + \frac{PL^2}{2}$$

Again integrating w.r.t 'x'

$$EI.y = \frac{-Px^3}{6} + \frac{PL^2x}{2} + C_2$$

$$at, x = L$$

$$y = 0$$

$$0 = \frac{-PL^3}{6} + \frac{PL^3}{2} + C_2$$

$$C_2 = \frac{-PL^3}{3}$$

$$EI.y = \frac{-Px^3}{6} + \frac{PL^2x}{2} - \frac{PL^3}{3}$$

Ankit Saxena (saxena01ankit@gmail.com)

Note

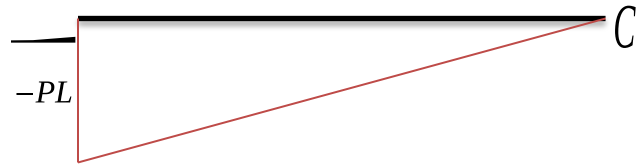
$$EI.y = \frac{-Px^3}{6} + \frac{PL^2x}{2} - \frac{PL^3}{3}$$

$$y_{\max} = y_C\Big)_{x=0} = -\frac{PL^3}{3EI}$$

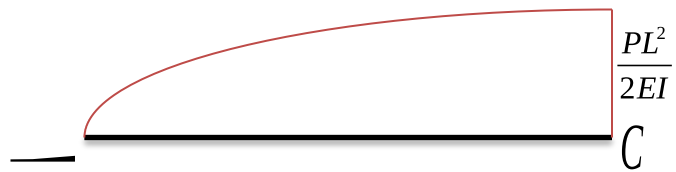
$$\theta_{\max} = \theta_C\Big)_{x=0} = \left(\frac{dy}{dx}\right)_{Max} = \frac{PL^2}{2EI}$$

1. The magnitude of the slope curve is slope of deflection curve.
2. The slope of slope curve is magnitude of bending moment.

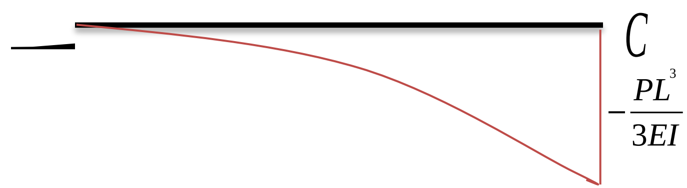
BMD



Slope Curve



Deflection Curve



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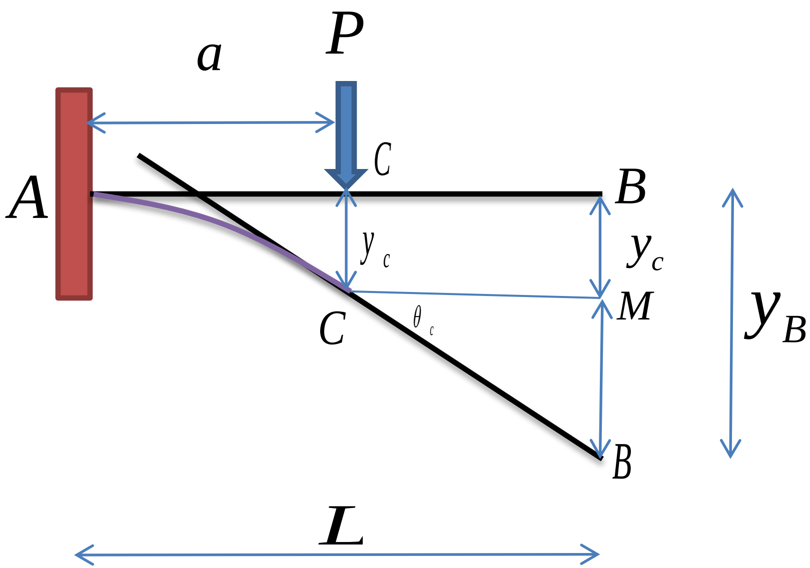
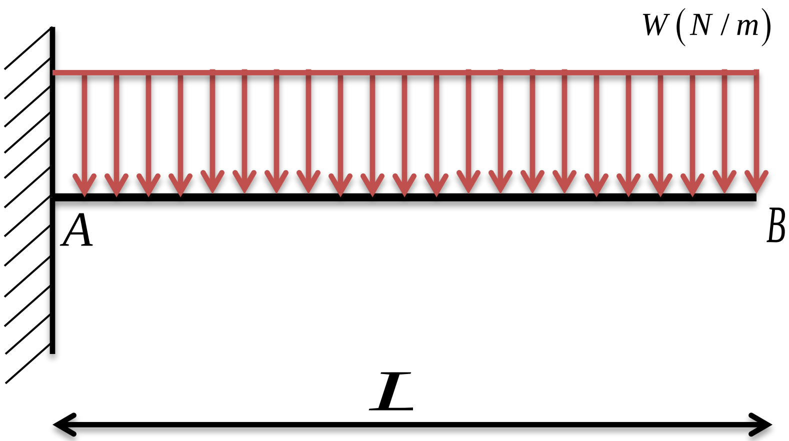
Case (2) Cantilever Beam concentrated Load not at free end



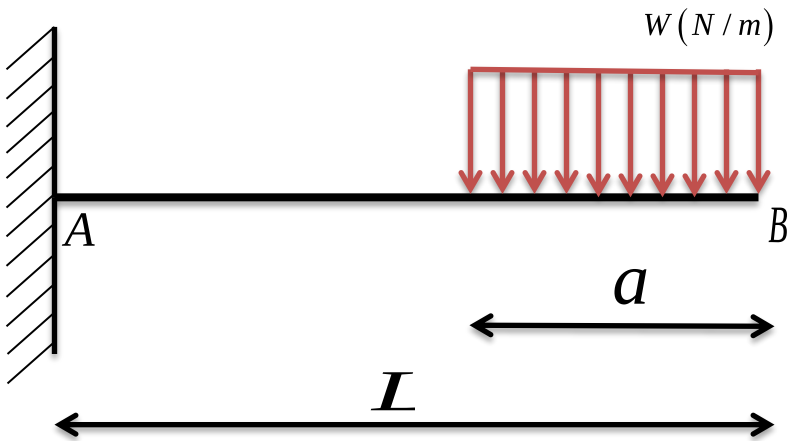
Case (3) Cantilever beam subjected to uniformly distributed load on whole span length



$$y_{\max} = y_B\Big)_{x=0} = -\frac{WL^4}{8EI}$$

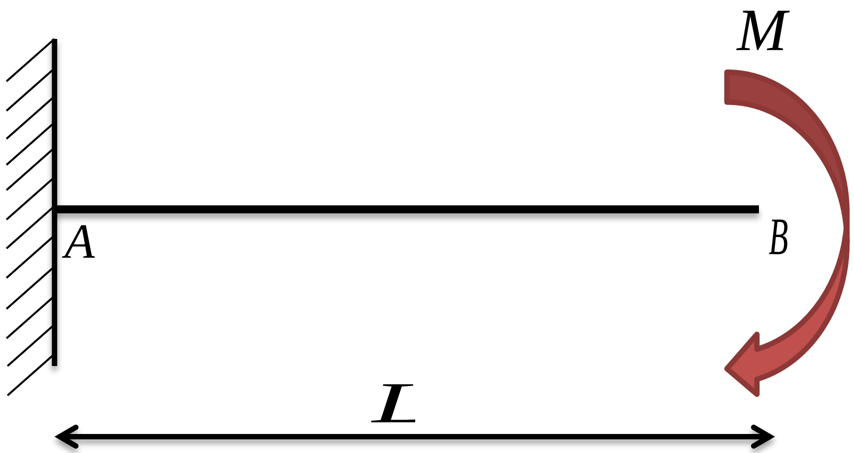
$$\theta_{\max} = \theta_B\Big)_{x=0} = \left(\frac{dy}{dx}\right)_{Max} = \frac{WL^3}{6EI}$$

Case (4) Cantilever beam subjected to uniformly distributed load on a part of span length



$$y = \frac{-WL^4}{8EI} + \frac{W(L-a)^4}{8EI} + \frac{W(L-a)^3 .a}{6EI}$$

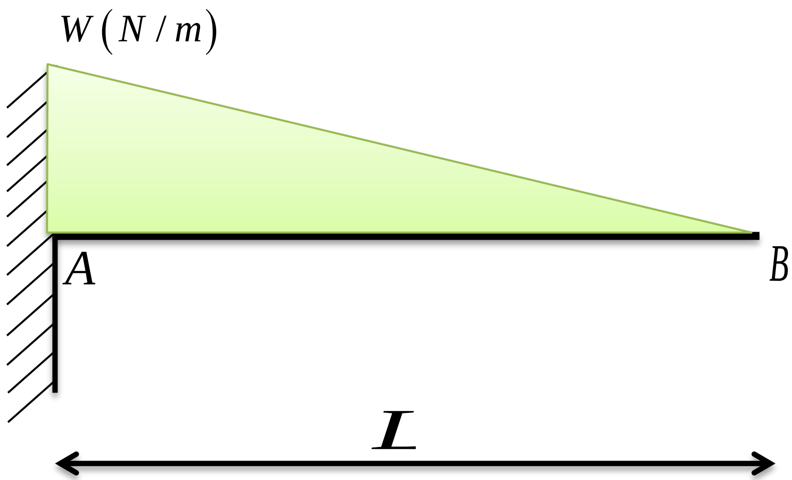
Case (5) Cantilever beam subjected to a couple at the free end.



$$y_{\max} = y_B\big)_{x=0} = \frac{-ML^2}{2EI}$$

$$\theta_{\max} = \theta_B\big)_{x=0} = \left(\frac{dy}{dx}\right)_{Max} = \frac{ML}{EI}$$

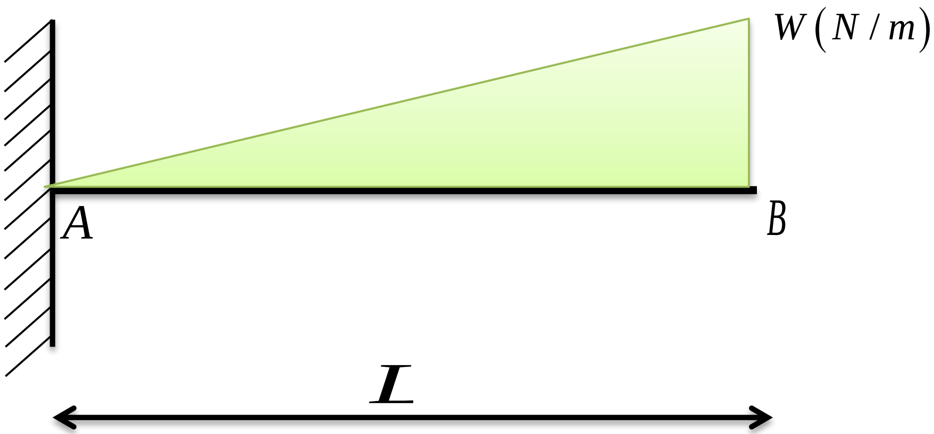
Case (6) Cantilever beam subjected to a uniformly varying load having zero intensity at the free end.



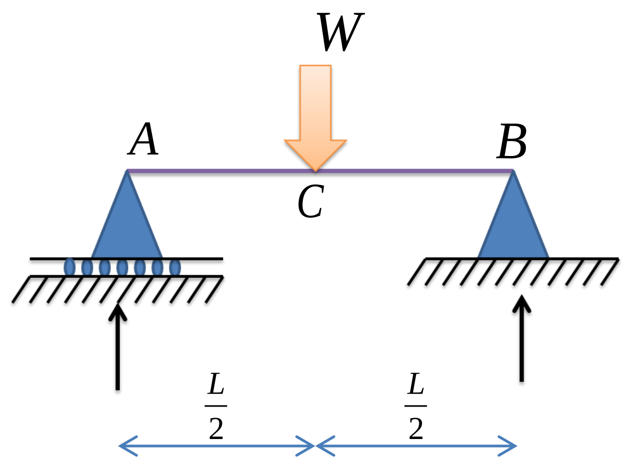
$$y_{\max} = y_B\big)_{x=0} = \frac{-WL^4}{30EI}$$

$$\theta_{\max} = \theta_B\big)_{x=0} = \left(\frac{dy}{dx}\right)_{Max} = \frac{WL^3}{24EI}$$

Case (7) Cantilever beam subjected to a uniformly varying load having zero intensity at the fixed end.



Simply Supported Beam



Symmetry in Bending moment diagram

1. Maximum deflection occurs at the centre or mid point of the beam axis.

2. At the mid point, point C [slope = 0]

$$x = \frac{L}{2}$$

$$y = y_{Max}$$

$$\theta = \frac{dy}{dx} = 0$$

3. At supports (A & B)

$$y = 0$$

$$\theta = \frac{dy}{dx} = \theta_{Max}$$

No symmetry in Bending moment diagram

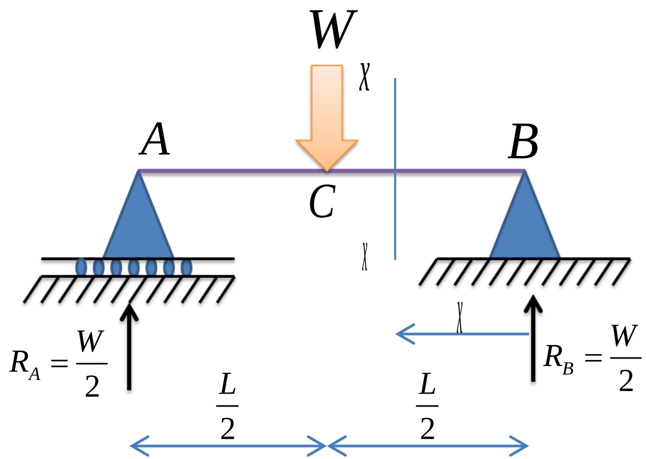
1. Maximum deflection occurs in a region between point of application of load and mid point.
2. The maximum slope is occurred at that support which is nearer to the line of action of force.

Case 1. Simply Supported beam subjected to a point load at mid span

Bending Equation for the section BC

$$M_{x-x} = \frac{Wx}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wx}{2}$$



On Integration w.r.t 'x'

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again integrating w.r.t 'x'

$$EI.y = \frac{Wx^3}{12} + C_1x + C_2 \ldots\ldots\ldots\ldots (2)$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$EI.y = \frac{Wx^3}{12} + C_1x + C_2 \ldots\ldots\ldots\ldots(2)$$

Apply boundary condition, At

$$x = \frac{L}{2}, \frac{dy}{dx} = 0$$

$$\&$$

$$x = 0 / L, y = 0$$

From equation (2)

$$C_2 = 0$$

From equation (1)

$$C_1 = \frac{-WL^2}{16}$$

Substitute the value of C_1 & C_2 in equation (1) & (2)

$$EI\frac{dy}{dx}=\frac{Wx^2}{4}+\frac{-WL^2}{16}\text{.....................}(3)$$

$$EI.y=\frac{Wx^3}{12}+\frac{-WL^2x}{16}\text{...................}(4)$$

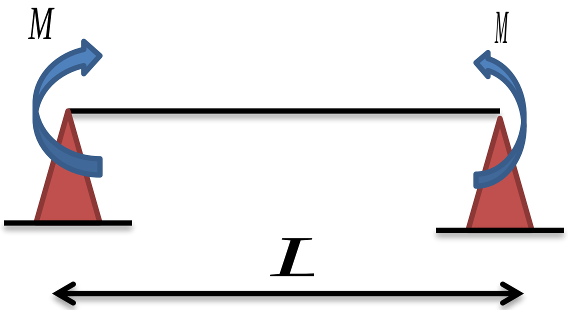
For maximum deflection and slope

$$y_{\max}=y_C\Big)_{x=\frac{L}{2}}=\frac{WL^3}{48EI}$$

$$\theta_{\max}=\theta_B\Big)_{x=0}=\theta_A\Big)_{x=L}=\left(\frac{dy}{dx}\right)_{Max}=\pm\frac{WL^2}{16EI}$$

Home Work

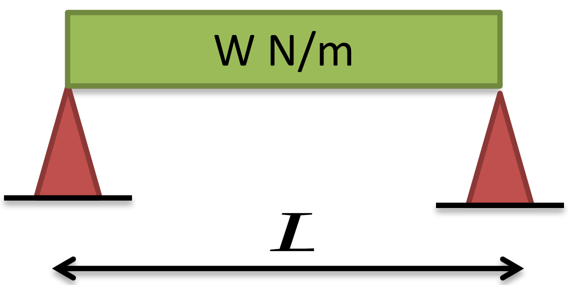
Case (1) S.S.B subjected to point moment at its both ends.



$$y_{\max} = y_C\Big)_{x=\frac{L}{2}} = \frac{ML^2}{8EI}$$

$$\theta_{\max} = \theta_B\Big)_{x=0} = \theta_A\Big)_{x=L} = \left(\frac{dy}{dx}\right)_{Max} = \pm\frac{ML}{2EI}$$

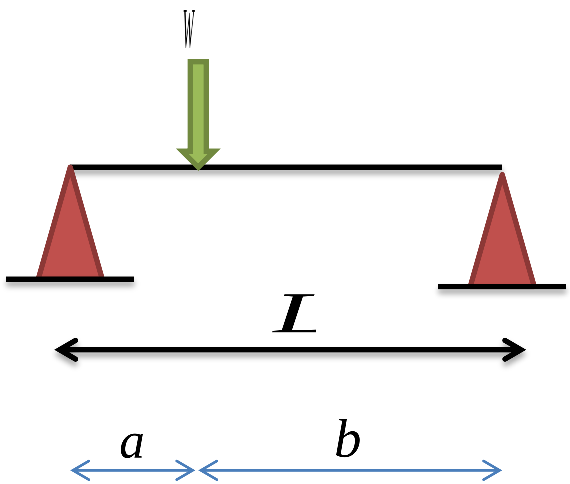
Case (2) S.S.B subjected to UDL (Uniformly distributed load) over it length.



$$y_{\max} = y_C\Big)_{x=\frac{L}{2}} = \frac{5WL^4}{384EI}$$

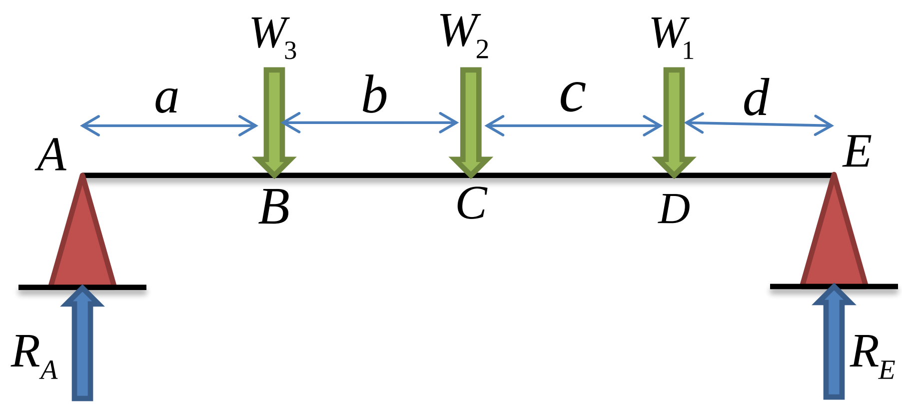
$$\theta_{\max} = \theta_B\Big)_{x=0} = \theta_A\Big)_{x=L} = \left(\frac{dy}{dx}\right)_{Max} = \pm\frac{WL^3}{24EI}$$

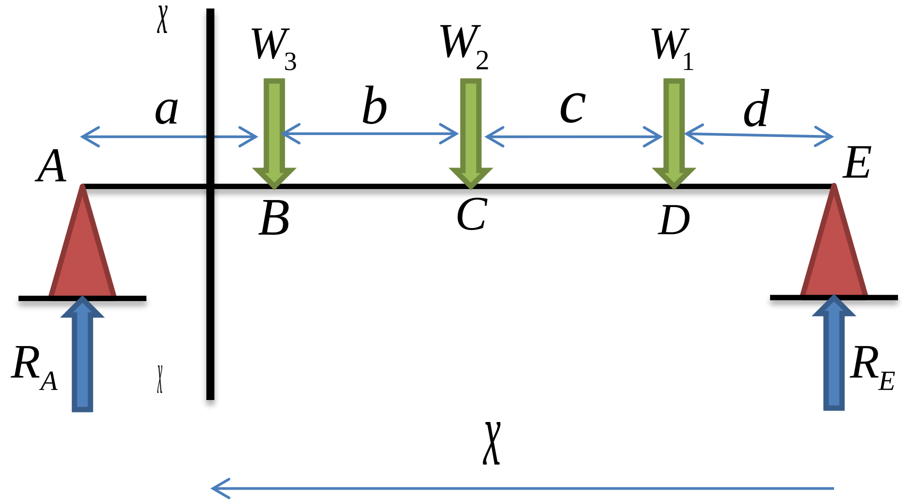
Case (2) S.S.B subjected to Eccentric concentrated load



Macaulay's Method

1. In Macaulay's method, a single equation is written for the bending moment for all the portions of the beam.
2. Same integration constants of integration are applicable for all portions.





$$M_{x-x} = R_E.x\big|_{DE} - W_1.(x-d)\big|_{CD} - W_2(x-c-d)\big|_{BC} - W_3(x-b-c-d)\big|_{AB}$$

<u>Note</u>

In the bending moment equation by substituting any value of x, if the form in the **bracket** become negative, delete that term completely.

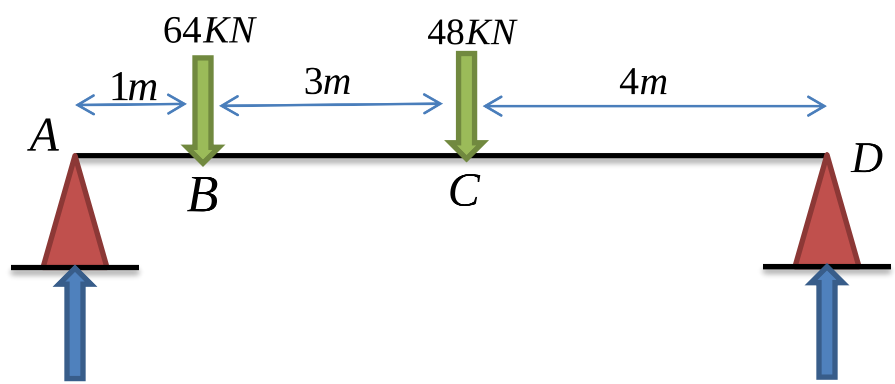
$$EI\frac{d^2y}{dx^2} = R_E.x\Big|_{DE} - W_1.\left(x-d\right)\Big|_{CD} - W_2\left(x-c-d\right)\Big|_{BC} - W_3\left(x-b-c-d\right)\Big|_{AB}$$

$$EI\frac{dy}{dx} = R_E.\frac{x^2}{2} + C_1\Bigg|_{DE} - W_1.\frac{\left(x-d\right)^2}{2}\Bigg|_{CD} - W_2\frac{\left(x-c-d\right)^2}{2}\Bigg|_{BC} - W_3\frac{\left(x-b-c-d\right)^2}{2}\Bigg|_{AB}$$

$$EI.y = R_E.\frac{x^3}{6} + C_1x + C_2\Bigg|_{DE} - W_1.\frac{\left(x-d\right)^3}{6}\Bigg|_{CD} - W_2\frac{\left(x-c-d\right)^3}{6}\Bigg|_{BC} - W_3\frac{\left(x-b-c-d\right)^3}{6}\Bigg|_{AB}$$

Q. A simply Supported beam carry 2 point loads 64 KN & 48 KN at B and C points. Find the deflection under each load.

Given E = 210 GPa and I = 180 * 10^6 mm^4.



Area Moment Method (Mohr's Theorem)

Statement [1]

The difference between the slope of any two point is equal to the area of (M/EI) diagram.

Statement [2]

The difference between the deflection of any two point is equal to the moment of area of (M/EI) diagram.

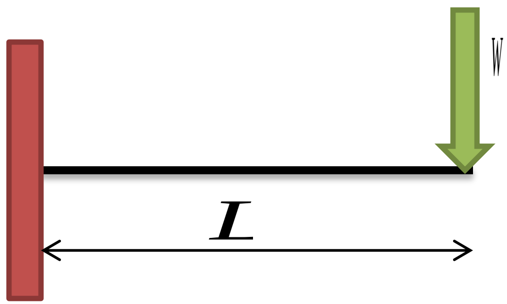
Note

1. For the two points, one point should be point of zero slope and other point should be point of non-zero slope.
2. The point of zero slope is known as reference point and point of non zero slope is known as origin point.
3. As per statement [2]

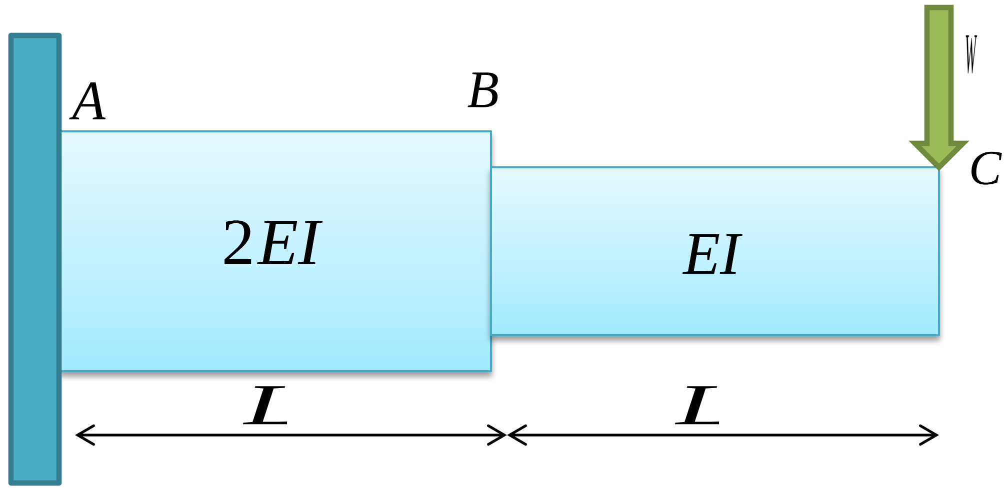
Difference b/w deflection of any two point = moment of area of [M/EI] diagram

= A * x of [M/EI] diagram

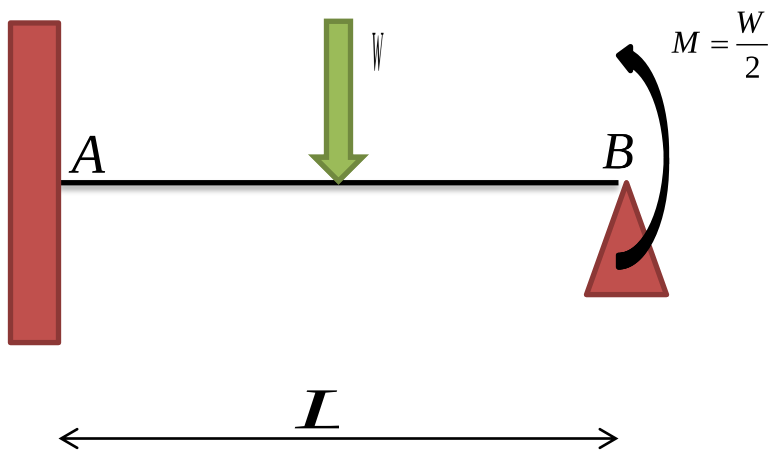
x = it is the distance b/w the centroid of area and the point of non zero slope point or origin point.


W
L

Q. For the cantilever beam as known in the figure determine the maximum slope and deflection. For section AC the flexural rigidity is 2EI and for CB section is EI.



Q. For the given propped cantilever beam as shown in the figure, determine the support reaction at the simple support B. W load is acting at mid point of the given beam.

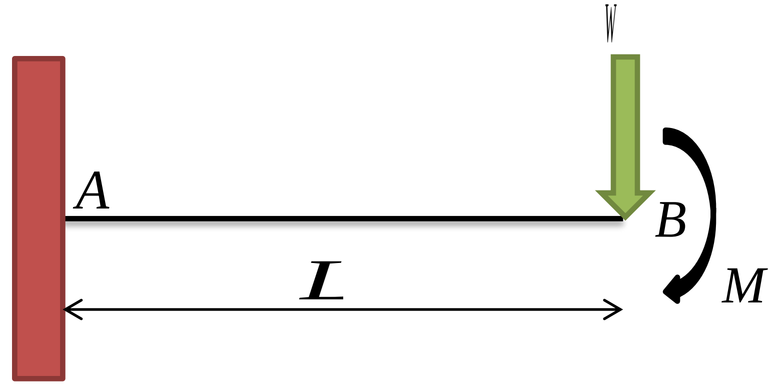


Castigliano's Theorem

If a structure is subjected to a number of external loads (or couples), the partial derivative of the total strain energy with respect with respect to any load (or couple) provides the deflection in the direction of that load (or couple).

U = strain energy [S.E] of beam due to bending moment

$$S.E = U = \int_0^L \frac{\left(M_{x-x}\right)^2 dx}{2\left(EI\right)_{x-x}}$$



Statement [1]

$$y_B = \frac{d}{dW}(S.E)$$

Statement [2]

$$\theta_B = \frac{d}{dM}(S.E)$$

Note - W is concentrated point load and M is the concentrated point moment.

Note

1. For calculating deflection at a point if there is no point load at that point, introduce a dummy point load at that point and do the complete calculation.
2. In the final step, put the dummy load value equal to the zero.

Gate [2014] / 2 Marks

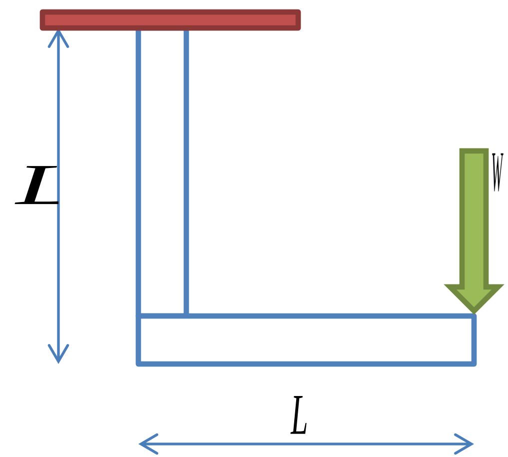
Q. A frame is subjected to a load P as shown in the figure. The frame has a constant flexural rigidity EI. The effect of axial load is neglected. The deflection at point A due to the applied load P is -

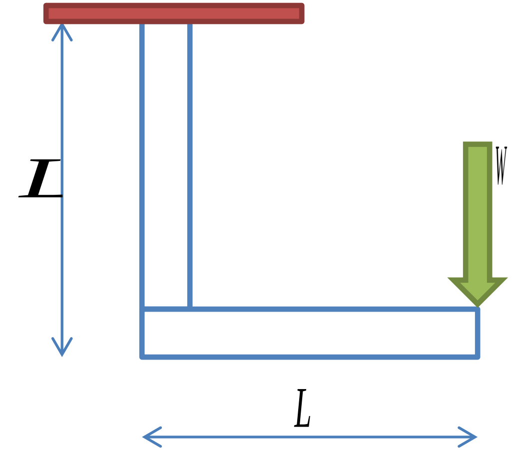
$(a)\ \frac{1}{3}\frac{PL^3}{EI}$

$(b)\ \frac{2}{3}\frac{PL^3}{EI}$

$(c)\ \frac{PL^4}{EI}$

$(d)\ \frac{4}{3}\frac{PL^3}{EI}$

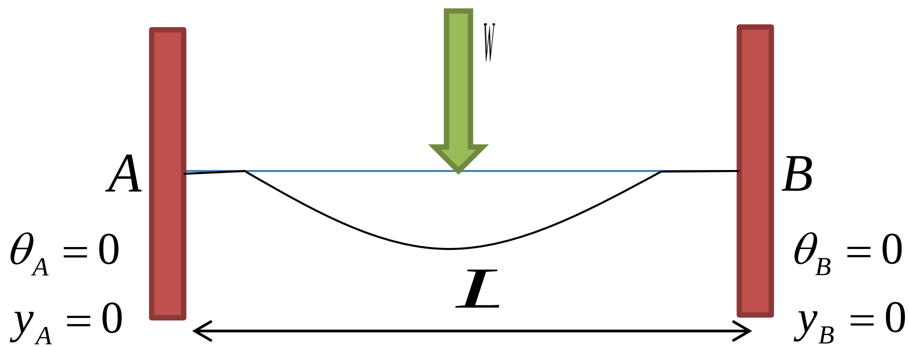



L
W
L

FIXED BEAM

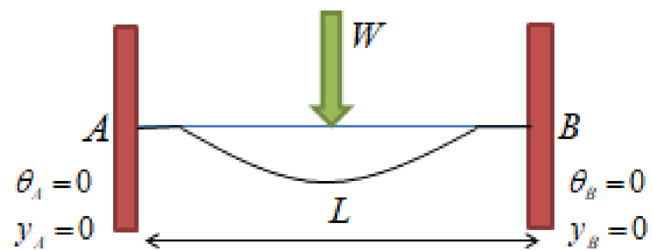
Fixed Beam

Symmetrical Loading

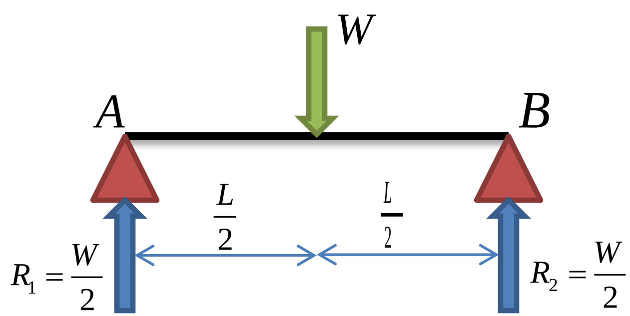


Case 1. Symmetrical loading on fixed beam

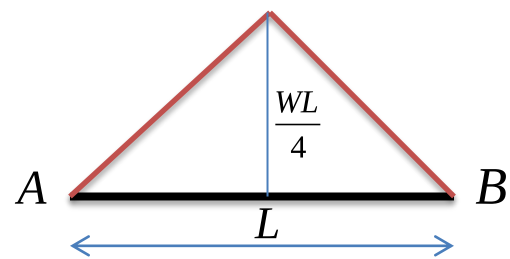
(A fixed beam having a concentrated point load at mid span)



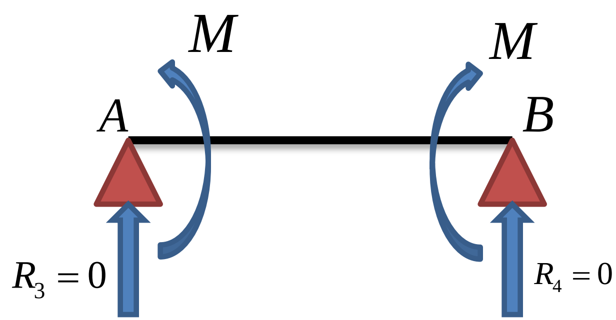
Step (1) A fixed beam is considered as a simply beam with the given loading condition.



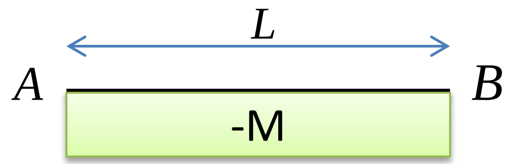
Step (2) Draw the free moment diagram.



Step (3) A fixed beam is considered as a simply supported beam having fixing moments at both the ends. Since it is a case of symmetrical loading so equal fixing moments are required at both ends.



Step (4) Draw the fixing moment diagram.



$\theta_B - \theta_A =$ area of (M/EI) Diagram = 0

Area of free bending moment diagram = area of fixing bending moment diagram

$$\frac{L}{2} \cdot \frac{WL}{4} = M.L$$

$$M = \frac{WL}{8}$$

Step (5) Maximum deflection will be at point C.

$y_C - y_A =$ moment of area of (M/EI) diagram about any end support.

$y_C - y_A =$ { moment of area of [M/EI] diagram for free moment diagram about any support} - { moment of area of [M/EI] diagram for fixing moment diagram about any support.

$$y_C - y_A = A_1 x_1 - A_2 x_2$$

A_1 area between A and C in free moment diagram.

x_1 Distance between centroid of area between A and C in free moment diagram to fixed support A.

A_2 area between A and C in fixing moment diagram.

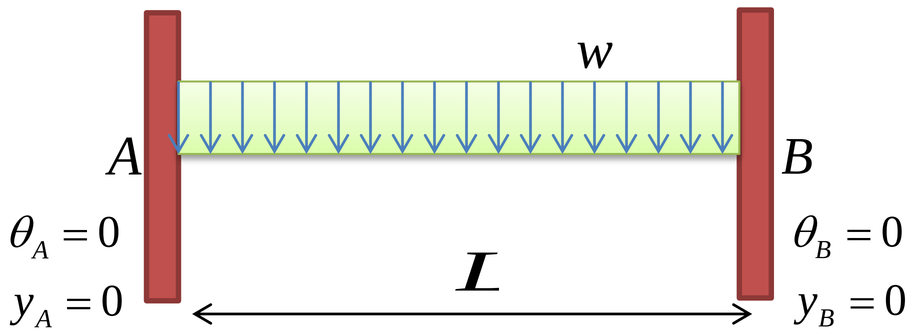
x_2 Distance between centroid of area between A and C in fixing moment diagram to fixed support A.

$$y = \frac{1}{EI}\left[\left(\frac{1}{2}\cdot\frac{WL}{4}\cdot\frac{L}{2}\right)\left(\frac{2}{3}\cdot\frac{L}{2}\right) - \left(\frac{WL}{8}\cdot\frac{L}{2}\right)\cdot\frac{L}{4}\right]$$

$$y = \frac{WL^3}{192EI}$$

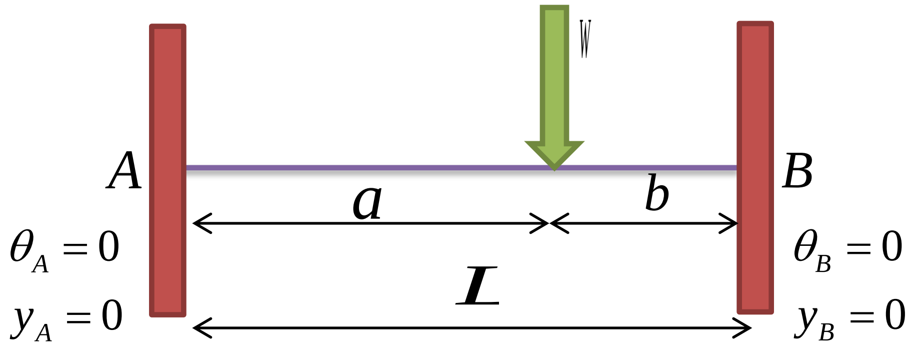
Ankit Saxena (saxena01ankit@gmail.com)

Q. Determine the maximum bending moment and the deflection of a beam of length L and flexural rigidity EI. The beam is fixed horizontally at both ends and carries a uniformly distributed load w over the entire length.

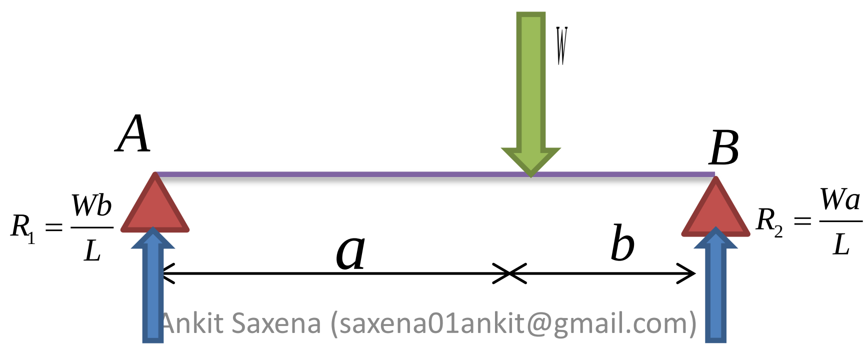


$$y = \frac{wL^4}{384EI}$$

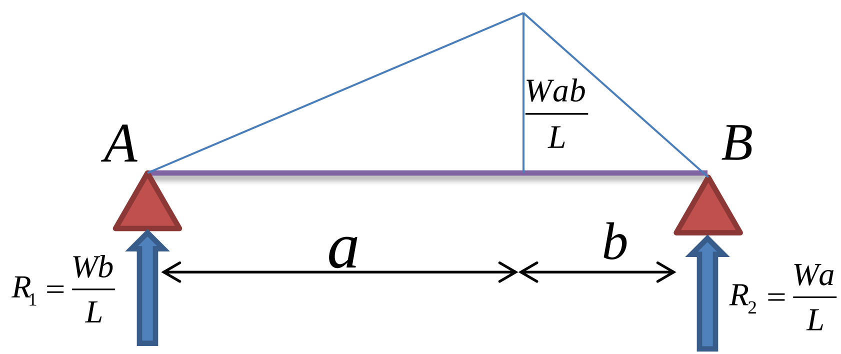
Unsymmetrical Loading



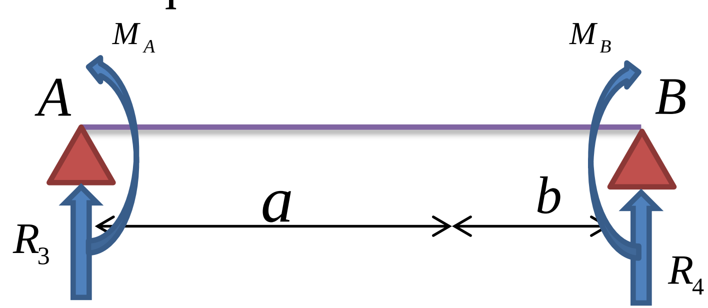
Step (1) A fixed beam is considered as a simply beam with the given loading condition.



Step (2) Draw free moment diagram.



Step (3) A fixed beam is considered as a simply supported beam having fixing moments at both the ends. Since it is a case of unsymmetrical loading so unequal fixing moments are required at both ends.



$$R_3 + R_4 = 0$$

$$\sum M_A = 0$$

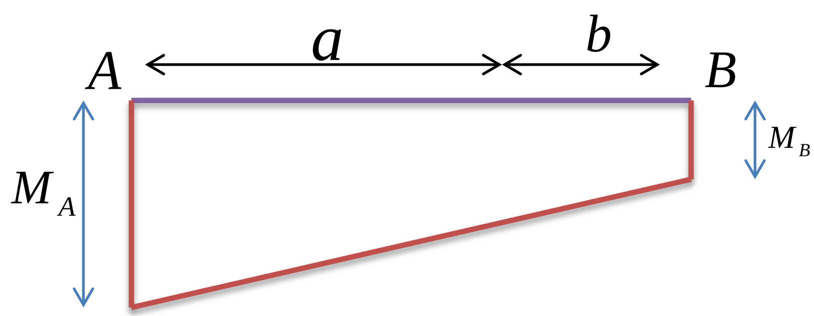
$$M_B - R_4 l - M_A = 0$$

$$R_4 = \frac{(M_B - M_A)}{L}$$

$$R_3 = \frac{(M_A - M_B)}{L}$$

Step (4) Draw the fixing moment diagram.

Let,



area of (M/EI) Diagram = 0

$$\frac{M_A + M_B}{2} L = \frac{1}{2} \frac{Wab}{L} . L$$

$$M_A + M_B = \frac{Wab}{L} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Step (5) deflection at points A & B is zero. So net moment of area in between A & B is also zero.

Moment of area in between A & B in free moment diagram = moment of area in between A & B in fixing moment diagram

$$A_1 x_1 = A_2 x_2$$

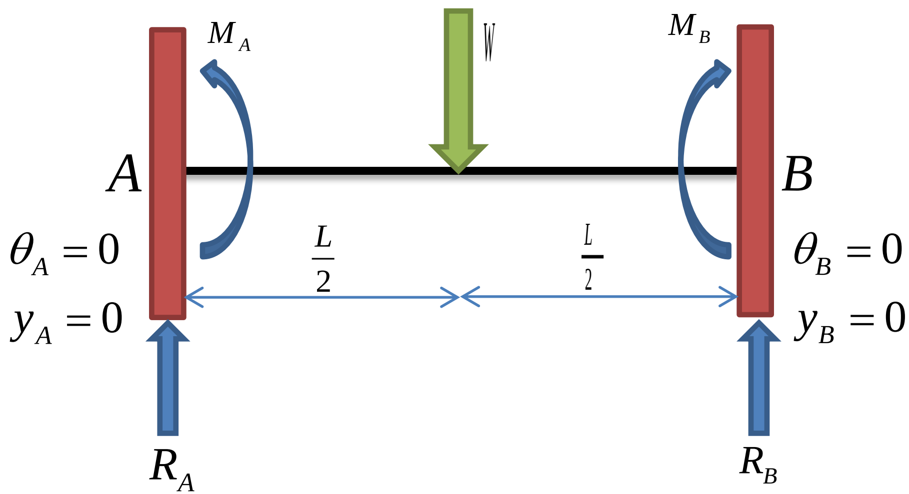
$$\left(\frac{1}{2}.a.\frac{Wab}{L}\right).\frac{2}{3}a + \left(\frac{1}{2}.b.\frac{Wab}{L}\right).\left(a + \frac{b}{3}\right) = \left(M_B.L\right).\frac{L}{2} + \frac{1}{2}L\left(M_A - M_B\right)\frac{L}{3} \qquad \text{...........(2)}$$

From equation (1) & (2)

$$M_A = \frac{Wab^2}{L^2}$$

$$M_B = \frac{Wa^2b}{L^2}$$

Macaulay's Method

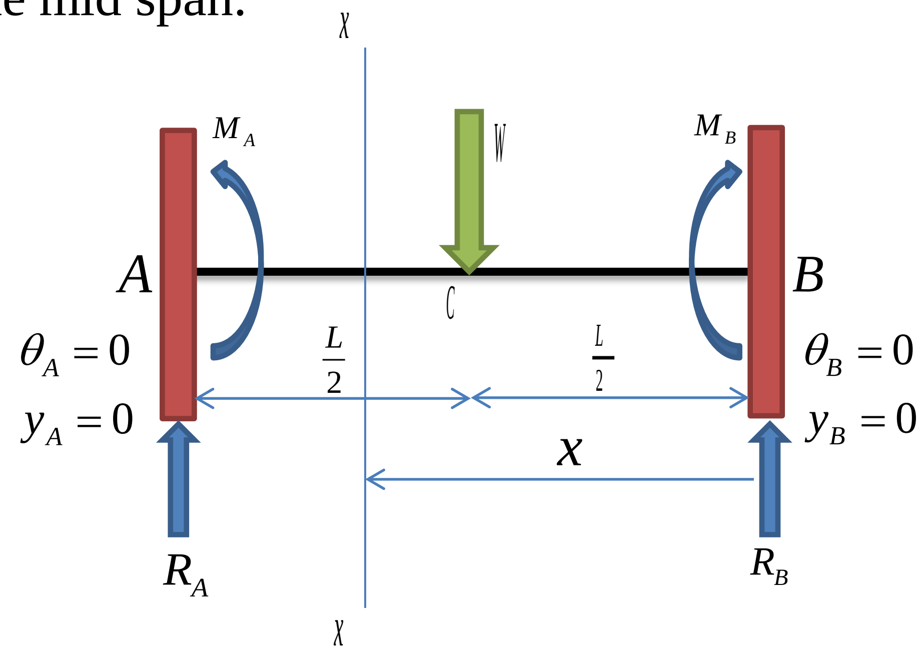


Due to symmetry,

$$R_A = R_B = R$$

$$M_A = M_B = M$$

Q. Determine the maximum bending moment and deflection of a beam of length L and flexural rigidity is EI. The beam is fixed horizontally at both ends and carries a concentrated load w at the mid span.



Writing general equation -

$$M_{x-x} = -M_B + R_B.x\Big|_{CB} - W\left(x - L/2\right)\Big|_{AC}$$

$$EI\frac{d^2y}{dx^2} = M_{x-x}$$

$$EI\frac{d^2y}{dx^2} = -M_B + R_B.x\Big|_{CB} - W\left(x - L/2\right)\Big|_{AC}$$

On integrating,

$$EI\frac{dy}{dx} = -M_B.x + R_B.\frac{x^2}{2} + A\Bigg|_{CB} - W\frac{\left(x - L/2\right)^2}{2}\Bigg|_{AC}$$

Ankit Saxena (saxena01ankit@gmail.com)

$$At, x = 0$$

$$\frac{dy}{dx} = 0$$

$$A = 0$$

$$EI\,\frac{dy}{dx} = -M_B.x + R_B.\frac{x^2}{2}\Bigg|_{CB} - W\,\frac{\left(x - L/2\right)^2}{2}\Bigg|_{AC}$$

Again, integrating w.r.t 'x'

$$EI.y = -M_B\,\frac{x^2}{2} + R_B.\frac{x^3}{6} + B\Bigg|_{CB} - W\,\frac{\left(x - L/2\right)^3}{6}\Bigg|_{AC}$$

$$At, x = 0$$

$$y = 0$$

$$B = 0$$

$$EI.y = -M_B\,\frac{x^2}{2} + R_B.\frac{x^3}{6}\Bigg|_{CB} - W\,\frac{\left(x - L/2\right)^3}{6}\Bigg|_{AC}$$

So the general equation of slope and deflection for the fixed beam for given loading condition is given as -

$$EI\frac{dy}{dx} = -M_B.x + R_B.\frac{x^2}{2}\bigg|_{CB} - W\frac{(x-L/2)^2}{2}\bigg|_{AC}$$

$$EI.y = -M_B\frac{x^2}{2} + R_B.\frac{x^3}{6}\bigg|_{CB} - W\frac{(x-L/2)^3}{6}\bigg|_{AC}$$

At x = L, y = 0 & dy/dx = 0

From the above 2 equations we get -

$$0 = -M_B.L + R_B.\frac{L^2}{2}\bigg|_{CB} - W\frac{L^2}{8}\bigg|_{AC} = -M_B.L + \frac{R_B L^2}{2} - \frac{WL^2}{8}$$

$$0 = -M_B\frac{L^2}{2} + R_B.\frac{L^3}{6}\bigg|_{CB} - W\frac{L^3}{48}\bigg|_{AC} = -M_B\frac{L^2}{2} + \frac{R_B L^3}{6} - \frac{WL^3}{48}$$

On solving these 2 equations the value of R_B and M_B is given as -

$$M_B = \frac{WL}{8}$$

$$R_B = \frac{W}{2}$$

If we take the section from the end A at a distance of x, we will get the R_A and M_A which value is given as -

$$M_A = \frac{WL}{8}$$

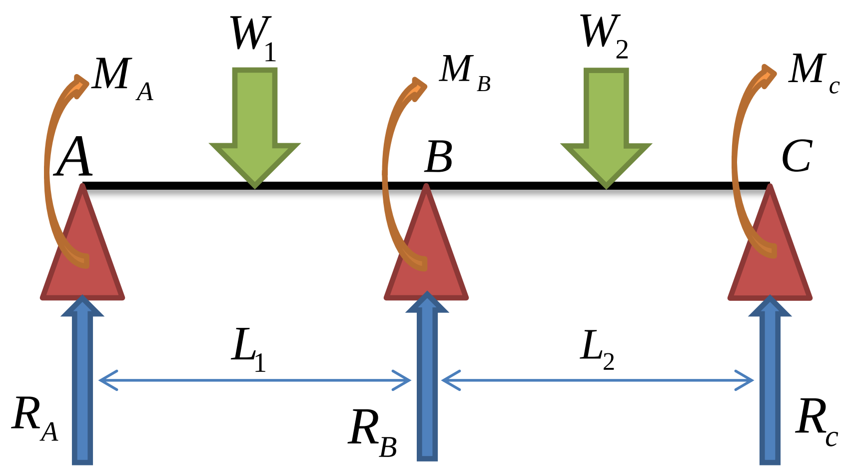
$$R_A = \frac{W}{2}$$

CONTINUOUS BEAM

Continuous Beam

1. In continuous beam we used 3 or more than 3 simple supports.

2. **The moment reactions only at the end supports are zero.**

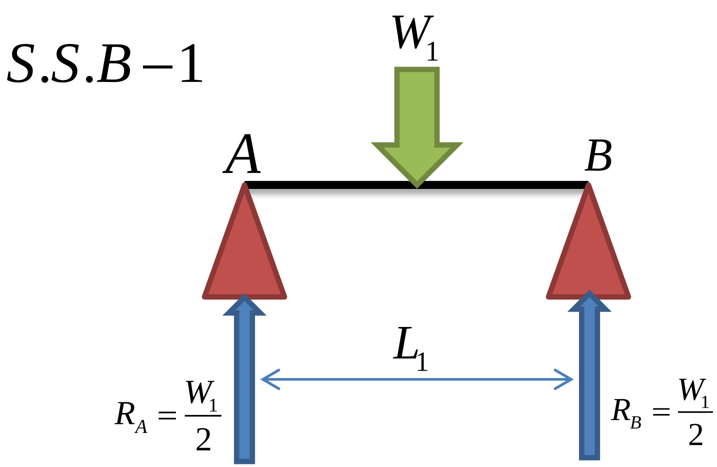


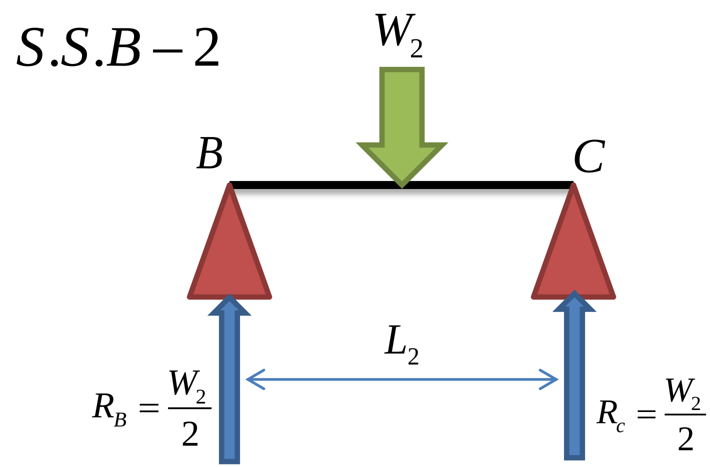
Note - [W1 & W2 are acting at mid point of AB & BC respectively.]

Clapeyron's Three-Moment Equation (Procedure)

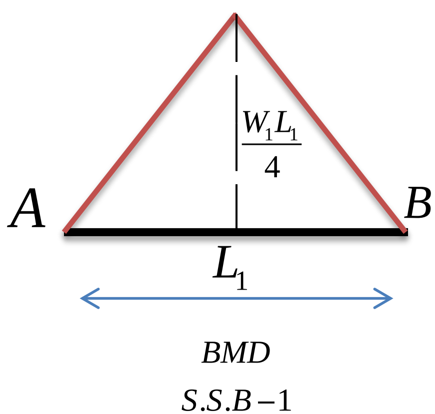
1. Between 3 consecutive supports continuous beam is treated as simply supported beam with given loading condition.

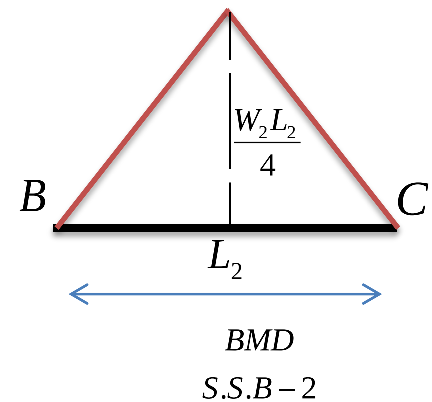
2. Calculate the support reactions.





3. Draw the Bending moment diagram for SSB-1 and SSB-2





4. Apply Clapeyron's equation

$$M_A L_1 + 2M_B\left(L_1 + L_2\right) + M_C L_2 = -\frac{6A_1\overline{x_1}}{L_1} - \frac{6A_2\overline{x_2}}{L_2}$$

5. In continuous beam moment reactions at the end support is zero. So moment reaction at A & C supports are zero.

$$M_A = 0$$
$$M_C = 0$$

6. A_1 = Area of BMD for SSB-1

A_2 = Area of BMD for SSB-2

7. $\overline{x_1}$ = Distance between centroid of area A1 to the left hand side support in BMD-1

. $\overline{x_2}$ = Distance between centroid of area A2 to the right hand side support in BMD-2

Applying the given conditions to Clapeyron's three moment equation

$$M_A = 0$$

$$M_C = 0$$

$$A_1 = \frac{1}{2}.L_1.\frac{W_1 L_1}{4} = \frac{W_1 L_1^{\ 2}}{8}$$

$$A_2 = \frac{1}{2}.L_2.\frac{W_2 L_2}{4} = \frac{W_2 L_2^{\ 2}}{8}$$

$$\overline{x_1} = \frac{L_1}{2}$$

$$\overline{x_2} = \frac{L_2}{2}$$

$$2M_B\left(L_1 + L_2\right) = -\frac{6.\frac{W_1 L_1^{\ 2}}{8}.\frac{L_1}{2}}{L_1} - \frac{6.\frac{W_2 L_2^{\ 2}}{8}.\frac{L_2}{2}}{L_2}$$

$$2M_B\left(L_1 + L_2\right) = -.\frac{3W_1 L_1^{\ 2}}{8} - .\frac{3W_2 L_2^{\ 2}}{8}$$

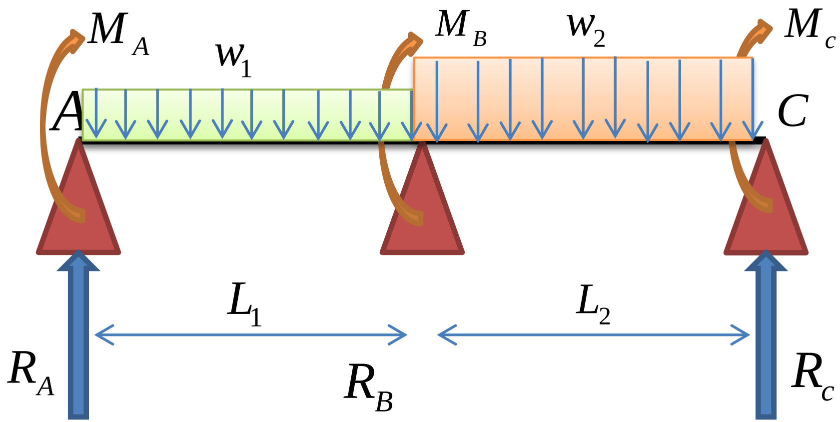
Case [1] If,

$$L_1 = L_2 = L$$
$$W_1 = W_2 = W$$

From the above equation

$$2M_B\left(L+L\right) = -.\frac{3WL^2}{8} - .\frac{3WL^2}{8}$$
$$M_B = -\frac{3}{16}.WL$$

Case [2] when continuous beam subjected to uniformly distributed load over its entire span length.



On simplification the term $-\frac{6A_1\overline{x_1}}{L_1}-\frac{6A_2\overline{x_2}}{L_2}$ has become,

$$-\frac{6A_1\overline{x_1}}{L_1}-\frac{6A_2\overline{x_2}}{L_2}=-\frac{w_1L_1^{\,3}}{4}-\frac{w_2L_2^{\,3}}{4}$$

SPRINGS

Spring

- Spring is a elastic member which deflect under the action of external load or couple.
- Due to deflection, spring store the energy and at the required time released the energy.

Note

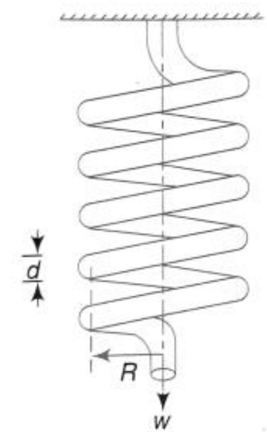
- ❑ **Stiffness or spring constant :** It is defined as the force required per unit deflection.
- ❑ **Solid length :** It is the length of a spring in the fully compressed state when the coils touch each other.

Helical Spring

"A helical spring is a piece of wire coil in the form of helix."

Helix- when a right angle triangle is wrapped around the circumference of a cylinder through its base, a helix profile is generated.

Helix angle - The angle made by plane of coil with the horizontal plane which is perpendicular to the axis of the spring is known as helix angle.



d = wire diameter (mm)

R = mean coil radius (mm)

D = mean coil diameter (mm)

Spring index:

It is the ratio of mean coil diameter [D] to the wire diameter [d]. It is denoted by 'C'.

$$C = \frac{D}{d}$$

Types of Helical Spring	
<u>Open coil helical spring</u>	<u>Closed coil helical spring</u>
1. Coils do not touch each other	1. Coils touch each other.
2. helix angle is generally greater than 10 degree.	2. helix angle generally very small generally less than 5 degree

Note

1. If there are,

Number of active coil = n

Length of spring = $\pi.D.n$

Closed coil helical spring

1. The helix angle is very small.
2. The coils may be assumed to be in a horizontal plane.
3. These spring may be acted upon by axial load or axial toque.
4. Due to axial load, there is axial extension may take place in a spring.
5. Due to axial torque, there is a change in the radius of curvature of the spring coils.

Note :- Due to axial torque, there is an angular rotation of the free end and the action is known as wind-up.

Closed coil helical spring under axial load

W = axial load (N)

D = mean coil diameter (mm)

R = mean coil radius (mm)

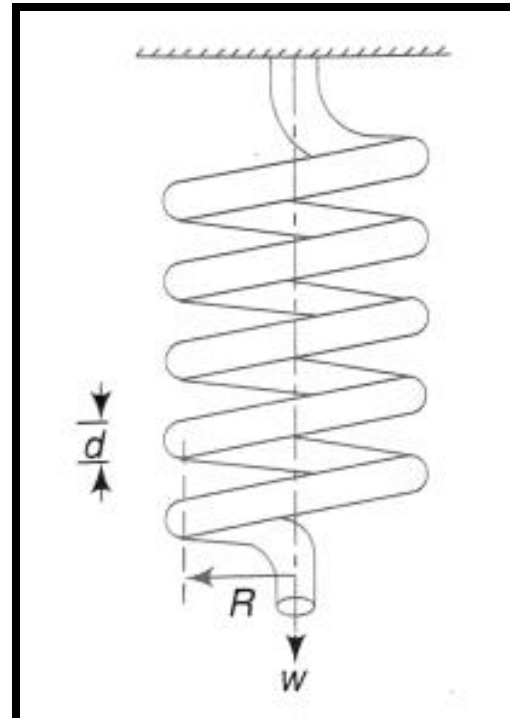
d = wire diameter (mm)

θ = total angle of twist along wire (radian)

δ = deflection of W along the spring axis (mm)

n = number of coils

L = length of wire



Torsional torque = W. D/2

Assumptions:

1. Neglect the effect of shear force.
2. Neglect the radius of curvature effect.

$$\tau = \frac{16T}{\pi d^3}$$

$$\tau = \frac{16.W.\frac{D}{2}}{\pi d^3}$$

Torsional shear stress $$\tau = \frac{8WD}{\pi d^3}$$

Ankit Saxena (saxena01ankit@gmail.com)

Angle of twist (θ)

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\theta = \frac{TL}{GJ}$$

$$= \frac{W.\dfrac{D}{2}.L}{G.\dfrac{\pi}{32}d^4}$$

$$\theta = \frac{16WDL}{G\pi d^4}$$

$$\theta = \frac{16WD\left(\pi Dn\right)}{G\pi d^4}$$

$$\theta = \frac{16WD^2L.n}{Gd^4}$$

Strain Energy [S.E]

$$S.E = \frac{1}{2}.\frac{T^2 L}{GJ}$$

$$= \frac{1}{2}.\frac{W^2.\frac{D^2}{4}.\pi D n}{G.\frac{\pi}{32}.d^4}$$

$$S.E = \frac{4W^2 D^3.n}{Gd^4}$$

Deflection of W along the shaft axis (mm) [δ]

$$\delta = \frac{d}{dW}\left(S.E\right)$$

$$\delta = \frac{d}{dW}\left(\frac{4W^2 D^3.n}{Gd^4}\right)$$

$$\delta = \frac{8WD^3 n}{Gd^4}$$

Note

Consider shear stress and radius of curvature effect.

Torsional shear stress is given as-

$$\tau = \frac{8wD}{\pi d^3}.K_w$$

Where K_w = Wahl's factor

$$K_w = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

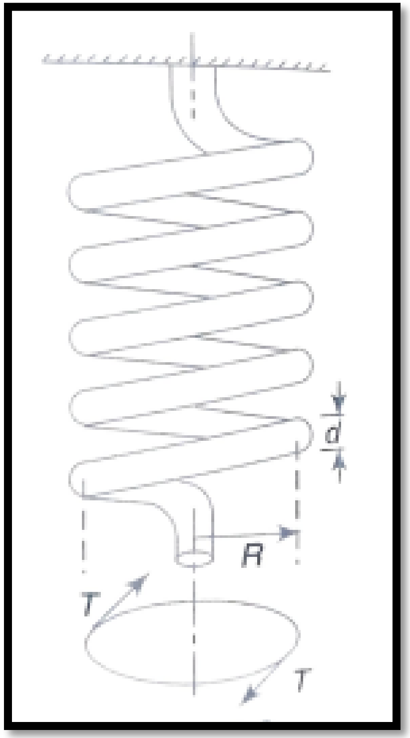
Where C = Spring index, it is given as - $C = \frac{D}{d}$

Ankit Saxena (saxena01ankit@gmail.com)

Closed oil helical spring subjected to axial torque

T = Axial torque

This axial torque become act as bending moment for spring wire.



$$S.E = \frac{1}{2}.\frac{M^2 L}{EI}$$

$$= \frac{1}{2}.\frac{T^2 L}{EI}$$

$$L = \pi.D.n$$

$$= \frac{1}{2}.\frac{T^2 \pi.D.n}{EI}$$

$$S.E = \frac{32T^2 Dn}{Ed^4}$$

Angle of twist (θ)

$$\theta = \frac{d}{dM}\left(S.E\right) = \frac{d}{dM}\left(\frac{32T^2Dn}{Ed^4}\right)$$

$$\theta = \frac{64TDn}{Ed^4}$$

A close-coiled helical spring is required to absorb $2.25*10^3$ joules of energy. Determine the diameter of the wire, the mean diameter of the spring and the number of coils necessary if

1. The maximum stress is not to exceed 400 MN/m^2
2. The maximum compression of the spring is limited to 250 mm
3. the mean diameter of the spring can be assumed to be eight times that of the wire

How would the answers change if appropriate Wahl factors are introduced?

For spring material G = 70 GN/m^2.

Q. Show that the ratio of extension per unit axial load to angular rotation per unit axial torque of a close-coiled helical spring is directly proportional to the square of the mean diameter, and hence that the constant of proportionality is 1/4 *(1 + ν).

If Poisson's ratio $\nu = 0.3$, determine the angular rotation of a close-coiled helical spring of mean diameter 80 mm when subjected to a torque of 3 N m, given that the spring extends 150 mm under an axial load of 250 N.

Springs in Series

1. When two springs of different stiffness are joined end to end, they are said to be connected in series.

2. For spring in series, the load is the same for both the springs whereas the deflection is the sum of deflection of each.

$$\delta = \delta_1 + \delta_2$$

$$\frac{W}{s} = \frac{W}{s_1} + \frac{W}{s_2}$$

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}$$

$$s = \frac{s_1 s_2}{s_1 + s_2}$$

Springs in Parallel

1. When two springs of different stiffness are joined in parallel, they have the common deflection and the load is the sum of load taken by each,

i.e., common deflection $\delta = \frac{W}{s} = \frac{W_1}{s_1} = \frac{W_2}{s_2}$

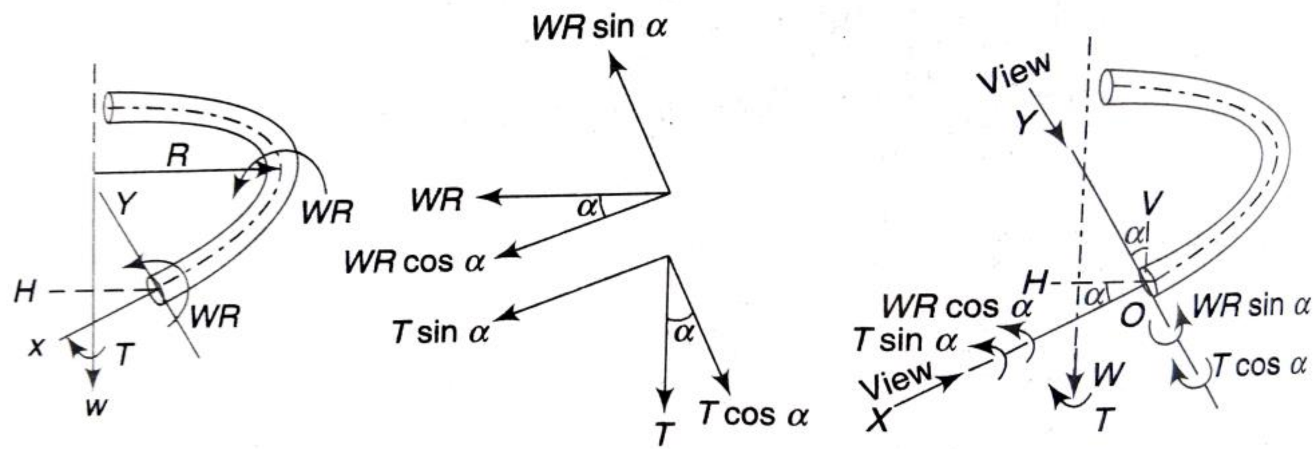
$$W = W_1 + W_2$$

$$s\delta = s_1\delta + s_2\delta$$

$$s = s_1 + s_2$$

Open coiled helical spring subjected to axial load (W) and axial Torque (T)

Due to this axial load (W) and axial torque (T) both, twisting couple and bending couple will act in spring wire.



The combined twisting couple is given as,

$$T' = WR\cos\alpha + T\sin\alpha$$

The combined bending couple is given as,

$$M = T\cos\alpha - WR\sin\alpha$$

Length of wire is given as,

$$L = \frac{\pi D n}{\cos\alpha} = \pi D n \sec\alpha$$

Total strain energy (U) is given as -

$$U = \frac{M^2 L}{2EI} + \frac{T^2 L}{2GJ}$$

$$U = \frac{\left(T\cos\alpha - WR\sin\alpha\right)^2 L}{2EI} + \frac{\left(WR\cos\alpha + T\sin\alpha\right)^2 L}{2GJ}$$

Axial Deflection

$$\delta = \frac{d}{dW}\left(U\right)$$

$$\delta = \frac{2L\left(T\cos\alpha - WR\sin\alpha\right)\left(-R\sin\alpha\right)}{2EI} + \frac{2L\left(WR\cos\alpha + T\sin\alpha\right)R\cos\alpha}{2GJ}$$

$$\delta = WR^2 L\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right) + TRL\left(\frac{1}{GJ} - \frac{1}{EI}\right)\sin\alpha\cos\alpha$$

Axial Rotation

$$\phi = \frac{d}{dT}(U)$$

$$\delta = \frac{2L(T\cos\alpha - WR\sin\alpha)\cos\alpha}{2EI} + \frac{2L(WR\cos\alpha + T\sin\alpha)\sin\alpha}{2GJ}$$

$$\delta = WRL\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right) + TL\left(\frac{\sin^2\alpha}{GJ} - \frac{\cos^2\alpha}{EI}\right)$$

Stresses

1. The combined twisting couple is given as,

Torsional shear stress is given as,

$$\tau = \frac{16T'}{\pi d^3}$$

2. The combined bending couple is given as,

Bending stress is given as,

$$\sigma_b = \frac{32M}{\pi d^3}$$

THE END