Engg. Mathematics IV

Notes of Unit IV

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Method of separation of variables:-

Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, u, p, q) = 0 , \text{———} (1)$$

where R, S and T are continuous functions of x and y.

$r = \frac{\partial^2 u}{\partial x^2}$, $s = \frac{\partial^2 u}{\partial x \partial y}$, $t = \frac{\partial^2 u}{\partial y^2}$ and

$u = u(x, y)$ (i.e u is function of x and y)

Let us assume that solution of (1) can be written as $\quad u(x,y) = X(x)\, Y(y)$, ——— (2)

where X and Y are the functions of x and y respectively.

(Obviously solution of (1) is a relation between the variables u (dependent) and independent variable x and y. Here we have assume that the solution can be put in the form as given in (2))

Differentiating eqn (2) partially w.r.s. to x and y, we get

$p = \frac{\partial u}{\partial x} = Y \frac{\partial X}{\partial x} = Y \frac{dX}{dx}$ ($\because$ X is function of only one variable x) ②

Similarly

$q = \frac{\partial u}{\partial y} = X \frac{dY}{dy}$, $r = Y \frac{d^2X}{dx^2}$, $t = X \frac{d^2Y}{dy^2}$

and $s = \frac{dY}{dy} \frac{dX}{dx}$

Eqn ① can also be written as

$RD^2 + SDD' + TD'^2 + f(x, y, u, D, D') = 0$

On substitution of values of r, s, t, p and q, we will have

$$\frac{f(D)X}{X} = \frac{f(D')}{Y} = \lambda \text{ (say)}, \quad \text{——— } ③$$

where $f(D)$ and $f(D')$ are quadratic function of D and D', respectively.

Equation ③ is such that it's L.H.S. is a function of x only and R.H.S. is a function of y only, both are equal only when they are equal to same constant λ (say).

Case-iii When $\lambda = 0$

Then in this case solution will be

$X = C_1 x + C_2$ and $T = C_3$ so

$$u(x,t) = (C_1 x + C_2) C_3 \quad \text{——— (7)}$$

Now we have three solutions, but according to given condition

i) Case iii) gives always zero solution because $u(x,t) = 0$ at $x = 0$. Therefore this condition is discard.

ii) In second case i) $u \to \infty$ as $t \to \infty$, so this also don't give solution.

Hence the appropriate solution is

$$\boxed{u(x,t) = (C_1 \cos \mu x + C_2 \sin \mu x) C_3 e^{-\frac{\mu^2}{K} t}} \quad \text{——— (8)}$$

Above solution is consistent with physical nature of problem.

Ques. Solve the 1-dimensional Heat equation

$$\frac{\partial^2 u}{\partial x^2} = K\frac{\partial u}{\partial t}, \quad 0<x<l, \ t>0,$$ with the

conditions that $u=0$ when $t\to\infty$ and $u(0,t) = u(l,t) = 0$, by using the method of separation of variables.

Solⁿ- Given equation is

$$\frac{\partial^2 u}{\partial x^2} = K\frac{\partial u}{\partial t}, \quad 0<x<l, \ t>0 \quad \text{---} (1)$$

Let the solution of (1) be

$$u(x,t) = X(x)T(t) \quad \text{---} (2)$$

Differentiating (2) partially w.r.s. to x and t, we get

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

On putting these values in (1), we will get

$$T\frac{d^2X}{dx^2} = KX\frac{dT}{dt}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{K}{T}\frac{dT}{dt} = \lambda \text{ (say)} \quad \text{---} (3)$$

Equating separately

$$\left.\begin{array}{l} \frac{1}{X}\frac{d^2X}{dx^2} = \frac{X''}{X} = \lambda \Rightarrow X'' - \lambda X \\ \frac{K}{T}\frac{dT}{dt} = \lambda \Rightarrow KT' - \lambda T = 0 \end{array}\right\} \quad \text{——— (4)}$$

Case i) When $\lambda = \mu^2$ ($\mu > 0$ real)

Eqn (4) becomes $X'' - \mu^2 X = 0$ and $T' - \frac{\lambda}{K}T = 0 \Rightarrow$

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

and $T = c_3 e^{\frac{\mu^2}{K}t}$

Thus solution $u(x,t) = X(x)\,T(t)$ becomes

$$u(x,t) = \left(c_1 e^{\mu x} + c_2 e^{-\mu x}\right) c_3 e^{\frac{\mu^2}{K}t} \quad \text{——— (5)}$$

Case ii) When $\lambda = -\mu^2$ ($\mu > 0$ real)

Solution of (4) becomes

$X = c_1 \cos \mu x + c_2 \sin \mu x$ and

$T = c_3 e^{-\frac{\mu^2}{K}t}$

Thus solution will be

$$u(x,t) = \left(c_1 \cos \mu x + c_2 \sin \mu x\right) c_3 e^{-\frac{\mu^2}{K}t} \quad \text{——— (6)}$$

Now we will use the conditions in eqn (8)

$$u(0,t) = u(l,t) = 0$$

$$u(0,t) = c_1 c_3 e^{-\mu^2/k\, t} = 0 \Rightarrow c_1 = 0,$$

$c_3 \neq 0$ because if $c_3 = 0$ then $u(x,t) = 0$ and in case of heat flow equation $u(x,t)$ denote the temperature so if $u=0$ it means temperature always remain zero, which is not possible.

Hence solution becomes

$$u(x,t) = c_2 c_3 \sin\mu x\, e^{-\frac{\mu^2}{k} t} \quad \text{——— (9)}$$

Again $u(l,t) = 0$

$$\Rightarrow \quad u(l,t) = c_2 c_3 \sin\mu l\, e^{-\frac{\mu^2}{k} t} = 0$$

$c_2 \neq 0$ $c_3 \neq 0$ (According as above) Also $e^{-\mu^2/k\, t} \neq 0$ because if $e^{-\frac{\mu^2}{k} t} = 0$ it means $t \to -\infty$ which is not given. So

$$\sin\mu l = 0 = \sin n\pi$$

$$\Rightarrow \mu = \frac{n\pi}{l} \qquad n \in I$$

Hence $u(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2 t}{Kl^2}}$

For more general solution of heat flow equation we will take $C_2 C_3 = b_n$ and $n \in I^+$

Hence

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2}{Kl^2} t}} \quad \text{--- (10)}$$

Above equation is general solution of 1-dimensional Heat-flow equation

Ques. Solve the P.D.E. by separation of variables method.

$u_{xx} = u_y + 2u$, $u(0,y) = 0$, $\frac{\partial u(0,y)}{\partial x} = 1 + e^{-3y}$

Solⁿ The given P.D.E. is

$$u_{xx} = u_y + 2u \quad \text{——— (1)}$$

Let us assume that solution of (1) be

$$u(x,y) = X(x)Y(y) \quad \text{——— (2)}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = Y\frac{d^2X}{dx^2} \quad , \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

$$u_y = \frac{\partial u}{\partial y} = X\frac{dY}{dy}$$

Putting these values in (2), we have

$$Y\frac{d^2X}{dx^2} = X\frac{dY}{dy} + 2XY$$

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 \quad \text{——— (3)}$$

Again $\frac{X''}{X} = \frac{Y'}{Y} + 2 = (\lambda)$ say ——— (4)

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{ In this question we don't need to choose one three cases of λ, because first case $\lambda=0$ contains no term of x, so it will not appropriate. Again in case $\lambda=-\mu^2$, solution will take periodic term and if $\lambda=\mu^2$, solution will take exponential term. Now according to the conditions given in our question $\frac{\partial z(0,y)}{\partial x} = 1+e^{-3y}$, solution must has to take exponential term. (Also if take solution with exponential term then it is to compare with the condition) }

Hence here we take $\lambda=\mu^2$ or λ^2 (itself) or simply λ.

From eqn (4)

$$\frac{X''}{X} = \frac{Y'}{Y}+2 = \mu^2 \quad (\mu>0 \text{ real}) \quad \text{--- (5)}$$

Equating separately

$$X''-\mu^2 X \Rightarrow X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$\frac{Y'}{Y} = -2+\mu^2 \Rightarrow Y = c_3 e^{(\mu^2-2)y}$$

Hence solution (2) will be

$$u(x,y) = (c_1 e^{\mu x} + c_2 e^{-\mu x})\, c_3 e^{(\mu^2-2)y} \quad \text{--- (6)}$$

Using given condition $u(0,y)=0$, This will implies that

$$(c_1+c_2)\,c_3\,e^{(\mu^2-2)y} = 0 \Rightarrow c_1 = -c_2 \text{ as } c_3 \neq 0$$

or $c_2 = -c_1$

So $$u(x,y) = \left(c_1 e^{\mu x} - c_1 e^{-\mu x}\right) c_3 e^{(\mu^2-2)y}$$

$$u(x,y) = c_1 c_3 \left(e^{\mu x} - e^{-\mu x}\right) e^{(\mu^2-2)y} \quad \text{——— (7)}$$

Again using $\dfrac{\partial u(0,y)}{\partial x} = 1+e^{-3y}$ is above

$$1+e^{-3y} = c_1 c_3 \mu\,(1+1)\,e^{(\mu^2-2)y}$$

$$2\mu c_1 c_3\, e^{(\mu^2-2)y} = 1+e^{-3y} \quad \text{——— (8)}$$

c_1 and c_3 are arbitrary constants so they have many values.

Eqn (7) can be written as

$$2\mu c_1 c_3\, e^{(\mu^2-2)y} = e^{0.y} + e^{-3y}$$

comparing with first term

$$2\mu c_1 c_3 = 1 \qquad \mu^2-2 = 0$$

$$\Rightarrow c_1 c_3 = \frac{1}{2\mu} \qquad \mu = \pm\sqrt{2}$$

$$c_1c_3 = \pm \frac{1}{2\sqrt{2}}$$

Also on comparing with second term

$2\mu c_1 c_3 = 1$ and $\mu^2 - 2 = -3 \Rightarrow \mu^2 = -1$

$c_1c_3 = \frac{1}{2\mu}$ and $\mu = \pm i$

$$c_1c_3 = \pm \frac{1}{2i}$$

Hence solution will be (by taking +ve. values)

$$u(x,y) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)e^{-3y}$$

$$= \frac{1}{2\sqrt{2}} \sinh\sqrt{2}x + \frac{1}{2i}\left(2i \sin x\right)e^{-3y}$$

$$\boxed{u(x,y) = \frac{1}{2\sqrt{2}} \sinh\sqrt{2}x + \sin x\, e^{-3y}}$$ Ans.

Illustrative questions -

Ques- A rod of length l with insulated sides is intially at a uniform temperature u_0. It's end's are suddenly cooled to 0°c and are kept at that temperature. Find the temperature $u(x,t)$.

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Solution - Since we have to find the temperature so we will take heat equation to solve it.

Equation of 1-dimensional heat flow

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c \text{ is constant}) \quad \text{——— (1)}$$

Given that ends are suddenly cooled to 0°c

so $u(0,t) = u(l,t) = 0$.

Also intial temperature is u_0 (given). It

means $u(x,0) = u_0$

The general solution of (1) will be

$$u(x,t) = (C_1 \cos \mu x + C_2 \sin \mu x)\, C_3 e^{-c^2\mu^2 t} \quad \text{——— (2)}$$

Now $u(0,t) = 0$

$$C_1 C_3 e^{-c^2\mu^2 t} = 0 \Rightarrow C_1 = 0 \quad (\because C_3 \neq 0$$

$$\text{So } u(x,t) = C_2 C_3 \sin \mu x\, e^{-c^2\mu^2 t} \quad \text{——— (3)}$$

Again $u(l,t) = 0$ in (2)

$$C_2 C_3 \sin \mu l\, e^{-c^2\mu^2 t} = 0$$

$C_2 \neq 0$ $C_3 \neq 0$ $e^{-c^2\mu^2 t} \neq 0$ (explanied in previous questions)

$\Rightarrow \sin \mu l = 0 = \sin n\pi$

$\Rightarrow \mu = \frac{n\pi}{l} \quad n \in I$

$$u(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \, e^{\frac{-c^2 n^2 \pi^2}{l^2} t}$$

For more general solution, we take

$c_2 c_3 = b_n$ and $n \in I^+$

$$\boxed{u(x,t) = \sum b_n \sin \frac{n\pi x}{l} \, e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}}$$ Ans

Note. In other questions except heat equation and wave equation, we don't need to take always 3 cases of constant term λ. We will take appropriate case for λ according to our given boundary and intial conditions.

Ques.- Solve P.D.E. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x,0) = 3 \sin n\pi x$, $u(0,t) = 0$, $u(l,t) = 0$, $0 < x < l$, $t > 0$.

Solution of two-dimensional Heat (Diffusion) Equation of separation of variables method:-

Two-dimensional Heat-flow equation is

$$\frac{\partial u}{\partial t} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\}, \text{ where } c \text{ is constant} \quad \text{——} (1)$$

In steady state $\frac{\partial u}{\partial t} = 0$, so from (1)

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad \text{——} (2)$$

Equ^n (2) is known as two-dimensional Laplace equation

(2) can also be written as

$\nabla^2 u = 0$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (for 2-dimension)

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (for 3-dimension)

Three-dimensional Heat-Flow equation is

$$\frac{\partial u}{\partial t} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} \quad \text{——} (3)$$

For steady state $\frac{\partial u}{\partial t} = 0$, so from (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{——} (4)$$

Eqn (4) is known as 3-dimensional Laplace equation.

(4) can also be written as $\nabla^2 u = 0$

Question – Solve two-dimensional heat flow equation by separation of variable method.

Soln Two-dimensional Heat flow equation be

$$\frac{\partial u}{\partial t} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} \quad \text{—— (1)}$$

Let us assume it's solution be

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \quad \text{—— (2)}$$

On partial differentiation of (2) w.r.t. to x, y and t respectively,

$$\frac{\partial u}{\partial t} = XY\frac{dT}{dt}, \quad \frac{\partial^2 u}{\partial x^2} = YT\frac{d^2X}{dx^2}, \quad \frac{\partial^2 u}{\partial y^2} = XT\frac{d^2Y}{dy^2}$$

On putting these values in (1), we will get

$$XY\frac{dT}{dt} = c^2\left\{YT\frac{d^2X}{dx^2} + XT\frac{d^2Y}{dy^2}\right\}$$

$$\frac{1}{c^2 T}T' = \frac{1}{X}X'' + \frac{1}{Y}Y''$$

Again

$$\frac{T'}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -l^2 = -m^2 - n^2 \text{ (say)} \quad \text{--- (3)}$$

$$\frac{T'}{c^2T} = -l^2, \quad \frac{X''}{X} = -m^2, \quad \frac{Y''}{Y} = -n^2$$

(where l, m and n are constants. Eqn (3) is true only when each term of it separately equal to respective constants.) So,

$$\frac{T'}{c^2 T} = -l^2 \Rightarrow \frac{T'}{T} = -c^2 l^2 \Rightarrow T(t) = c_5 e^{-c^2 l^2 t}$$

$$\frac{X''}{X} = -m^2 \Rightarrow X'' + m^2 X = 0 \Rightarrow X(x) = C_1 \cos mx + C_2 \sin mx$$

$$\frac{Y''}{Y} = -n^2 \Rightarrow Y'' + n^2 Y = 0 \Rightarrow Y(y) = C_3 \cos ny + C_4 \sin ny$$

Hence $u(x,y,t) = X(x)\, Y(y)\, T(t)$

$$u(x,y,t) = \{C_1 \cos mx + C_2 \sin mx\}\{C_3 \cos ny + C_4 \sin ny\} C_5 e^{-c^2 l^2 t}$$

$$\boxed{u(x,y,t) = \{C_1 \cos mx + C_2 \sin mx\}\{C_3 \cos ny + C_4 \sin ny\} C_5 e^{-c^2(m^2+n^2)t}}$$

Since $X(x)$ and $Y(y)$ can also be written as

$$X(x) = C_1 \cos(mx + \varepsilon_m) \quad \text{and} \quad Y(y) = C_2 \cos(nx + \varepsilon_n)$$

Hence

$$u(x,y,t) = C_1 \cos(mx + \varepsilon_m)\, C_2 \cos(nx + \varepsilon_n)\, C_5 e^{-c^2(m^2+n^2)t}$$

$$u(x,y,t) = C_1 C_2 C_5 \cos(mx + \varepsilon_m) \cos(nx + \varepsilon_n)\, e^{-c^2(m^2+n^2)t}$$

For more general solution

$$\boxed{u(x,y,t) = \sum_m \sum_n C_m C_n \cos(mx+\epsilon_m)\cos(nx+\epsilon_n)\, e^{-c^2(m^2+n^2)t}}$$

Question- A rectangular plate with insulated surface is 8 cm wide and so long compared to it's width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y=0$ is given by

$$u(x,0) = 100 \sin\frac{\pi x}{8}, \quad 0<x<8,$$

while the two long edges $x=0$ and $x=8$ as well as the other short edge are kept at 0°C, show that the steady state temperature at any point of the plate is given by

$$u(x,y) = 100\, e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}$$

Solution- We have to prove that the steady state temperature at any point of the plate be.

$$u(x,y) = 100\, e^{-\frac{\pi y}{8}} \sin\frac{\pi x}{8}$$

So we will take two-dimensional heat-flow equation in steady state or two-dimensional Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (1)}$$

It's solution be

$$u(x,y) = (C_1 \cos \mu x + C_2 \sin \mu x)(C_3 e^{\mu y} + C_4 e^{-\mu y}) \quad \text{——— (2)}$$

Boundary conditions are given.
Since temperature along edges $x=0$ and $x=8$ are kept at $0°c$ so

$u(0,y) = 0$ and $u(8,y) = 0$, also

$\lim_{y \to \infty} u(x,y) = 0$ {as given length may be considered infinite, also other short edges except $x=0$ and $x=8$ are also kept at $0°c$}

and $u(x,0) = 100 \sin \frac{\pi x}{8}$, $0 < x < 8$

from (2) $u(0,y) = 0 \Rightarrow$

$$0 = C_1 \{ C_3 e^{\mu y} + C_4 e^{-\mu y} \} \Rightarrow C_1 = 0$$

So $u(x,y) = C_2 \sin \mu x \{ C_3 e^{\mu y} + C_4 e^{-\mu y} \}$ ——— (3)

Again $u(8,y) = 0 \Rightarrow$

$$C_2 \sin 8\mu \{ C_3 e^{\mu y} + C_4 e^{-\mu y} \} = 0$$

$C_2 \neq 0 \Rightarrow \sin 8\mu = 0 = \sin n\pi$

$\Rightarrow \mu = \frac{n\pi}{8}$, $n \in I$

from (3)

$$u(x,y) = C_2 \sin\frac{n\pi x}{8}\left(C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right) \quad \text{--- (4)}$$

Again $\lim_{y\to\infty} u(x,y) = 0$

$$\Rightarrow \quad 0 = \lim_{y\to\infty} C_2 \sin\frac{n\pi x}{8}\left\{C_3 e^{\frac{n\pi y}{8}} + C_4 e^{-\frac{n\pi y}{8}}\right\}$$

$$0 = C_2 \sin\frac{n\pi x}{8}\left\{C_3 \underbrace{\lim_{y\to\infty} e^{\frac{n y \pi}{8}}}_{\infty} + C_4 \underbrace{\lim_{y\to\infty} e^{-\frac{n\pi y}{8}}}_{0}\right\}$$

Above is true only when $C_3 = 0$

Hence solution will be

$$u(x,y) = C_2 C_4 \sin\frac{n\pi x}{8}\, e^{-\frac{n\pi y}{8}}$$

Put $C_2 C_4 = b_n$

$$u(x,y) = b_n \sin\frac{n\pi x}{8}\, e^{-\frac{n\pi y}{8}} \quad \text{--- (5)}$$

Again since $u(x,0) = 100 \sin\frac{\pi x}{8}$, so from (5)

$$100 \sin\frac{\pi x}{8} = b_n \sin\frac{n\pi x}{8}$$

$$\Rightarrow \quad b_n = 100, \quad n = 1$$

Thus solution is

$$\boxed{u(x,y) = 100 \sin\frac{\pi x}{8}\, e^{-\frac{\pi y}{8}}}$$

— Proved Answer.

Solution of Laplace equation in two-dimension-

Laplace equation in 2-dimension is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Let solution of (1) be

$$u(x,y) = X(x)Y(y) \quad \text{--- (2)}$$

On partial differentiation of (2) and then values putting in (1), we obtained

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Again $\dfrac{X''}{X} = \dfrac{-Y''}{Y} = \lambda$ --- (3)

Case-i, When $\lambda = \mu^2$

$$X'' - \mu^2 X = 0 \Rightarrow X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$Y'' + \mu^2 Y = 0 \Rightarrow Y = c_3 \cos \mu y + c_4 \sin \mu y$$

$$u(x,y) = (c_1 e^{\mu x} + c_2 e^{-\mu x})(c_3 \cos \mu y + c_4 \sin \mu y) \quad \text{--- (4)}$$

Case-ii When $\lambda = -\mu^2$ ($\mu > 0$ real)

$$X'' + \mu^2 X = 0 \Rightarrow X = c_1 \cos \mu x + c_2 \sin \mu x$$

$$Y'' - \mu^2 Y = 0 \Rightarrow Y = c_3 e^{\mu y} + c_4 e^{-\mu y}$$

$$u(x,y) = XY = (c_1 \cos \mu x + c_2 \sin \mu x)(c_3 e^{\mu y} + c_4 e^{-\mu y}) \quad \text{--- (5)}$$

Case-iii when $\lambda = 0$

$$X'' = 0, \quad Y'' = 0$$

$$\Rightarrow \quad X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

$$\Rightarrow \quad u = (c_1 x + c_2)(c_3 y + c_4) \quad \text{--- (6)}$$

The above three solutions are the solutions of eqn (1), but we will choose the solution of among these three which is consitent with the physical nature and boundary conditions.

Ques Solve by separation of variables method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$ with the boundary conditions

$$u(0,y) = u(l,y) = u(x,0) = 0 \text{ and}$$

$$u(x,a) = \sin \frac{n\pi x}{l}$$

Sol We have 3 solutions of 2-dimensional Laplace equation

$$u = (c_1 e^{\mu x} + c_2 e^{-\mu x})(c_3 \cos \mu y + c_4 \sin \mu y) \quad \text{--- (1)}$$

$$u = (c_1 \cos \mu x + c_2 \sin \mu x)(c_3 e^{\mu y} + c_4 e^{-\mu y}) \quad \text{--- (2)}$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \quad \text{--- (3)}$$

Using the boundary conditions in eqn (3)

$$u(x,0) = u(0,y) = u(l,y) = 0$$

$$u(l,y) = (c_1 l + c_2)(c_3 y + c_4) = 0$$

$$\Rightarrow c_2 = -c_1 l$$

Hence

$$u = c_1(x-l)(c_3 y + c_4)$$

Again $u(x,0) = 0$

$$u = c_1(x-l)c_4 = 0$$

$$\Rightarrow c_1 c_4 (x-l) = 0$$

$$\Rightarrow \text{either } c_1 = 0 \text{ or } c_4 = 0 \text{ or } x = l$$

if $c_1 = 0$ or $x = l$, then $u = 0$.

and if $c_4 = 0$, then

$$u = c_1 c_3 (x-l) y \quad \text{--- (4)}$$

Again $u(0,y) = 0 \Rightarrow -l c_1 c_3 y = 0$

In every case either $c_1 = 0$ or $c_3 = 0$ or $y = 0$ solution will $u = 0$.

This discard the solution ③, because we have to find non-zero solution.

Again from eqn ①

$$u = (c_1 e^{\mu x} + c_2 e^{-\mu x})(c_3 \cos \mu y + c_4 \sin \mu y)$$

$u(x, 0) = 0$

$$0 = (c_1 e^{\mu x} + c_2 e^{-\mu x}) c_3 \Rightarrow c_3 = 0$$

Therefore $u = c_4 \sin \mu y (c_1 e^{\mu x} + c_2 e^{-\mu x})$

$u(0, y) = 0$

$$0 = c_4 \sin \mu y (c_1 + c_2)$$

$\Rightarrow$ either $c_4 = 0$ or $c_1 + c_2 = 0$

we will take $c_1 + c_2 = 0$

Therefore

$$u = c_1 c_4 \sin \mu y (e^{\mu x} + e^{-\mu x}) \quad \text{—⑤}$$

$u(l, y) = 0$

$$0 = c_1 c_4 \sin \mu y (e^{\mu l} - e^{-\mu l})$$

$\Rightarrow$ either $c_1 = 0$ or $c_4 = 0$ or $e^{\mu l} - e^{-\mu l} = 0$

If $e^{\mu l} - e^{-\mu l} = 0$

$\Rightarrow e^{\mu l} = e^{-\mu l}$

$\Rightarrow e^{2\mu l} = 1 = e^{0}$

$\Rightarrow 2\mu l = 0 \quad (\because l \neq 0$

$\Rightarrow \mu = 0$ (which is not possible because solution ① is correspond to the case $\lambda = \mu^2 (\lambda > 0, \mu > 0)$)

Hence either $c_1 = 0$ or $c_4 = 0$

(Also if $\sin \mu y = 0 = \sin n\pi \Rightarrow y = \frac{n\pi}{\mu}$, means y is constant, which also not possible)

Hence from eqn ⑤

$u = 0$, which again a non-zero solution.

Again from eqn ②

$$u = (c_1 \cos \mu x + c_2 \sin \mu x)(c_3 e^{\mu y} + c_4 e^{-\mu y}) \quad \text{—⑥}$$

$u(x, 0) = 0$

$$0 = (c_1 \cos \mu x + c_2 \sin \mu x)(c_3 + c_4)$$

$\Rightarrow c_3 = -c_4$ or $c_4 = -c_3$

Hence $\quad u = c_3 (c_1 \cos \mu x + c_2 \sin \mu x)(e^{\mu y} + e^{-\mu y})$

—⑦

Again $u(0,y)=0$ in (6)

$$0 = c_1(c_3 e^{\mu y} + c_4 e^{-\mu y}) \Rightarrow c_1 = 0$$

Hence (6) takes form

$$u = c_2 \sin \mu x (c_3 e^{\mu y} + c_4 e^{-\mu y})$$

Again using $c_1 = 0$ in (7)

$$u = c_2 c_3 \sin \mu x (e^{\mu y} - e^{-\mu y}) \quad \text{—— (8)}$$

Again $u(l,y)=0$

$$0 = c_2 c_3 \sin \mu l (e^{\mu y} - e^{-\mu y})$$

$$c_2 \neq 0, \; c_3 \neq 0 \quad e^{\mu y} - e^{-\mu y} \neq 0$$

$$\Rightarrow \sin \mu l = 0 = \sin n\pi$$

$$\Rightarrow \mu = \frac{n\pi}{l} \quad n \in I$$

Hence (8)

$$u = c_2 c_3 \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}} \right)$$

put $c_2 c_3 = b_n$

$$u = b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}} \right) \quad \text{—— (9)}$$

Again $u(x,a) = \sin \frac{n\pi x}{l}$

$$\sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}\left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right)$$

$$\Rightarrow b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}} = \frac{1}{2\sinh\frac{n\pi a}{l}}$$

Hence

$$u(x,y) = \frac{e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}}{2\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l}$$

$$\boxed{u(x,y) = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\frac{n\pi a}{l}} \sin\frac{n\pi x}{l}} \quad \text{—— (10)}$$

Note The above solution shows that only eq^n (2) is consitent with the boundary condition, Eq^n (2) is

$$\boxed{u(x,y) = (C_1\cos\mu x + C_2\sin\mu x)(C_3 e^{\mu y} - C_4 e^{-\mu y})} \quad \text{—— (11)}$$

This is the requried solution of 2-dimensional Laplace eq^n.

Note Oftenly we use eq^n (11) as general solution for 2-dimensional Laplace equation.

Two-dimensional Laplace equation in polar Cordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

This equation can be obtained from eqn

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or $u_{xx} + u_{yy} = 0$ by putting

$x = r\cos\theta$ and $y = r\sin\theta$

Solution of One-dimensional Wave equation by method of separation of variables—

The one-dimensional P.D.E. of wave produced in a string of finite length l which is fastened at it's ends at $x=0$ and $x=l$. The string is displaced and then released to vibrate in the X-T plane.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} (1)$$

, where C is constant.

Boundary conditions are

$$u(0,t) = u(l,t) = 0 \qquad 0<x<l \;, \; t>0$$

Also the intial conditions are —(2)

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u(x,0)}{\partial t} = 0$$

Let us assume solution of (1) be

$$u(x,t) = X(x)\, T(t) \quad \text{——} (3)$$

Differentiating partially eqn w.r. to x and t

$$\frac{\partial^2 u}{\partial t^2} = X \frac{d^2 T}{dt^2} = X T''$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{d^2 X}{dx^2} = T X''$$

On putting these values in eqn 1 we will get

$$X T'' = c^2 T X''$$

$$\Rightarrow \frac{1}{c^2 T} T'' = \frac{1}{X} X'' = \lambda \text{ (say)} \quad \text{—— (4)}$$

Case-i) When $\lambda = \mu^2$ ($\mu > 0$ real)

Equation (4) becomes

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \mu^2$$

$$\Rightarrow \frac{T''}{c^2 T} = \mu^2 \Rightarrow T'' - \mu^2 c^2 T = 0$$

$$T(t) = C_3 e^{c\mu t} + C_4 e^{-c\mu t}$$

Also $X'' - \mu^2 X = 0$

$$\Rightarrow X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$$

Solution $u(x,t) = X(x)\,T(t)$ becomes

$$u(x,t) = \left\{C_1 e^{\mu x} + C_2 e^{-\mu x}\right\}\left\{C_3 e^{c\mu t} + C_4 e^{-c\mu t}\right\} \quad \text{—— (5)}$$

Case-ii) When $\lambda = -\mu^2$

Then solution of ④ becomes

$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$

$T(t) = c_3 \cos c\mu t + c_2 \sin c\mu t$

Now $u(x,t) = X(x) T(t)$ will be

$$u(x,t) = \{c_1 \cos \mu x + c_2 \sin \mu x\}\{c_3 \cos c\mu t + c_4 \sin c\mu t\} \quad \text{⑥}$$

Case-iii) When $\lambda = 0$

Then solutions of ④ are

$X(x) = c_1 x + c_2$ and $T(t) = c_3 t + c_4$

Hence $u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$ ⑦

The above three equations ⑤, ⑥ and ⑦ are the solutions of equation ①. but since ① is wave equation so we have to choose the solution among these three which is consistent with physical nature of problem. Since string is vibrating so we must have to choose the solution which is periodic. In eqn ⑥ u is periodic function of x and t, so ⑥ is appropriate solution.

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Hence solution will be

$$\boxed{u(x,t) = \{c_1 \cos \mu x + c_2 \sin \mu x\}\{c_3 \cos \mu ct + c_4 \sin \mu ct\}}$$

Now we will use boundary conditions

$u(0,t) = u(l,t) = 0$

$c_1 \{c_3 \cos c\mu t + c_4 \sin c\mu t\} = 0$

$\because$ $c_3 \cos c\mu t + c_4 \sin c\mu t \neq 0$ because if this expression is zero, then $u(x,t)$ will be zero, so $c_1 = 0$

Now $u(x,t) = c_2 \sin \mu x \{c_3 \cos c\mu t + c_4 \sin c\mu t\}$

Again $u(l,t) = 0$

$c_2 \sin \mu l \{c_3 \cos \mu ct + c_4 \sin \mu ct\} = 0$

$c_2 \neq 0$ and $\{c_3 \cos \mu ct + c_4 \sin \mu ct\} \neq 0$

$\Rightarrow \sin \mu l = 0 = \sin k\pi$

$\Rightarrow \mu = \frac{k\pi}{l}$ $\quad k \in I$

Hence solution will be

$u(x,t) = c_2 \sin \frac{k\pi}{l} x \left\{ c_3 \cos \frac{k\pi ct}{l} + c_4 \sin \frac{k\pi ct}{l} \right\}$

$u(x,t) = c_2 \left\{ c_3 \cos \frac{k\pi ct}{l} + c_4 \sin \frac{k\pi ct}{l} \right\} \sin \frac{k\pi}{l} x$

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$$u(x,t) = \left\{ C_2C_3 \cos\frac{k\pi ct}{l} + C_2C_4 \sin\frac{k\pi ct}{l} \right\} \sin\frac{k\pi x}{l}$$

For writing most general solution of (1) we will take $C_2C_3 = a_k$ and $C_2C_4 = b_k$ and hence

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \left\{ a_k \cos\frac{k\pi ct}{l} + b_k \sin\frac{k\pi ct}{l} \right\} \sin\frac{k\pi x}{l}} \quad \text{---} (9)$$

Now we will apply intial conditions of (2)

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t} = 0 \quad \text{at } t=0$$

$$f(x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi x}{l} \quad \text{---} (10)$$

and

$$0 = \left[\sum_{k=1}^{\infty} \left\{ -a_n \sin\frac{k\pi ct}{l} \times \frac{k\pi c}{l} + b_n \frac{k\pi c}{l} \cos\frac{k\pi ct}{l} \right\} \sin\frac{k\pi x}{l} \right]_{t=0}$$

$$0 = \sum_{k=1}^{\infty} b_k \frac{k\pi c}{l} \sin\frac{k\pi x}{l}$$

$$\Rightarrow b_k = 0$$

Hence $\boxed{u(x,t) = \sum_{k=1}^{\infty} a_k \cos\frac{k\pi ct}{l} \sin\frac{k\pi x}{l}}$ — (11)

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Now from (10)

$$f(x) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{l}x$$

Multiplying both side by $\sin\frac{k\pi}{l}x$ and integrating between 0 to x.

$$\sin\frac{k\pi}{l}x \cdot f(x) = \sum_{n=1}^{\infty} a_k \sin^2\frac{k\pi}{l}x$$

$$\quad " \quad = \frac{1}{2}\sum_{k=1}^{\infty} a_k\left(1 - \cos\frac{2k\pi x}{l}\right)$$

Now on integration

$$\int_0^l \sin\frac{k\pi}{l}x\, f(x)\,dx = \frac{1}{2}\sum_{k=1}^{\infty} a_k \int_0^l \left(1 - \cos\frac{2k\pi x}{l}\right)dx$$

$$" = \frac{1}{2}\sum_{k=1}^{\infty} a_k \left\{x - \sin\frac{2k\pi x}{l} \times \frac{l}{2k\pi}\right\}_0^l$$

$$" = \frac{1}{2}\sum_{k=1}^{\infty} a_k \left\{l\right\}$$

$$\int_0^l f(x)\sin\frac{k\pi x}{l}\,dx = \sum_{k=1}^{\infty} a_k \frac{l}{2} = \frac{l}{2}\sum_{k=1}^{\infty} a_k$$

$$\Rightarrow \boxed{a_k = \frac{2}{l}\int_0^l f(x)\sin\frac{k\pi x}{l}\,dx} \quad \text{--- (12)}$$

Solution is

$$\boxed{u(x,t) = \sum a_k \cos\frac{k\pi ct}{l}\sin\frac{k\pi}{l}x} \quad \text{--- (13)}$$

where a_k is given by (12)

Solution of two-dimensional wave equation by the method of separation of variables –

The equation of wave produced in a rectangular membrane stretched in

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} \quad \text{---} (1)$$

where c is constant.

Let it's solution be

$$u(x,y,t) = X(x)Y(y)T(t) \quad \text{---} (2)$$

Differentiating (2) partially w.r.s. to x, y and t

$$\frac{\partial^2 u}{\partial t^2} = XY\frac{d^2T}{dt^2} = XYT''$$

$$\frac{\partial^2 u}{\partial x^2} = YT\frac{d^2X}{dx^2} = YTX''$$

$$\frac{\partial^2 u}{\partial y^2} = XT\frac{d^2Y}{dy^2} = XYY''$$

$$XYT'' = c^2\left\{YTX'' + XTY''\right\}$$

$$\frac{1}{T}T'' = c^2\left\{\frac{1}{X}X'' + \frac{1}{Y}Y''\right\}$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{X}X'' + \frac{1}{Y}Y'' = -l^2 = -m^2 - n^2$$

$$\begin{matrix} \| & & \| & & \| \\ -l^2 & = & -m^2 & - & -n^2 \end{matrix}$$

(3)

[discussed while solving heat equation of 2-dimension in previous lecture]

Now

$T'' + l^2c^2T = 0 \Rightarrow C_5 \cos lct + C_6 \sin lct$

$X'' + m^2X = 0 \Rightarrow C_1 \cos mx + C_2 \sin mx$

$Y'' + n^2Y = 0 \Rightarrow C_3 \cos ny + C_4 \sin ny$

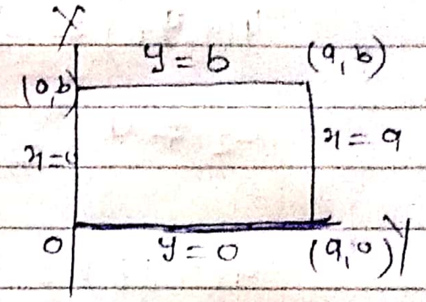
Hence $u(x,y,t) = XYT$ becomes

$u(x,y,t)$
$= \{C_1 \cos mx + C_2 \sin mx\}\{C_3 \cos ny + C_4 \sin ny\}\{C_5 \cos lct + C_6 \sin lct\}$ (4)

$$\boxed{u(x,y,t) = \{C_1 \cos mx + C_2 \sin mx\}\{C_3 \cos ny + C_4 \sin ny\}\{C_5 \cos \sqrt{m^2+n^2}\, ct + C_6 \sin \sqrt{m^2+n^2}\, ct\}}$$

(5)

Now let us suppose that membrance is stretched between the lines $x=0$, $x=a$, $y=0$ and $y=b$.



The boundary conditions are

i) $u(0,y,t) = 0 \quad (x=0)$

ii) $u(a,y,t) = 0 \quad (x=a)$

iii) $u(x,0,t) = 0 \quad (y=0)$

iv) $u(x,b,t) = 0 \quad (y=b)$ for every t in (4)

Applying the condition i), in (4)

$$u(0,y,t) = c_1\{c_3 \cos ny + c_4 \sin ny\}\{c_5 \cos lct + c_6 \sin lct\}$$

$$= 0$$

$\Rightarrow c_1 = 0$

So

$$u(x,y,t) = c_2 \sin mx \{c_3 \cos ny + c_4 \sin ny\}\{c_5 \cos lct + c_6 \sin lct\} \quad \text{(5)}$$

Applying condition ii) in (5)

$u(a,y,t) = 0$

$\Rightarrow c_2 \sin ma \{c_3 \cos ny + c_4 \sin ny\}\{c_5 \cos lct + c_6 \sin lct\} = 0$

$\Rightarrow c_2 \neq 0$ also $\{c_3 \cos ny + c_4 \sin ny\} \neq 0$ and $\{c_5 \cos lct + c_6 \sin lct\} \neq 0$ because if these are zero then $u(x,y,t) = 0$

Hence $\sin ma = 0 = \sin k\pi \quad k \in I$

$$\Rightarrow m = \frac{k\pi}{a}$$

Applying condition iii) in (5) and putting $m = \frac{k_1\pi}{a}$

$u(x,0,t) = C_2 \sin\frac{k_1\pi x}{a}\{C_3\}\{C_5\cos lct + C_6 \sin lct\} = 0$

$C_2 \neq 0 \Rightarrow C_3 = 0$

$u(x,y,t) = C_2C_4 \sin\frac{k_1\pi x}{a} \sin ny\{C_5 \cos lct + C_6 \sin lct\}$

Again applying condition iv) in above

$u(x,b,t) = 0$

$\Rightarrow C_2C_4 \sin\frac{k_1\pi x}{a} \sin nb\{C_5 \cos lct + C_6 \sin lct\} = 0$

$C_2 \neq 0 \quad C_4 \neq 0$ and $\{C_5 \cos lct + C_6 \sin lct\} \neq 0$

$\Rightarrow \sin nb = 0 = \sin k_2\pi \quad k_2 \in I$

$\Rightarrow n = \frac{k_2\pi}{b}$

Hence

$u(x,y,t) = C_2C_4 \sin\frac{k_1\pi x}{a} \sin\frac{k_2\pi y}{b}\{C_5 \cos lct + C_6 \sin lct\}$ (6)

~~For general solution put~~ $C_5C_2C_4 = A_{k_1k_2}$

and $C_2C_4C_6 = B_{k_1k_2}$ and

$k_1, k_2 \in I^+$

$l = \sqrt{m^2+n^2}$

$\Rightarrow l = \pi\sqrt{\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}}$

$p = \pi c\sqrt{\frac{k_1^2}{a^2}+\frac{k_2^2}{b^2}}$

Putting these values in above equation we get

$$\boxed{u(x,y,t)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}\left\{A_{k_1k_2}\cos pt+B_{k_1k_2}\sin pt\right\}} \quad \text{—— (7)}$$

Let us suppose that membrane starts from the rest with intial position $u(x,y,t)=f(x,y)_{t=0}$ i.e $u(x,y,0)=f(x,y)$

Also $\dfrac{\partial u(x,y,0)}{\partial t}=0$ ($\because$ string is intially at rest)

Using these in above

$$\left(\frac{\partial u}{\partial t}\right)_{t=0}=\left[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}\left\{-pA_{k_1k_2}\sin pt+pB_{k_1k_2}\cos pt\right\}\right]_{t=0}=0$$

$$\Rightarrow \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}\,p\,B_{k_1k_2}=0$$

$$\Rightarrow B_{k_1k_2}=0$$

So $$\boxed{u(x,y,t)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}A_{k_1k_2}\cos pt\,\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}} \quad \text{—— (8)}$$

Again $u(x,y,0)=f(x,y)$

$$\boxed{f(x,y)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}A_{k_1k_2}\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}} \quad \text{—— (9)}$$

Multiplying both sides by $\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}$

$$f(x)\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b} = \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} A_{k_1k_2}\sin^2\frac{k_1\pi x}{a}\sin^2\frac{k_2\pi y}{b}$$

$$= \frac{1}{4}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} A_{k_1k_2}\left\{1-\cos\frac{2k_1\pi x}{a}\right\}\left\{1-\cos\frac{2k_2\pi y}{b}\right\}$$

Integrate above between the limits 0 to a for x and 0 to b for y

$$\int_0^a\int_0^b f(x)\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}\,dx\,dy =$$

$$\frac{1}{4}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\int_0^a\int_0^b A_{k_1k_2}\left\{1-\cos\frac{2k_2\pi y}{b}-\cos\frac{2k_1\pi x}{a}+\cos\frac{2k_1\pi x}{a}\cos\frac{2k_2\pi y}{b}\right\}dx\,dy$$

$$= \frac{1}{4}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} A_{k_1k_2}\left\{\int_0^a\int_0^b dx\,dy\right\} - \int_0^a\int_0^b\cos\frac{2k_2\pi y}{b}\,dx\,dy$$

$$- \int_0^a\int_0^b\cos\frac{2k_1\pi x}{a}\,dx\,dy + \int_0^a\int_0^b\cos\frac{2k_1\pi x}{a}\cos\frac{2k_2\pi y}{b}\,dx\,dy$$

$$= \frac{1}{4}\sum\sum A_{k_1k_2}\left\{ab - \int_0^a\left\{\sin\frac{2k_2\pi y}{b}\right\}_0^b dx - \int_0^b\left\{\sin\frac{2k_1\pi x}{a}\right\}_0^a\right.$$

$$+ \int_0^a\cos\frac{2k_1\pi x}{a}\left\{\sin\frac{2k_2\pi y}{b}\right\}_0^b dx$$

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$$= \frac{1}{4}\sum\sum A_{k_1k_2}\, ab - 0 - 0 + \int_0^a 0\, dx$$

$$\int_0^a\int_0^b f(x)\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi}{b}y\, dx\, dy = \frac{1}{4}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} A_{k_1k_2}\, ab$$

On comparison

$$\boxed{A_{k_1k_2} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi y}{b}\, dx\, dy} \quad \text{—— (10)}$$

solution is

$$\boxed{u(x,y,t) = \sum\sum A_{k_1k_2}\sin\frac{k_1\pi x}{a}\sin\frac{k_2\pi}{b}y\cos pt}$$

where $A_{k_1k_2}$ given above (10) and —— (11)

$$p = \pi c\sqrt{\frac{k_1^2}{a^2} + \frac{k_2^2}{b^2}}$$

Illustrative questions based on 1-dimensional and 2-dimensional wave equations —

Ques:- A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in rest position given by $u = u_0 \sin^3 \frac{\pi x}{l}$. If string is released from rest position, find the displacement $u(x,t)$.

Sol^n Since we have to find displacement when string is stretched and then released, so we will use wave equation.

We have 1-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} (1) \, ,$$

with boundary conditions

$$u(0,t) = u(l,t) = 0$$

and initial condition

$$\left.\begin{aligned} &u(0,t) = u(l,t) = 0 \\ &\frac{\partial u(x,0)}{\partial t} = 0 \quad \text{and} \quad u(x,0) = u_0 \sin^3 \frac{\pi x}{l} \end{aligned}\right\} \text{——} (2)$$

Solution of (1) will be

$$u(x,t) = \{C_1 \cos \mu x + C_2 \sin \mu x\}\{C_3 \cos \mu c t + C_4 \sin \mu c t\}$$

{This has proved in eq^n (8) of 1-dimensional wave eq^n.}

Using conditions $u(0,t)=0$

$$c_1\{c_3\cos\mu ct + c_4 \sin\mu ct\}=0 \Rightarrow c_1=0$$

Hence

$$u(x,t) = c_2 \sin\mu x\{c_3\cos\mu ct + c_4\sin\mu ct\} \quad \text{--- (3)}$$

Again
$u(l,t)=0$

$$0 = c_2 \sin\mu l\{c_3\cos\mu ct + c_4\sin\mu ct\}$$

$$c_2 \neq 0 \text{ and } \{c_3\cos\mu ct + c_4\sin\mu ct\} \neq 0$$

$$\Rightarrow \sin\mu l = 0 = \sin n\pi$$

$$\Rightarrow \mu = \frac{n\pi}{l}, \quad n\in I^+ \text{ (proved in previous lectures)}$$

Hence

$$u(x,t) = \{c_2c_3 \sin\mu l \cos\mu ct + c_2c_4\sin\mu l \sin\mu ct\}\sin\mu x \quad \text{--- (4)}$$

Using above

$$u(x,t) = \left\{c_2c_3\left\{\cos\frac{n\pi ct}{l} + c_2c_4\sin\frac{n\pi ct}{l}\right\}\right\}\sin\frac{n\pi x}{l}$$

$$\frac{\partial u}{\partial t} = \frac{n\pi c}{l}c_2\left\{-c_3\sin\frac{n\pi ct}{l} + c_4\cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{at } t=0$$

$$0 = c_2c_4\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

$c_2 \neq 0$ already $\Rightarrow c_4 = 0$

Hence solution will be

$$u(x,t) = c_2 c_3 \cos\frac{m\pi c t}{l} \sin\frac{m\pi x}{l} \quad \text{---} (5)$$

For more general

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi c t}{l} \sin\frac{n\pi x}{l} \quad \text{---} (6)$$

Again $u(x,0) = u_0 \sin^3\frac{\pi x}{l}$

Using it in above

$$u_0 \sin^3\frac{\pi x}{l} = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\because \sin\frac{3\pi x}{l} = 3\sin\frac{\pi x}{l} - 4\sin^3\frac{\pi x}{l}$$

From above

$$\frac{u_0}{4}\left\{3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{3u_0}{4}\sin\frac{\pi x}{l} - \frac{u_0}{4}\sin\frac{3\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \cdots$$

On comparison

$b_1 = \frac{340}{4}$, $b_3 = -\frac{40}{4}$, $b_2 = b_4 = b_5 = \cdots = 0$

Hence solution is (from ⑥)

$$\boxed{u(x,t) = \frac{340}{4} \sin\frac{\pi x}{l} \cos\frac{\pi c t}{l} - \frac{40}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi c t}{l}}$$

Ans.

Ques. If a string of length l is intially at rest in equilibrium position and each of points is given velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, find the displacement $y(x,t)$.

Ques. If tightly stretched string with fixed ends points $x=0$ and $x=\pi$ is intially at rest in its equilibrium position. If it is set vibrating by giving to each of it's points an intial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$

then find the displacement at any point of string at any time t.

Note- Boundary conditions are same in the above two questions according as in question ①, only intial conditions are changed, which is clearly given.

Question - Find the deflection $u(x,y,t)$ of the square membrane with $a=b=1$, if the intial velocity is zero and the intial deflection $f(x,y) = A \sin \pi x \sin 2\pi y$.

Sol^n^ - The vibration of square membrane is governed by two - dimensional wave eq^n^

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} \quad \text{—— (1)}$$

Here the boundary conditions are

$$u(0,y,t) = u(1,y,t) = 0$$
$$u(x,0,t) = u(x,1,t) = 0$$

Intial conditions are

$$u(x,y,0) = A \sin \pi x \sin 2\pi y \quad \text{and}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{(Given that intial velocity is zero)}$$

Solution of (1) will be

$$u(x,y,t) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} A_{k_1 k_2} \sin \frac{k_1 \pi x}{a} \sin \frac{k_2 \pi y}{b} \cos pt \quad \text{—— (2)}$$

where $p = \pi c \sqrt{\frac{k_1^2}{a^2} + \frac{k_2^2}{b^2}}$

and

$$A_{k_1 k_2} = \frac{4}{ab}\int_0^a \int_0^b f(x,y) \sin\frac{k_1 \pi x}{a} \sin\frac{k_2\pi y}{b}\, dx\, dy \quad \text{--- (3)}$$

Here $a=b=1$ and $f(x,y) = A \sin\pi x \sin 2\pi y$.

So, $p = \pi c\sqrt{k_1^2+k_2^2}$, Also

$$A_{k_1 k_2} = 4\int_0^1\int_0^1 A \sin\pi x \sin 2\pi y \sin k_1\pi x \sin k_2\pi y\, dx dy \quad \text{--- (4)}$$

$$= 4\left\{\int_0^1 A\sin\pi x \sin k_1\pi x \int_0^1 \frac{2\sin 2\pi y \sin k_2\pi y}{2}\, dy\right\} dx$$

$$= \frac{4}{2}\left\{\int_0^1 A \sin\pi x \sin k_1\pi x \int_0^1 [\cos(k_2-2)\pi y - \cos(k_2+2)\pi y] dy\right\} dx$$

$$= 2A\int_0^1 \sin\pi x \sin k_1\pi x \left[\frac{\sin(k_2-2)\pi y}{k_2-2} - \frac{\sin(k_2+2)\pi y}{k_2+2}\right]_0^1 dx$$

$= 0$ when $k_2 \neq 2$

$A_{k_1,1} = A_{k_1,2} = A_{k_1,3} = \cdots = 0$ except $k_2 = 2$

So we will separately find $A_{k_1,2}$ from (4)

$$A_{k_1 2} = 4A\int_0^1\int_0^1 \sin\pi x \sin 2\pi y \sin k_1\pi x \sin 2\pi y\, dx dy$$

$$= 2A\int_0^1 \sin\pi x \sin k_1\pi x\, 2\sin^2 2\pi y\, dx\, dy$$

$$= 2A \int_0^1 \int_0^1 \sin \pi x \sin k_1 \pi x \,(1 - \cos 4\pi y)\, dx\, dy$$

$$= 2A \left\{ \int_0^1 \sin \pi x \sin k_1 \pi x \int_0^1 (1 - \cos 4\pi y)\, dy \right\} dx$$

$$= 2A \left\{ \int_0^1 \sin \pi x \sin k_1 \pi x \left\{ y - \frac{\sin 4\pi y}{4\pi} \right\}_0^1 dx \right.$$

$$= 2A \left\{ \int_0^1 \sin \pi x \sin k_1 \pi x \, dx \right.$$

$$= A \int_0^1 2 \sin \pi x \sin k_1 \pi x \, dx$$

$$= A \int_0^1 \left\{ \cos (k_1 - 1)\pi x - \cos (k_1 + 1)\pi x \right\} dx$$

$$= A \left[\frac{\sin (k_1 - 1)\pi x}{k_1 - 1} - \frac{\sin (k_1 + 1)\pi x}{k_1 + 1} \right]_0^1$$

$= 0$ excep $k_1 \neq 1$

$A_{21} = A_{22} = A_{32} = \text{----} = 0$

So we will separately find A_{12} from (4)

$$A_{12} = 4A \int_0^1 \int_0^1 \sin \pi x \sin 2\pi y \sin \pi x \sin 2\pi y \, dx\, dy$$

$$A_{12} = A\left\{\int_0^1 2\sin^2\pi x \int_0^1 2\sin^2 2\pi y \, dy\right\} dx$$

$$= A\left\{\int_0^1 2\sin^2\pi x \int_0^1 (1-\cos 4\pi y)\, dy\right\} dx$$

$$= A\int_0^1 2\sin^2\pi x \left\{ y - \frac{\sin 4\pi y}{4\pi}\right\}_0^1 dx$$

$$= A\int_0^1 2\sin^2\pi x \, dx = \int_0^1 (1-\cos 2\pi x)\, dx$$

$$= A\left[x - \frac{\sin 2\pi x}{2\pi}\right]_0^1$$

$$A_{12} = A$$

Hence from (2)

$$u(x,y,t) = A_{11}\sin\pi x \sin\pi y \cos pt + A_{12}\sin\pi x \sin 2\pi y \cos pt$$
$$+ \cdots\cdots + A_{21}\sin 2\pi x \sin\pi y \cos pt$$
$$+ A_{22}\sin 2\pi x \sin 2\pi y \cos pt + \cdots$$

all $A_{k_1k_2} = 0$ except $A_{12} = A$ (non-zero)

$u(x,y,t) = A\sin\pi x \sin 2\pi y \cos pt$, where $p = \pi c\sqrt{k_1^2+k_2^2}$

$$\boxed{u(x,y,t) = A\cos\pi c\sqrt{5}t \sin\pi x \sin 2\pi y}$$

$p = \pi c\sqrt{1+4}$
$= \pi c\sqrt{5}$

Note Equation (11) of two dimensional wave equation is used only when boundary and intial conditions are same as given in this derivation. Since in ques (4) boundary and intial conditions are same as taken in derivation of wave eq^n so we have use directly the solution.

Question. A tightly stretched unit square membrane starts vibrating from rest and its intial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is

$$k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\pi c t).$$

Questions - Solve by separation of variables method

i) $4\dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial x} = 3z$, $z = 3e^{-x} - e^{-5x}$ at $t = 0$

ii) $\dfrac{\partial z}{\partial x} = 4\dfrac{\partial z}{\partial y}$, $z(0, y) = 8e^{-3y}$

iii) $\dfrac{\partial z}{\partial t} + z\dfrac{\partial z}{\partial x} = 0$, $z(x,0) = \alpha + \beta x$. If $z(x,t) = 1$ along characteristic $x = t+1$. Then

a) $\alpha = 1, \beta = 1$ b) $\alpha = 2, \beta = 0$ c) $\alpha = 0, \beta = 0$

d) $\alpha = 0, \beta = 1$

iv) Solve $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

v) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$

vi) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = 0$