LAPLACE TRANSFORM

Definition: Let $f(t)$ be function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Provided the integral exists; is called Laplace Transform of $f(t)$.

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) \cdot dt}$$

Sufficient Condition- For Existence of Laplace Transform.

1)- $f(t)$ should be Continuous, or piecewise Continuous on the sub-interval of $(0,\infty)$

2) $f(t)$ should be exponential order.

i.e. $|f(t)| \leq Me^{\alpha t}$

where, $\alpha > 0$ is known as exponential Order

i.e $\lim\limits_{t\to\infty} \left[F(t)\, e^{-\alpha t}\right] \longrightarrow$ finite for $t > \alpha$

and $f(t)$ is Continuous then Laplace Transform of $f(t)$ $\int_0^\infty e^{-st} f(t)\, dt$ exist.

Note:- Above Condition are Sufficient not necessary.

Important Example:

(1) $L\{1\} = \frac{1}{s}$

(2) $L\{e^{at}\} = \frac{1}{s-a}$: $(s>a)$

(3) $L\{t^n\} = \frac{n!}{s^{n+1}}$ $n \in \mathbb{N} \cup \{0\}$

(4) $L\{\sin at\} = \frac{a}{s^2+a^2}$: $(s>0)$

(5) $L\{\cos at\} = \frac{s}{s^2+a^2}$: $(s>0)$

(6) $L\{\sinh at\} = \frac{a}{s^2+a^2}$ $(s>0)$

(7) $L\{\cosh at\} = \frac{s}{s^2-a^2}$ $(s>0)$

Proof of Some.

Sol^n:

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

or $$= \int_0^\infty e^{-st} \cdot t^n\, dt$$

Now put
$st = x$.
$s\,dt = dx$
$t = 0 \Rightarrow x = 0$
$t = \infty \Rightarrow x = \infty$

or $$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

or $$= \int_0^\infty \frac{e^{-x} \cdot x^n}{s^n} \cdot \frac{dx}{s}$$

or $$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

Gamma function -

$$\because \Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}$$

or, $$= \frac{1}{s^{n+1}} \cdot \Gamma(n+1)$$

and, $\Gamma(n+1) = n!$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

2). $L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at}\, dt$ $\qquad s>a$

$= \int_0^\infty e^{-(s-a)t}\, dt$

Put
$(s-a).t = x$
$(s-a)\, dt = dx$

$= \int_0^\infty e^{-x} \frac{dx}{(s-a)}$

$t=0 \Rightarrow x=0$
$t=\infty \Rightarrow x=\infty$

$= \frac{1}{s-a}\left[\frac{e^{-x}}{-1}\right]_0^\infty$

$e^{-\infty} \to 0$
if $s-a>0$

$= \frac{1}{s-a}\left[0+1\right]$

a. $\boxed{L\{e^{at}\} = \frac{1}{s-a}}$ $\quad s>a$

if $s<a$
then, $e^{-(s-a)t} \to e^{+(n)t}$
as $t\to\infty$, $e^{-(s-a)t} \to \infty$
then Laplace doesn't exist.

Similarly, we can find.

$\sinh(at) = \frac{e^{at}-e^{-at}}{2}$ $\qquad \cosh(at) = \frac{e^{at}+e^{-at}}{2}$

$\sin(at) = \frac{e^{iat}-e^{-iat}}{2i}$ $\qquad \cos(at) = \frac{e^{iat}+e^{-iat}}{2}$

$i^2 = -1$

change in above formulae and
use $L\{e^{at}\} = \frac{1}{s-a}$, we can find the Result.

Do Yourself.

Exp:.

Find Laplace Transform of-

$$f(t) = \begin{cases} t & 1 < t < 2 \\ 4-t & 2 < t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Sol:. From the definition of Laplace-

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st}.f(t)\,dt$$

or $$F(s) = \int_0^1 e^{-st}.f(t)dt + \int_1^2 e^{-st}.f(t)dt + \int_2^3 e^{-st}.f(t)dt + \int_3^\infty e^{-st}.f(t)dt$$

or $$F(s) = 0 + \int_1^2 e^{-st}.t\,dt + \int_2^3 e^{-st}.(4-t)\,dt + 0$$

or $$F(s) = \int_1^2 e^{-st}.t\,dt + \int_2^3 (4-t).e^{-st}\,dt$$

By Using Integral By-Part.

or $$F(s) = \left[\frac{t.e^{-st}}{-s} - \int \frac{e^{-st}}{-s}dt\right]_1^2 + \left[\frac{(4-t)e^{-st}}{-s} - \int \frac{-1.e^{-st}}{-s}dt\right]_2^3$$

or $$F(s) = \left[\frac{te^{-st}}{-s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\right]_1^2 + \left[\frac{(4-t)e^{-st}}{-s} - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\right]_2^3$$

$$= \left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - \frac{1.e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right]$$

or $F(s) = \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2 + \left[\frac{(t-1)e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_2^3$

or $F(s) = \left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right] + \left[\frac{e^{-3s}}{-s} + \frac{e^{-3s}}{s^2} + \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right]$

$$F(s) = \frac{-2e^{-2s}}{s^2} + \left(\frac{1}{s}+\frac{1}{s^2}\right)e^{-s} + \left(\frac{1}{s^2}-\frac{1}{s}\right)e^{-3s}$$

$$\boxed{F(s) = \frac{(s+1)}{s^2}.e^{-s} - \frac{2}{s^2}e^{-2s} + \frac{(1-s)}{s^2}e^{-3s}}$$

Exercise:

1). $f(t) = \begin{cases} \cos 3t & 0 < t < 2 \\ 0 & t > 2 \end{cases}$

2) $f(t) = \begin{cases} 4 & 0 < t < 2 \\ 2t & t > 2 \end{cases}$

Properties of Laplace Transform.

(1) $L\{a f_1(t) \pm b f_2(t)\} = a L\{f_1(t)\} \pm b L\{f_2(t)\}$

$\Rightarrow \int_0^\infty e^{-st}[a f_1(t) \pm b f_2(t)]dt = a\int_0^\infty e^{-st} f_1(t)\,dt \pm b\int_0^\infty e^{-st} f_2(t)\,dt$

↳ Linear Property.

(2) Change of Scale Property

If $L\{f(t)\} = F(s)$ then $\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

$\Rightarrow L\{f(at)\} = \int_0^\infty e^{-st} \cdot f(at)\,dt$,

$at = u$.
$a\,dt = du$
$t=0 \Rightarrow u=0$
$t=\infty \Rightarrow u=\infty$

$= \int_0^\infty e^{-\frac{su}{a}} \cdot f(u)\,\frac{du}{a}$

$= \frac{1}{a}\int_0^\infty e^{-Su} \cdot F(u)\,du$

$S = \frac{s}{a}$.

$= \frac{1}{a}F(S) = \frac{1}{a}F\left(\frac{s}{a}\right)$.

so, $\boxed{L\{f(at\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

Exp:- $L\{(6t)^4\} = ?$

as, $L\{t^4\} = \frac{4!}{s^5}$ then, $L\{(6t)^4\} = \frac{1}{6}\cdot\frac{4!}{\left(\frac{s}{6}\right)^5} = \frac{6^5 \cdot 4!}{6 \cdot s^5} = \frac{6^4 \cdot 4!}{s^5}$

(3) If $f(t)$ is defined for $t>0$ and $f(t)$ is sectionally continuous and of exponetial order then $F(s) \longrightarrow 0$ as $s \to \infty$

(4) **First shifting property:-**

if $L\{f(t)\} = F(s)$

then $L\{e^{at} f(t)\} = F(s-a)$

$$\Rightarrow \quad L\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} \cdot f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot f(t)\, dt$$

$$= F(s-a).$$

Exp:- $L\{e^{2t} \cdot \sin 4t\}$.

as, $L\{\sin 4t\} = \dfrac{4}{s^2+(4)^2} = \dfrac{4}{s^2+16}$

so, by shifting property -

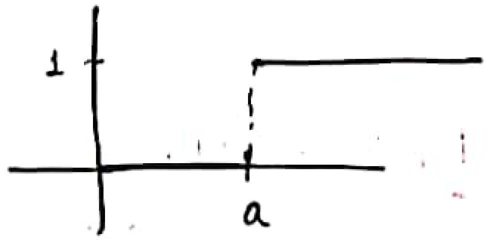
$$L\{e^{2t} \sin 4t\} = \frac{4}{(s-2)^2+16}.$$

5 **Unit Step function (Heaviside's).**

$$U(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$

or.

$$U(t-a) = \begin{cases} 0 & 0<t<a \\ 1 & t \geq a \end{cases}$$



Now, $\boxed{L\{U(t-a)\} = \dfrac{e^{-sa}}{s}}$

sol

$$L\{U(t-a)\} = \int_0^\infty e^{-st}.\, U(t-a) \cdot dt$$

$$= \int_0^a e^{-st} \times 0 \times dt + \int_a^\infty e^{-st}.\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \left[0 + \frac{e^{-as}}{s}\right] = \frac{e^{-as}}{s}$$

* 6 .

If $L\{f(t)\} = F(s)$ (Second shifting Property)

and $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & 0<t<a \end{cases}$

then, $\boxed{L\{g(t)\} = e^{-as}.\, F(s)}$

Initial and Final Value Theorem -

(a) Initial Value Theorem:-

If $L\{f(t)\} = F(s)$

then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} [sF(s)]$ Provided the limit exist

Proof: As we known that -

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

$$\lim_{s\to\infty} \int_0^\infty e^{-st}.f'(t)\,dt = \lim_{s\to\infty} [sF(s) - f(0)]$$

or $$\int_0^\infty \left(\lim_{s\to\infty} e^{-st}\right) f'(t)\,dt = \lim_{s\to\infty} [sF(s)] - f(0)$$

$$0 = \lim_{s\to\infty} [sF(s)] - f(0)$$

or, $$\lim_{s\to\infty} [sF(s)] = f(0) = \lim_{t\to 0} f(t).$$

$$\Rightarrow \boxed{\lim_{s\to\infty} [sF(s)] = \lim_{t\to 0} f(t).}$$

2) Final Value Theorem:-

$$L\{f(t)\} = F(s)$$

$$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$$ Provided the limit exists

So

$$L\{f'(t)\} = sF(s) - f(0)$$

$$\therefore \int_0^\infty e^{-st} \cdot f'(t)\, dt = sF(s) - f(0)$$

$$\lim_{s\to 0} \int_0^\infty e^{-st} \cdot f'(t)\, dt = \lim_{s\to 0} [sF(s) - f(0)]$$

or $$\int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) \cdot f'(t)\, dt = \lim_{s\to 0}[sF(s)] - f(0)$$

or $$\int_0^\infty f'(t)\, dt = \lim_{s\to 0}[sF(s)] - f(0)$$

$$\Rightarrow [f(t)]_0^\infty = \lim_{s\to 0}[sF(s)] - f(0)$$

$$\Rightarrow \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0}[sF(s)] - f(0)$$

$$\Rightarrow \boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0}[s F(s)].}$$

Laplace Transform of Error function

Error function is defined as -

$$\text{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx.$$

then,

$$\boxed{L\{\text{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}} = F(s)$$

Now,

$$L\{\text{erf}\, a\sqrt{t}\} = L\{\text{erf}\sqrt{a^2 t}\} = \frac{1}{a^2} F\left(\frac{s}{a^2}\right) =$$

Exp. Find $L\{\text{erf}\, 2\sqrt{t}\}$

Since,

$$L\{\text{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}} = F(s)$$

$$= L\{\text{erf}\sqrt{4t}\}$$

$$= \frac{1}{4} \cdot \frac{1}{\left(\frac{s}{4}\right)\sqrt{\frac{s}{4}+1}}$$ by change of Scale Property.

$$= \frac{1}{4} \times \frac{4 \times 2}{s\sqrt{s+4}} = \frac{2}{s\sqrt{s+4}}$$

Laplace Transform of Bessel function $J_0(x)$ and $J_1(x)$

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

and, $L\{J_0(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{\frac{s^2}{a^2}+1}}$$

$$= \frac{1}{a}\cdot\frac{a}{\sqrt{s^2+a^2}}$$

so, $L\{J_0(at)\} = \dfrac{1}{\sqrt{s^2+a^2}}$

$$L\{J_1(x)\} = -L\{J_0'(x)\} = -\left[sL\{J_0(x)\} - J_0(0)\right]$$

$$L\{J_1(x)\} = -\left[\frac{s}{\sqrt{s^2+1}} - 1\right]$$

$$\boxed{L\{J_1(x)\} = 1 - \frac{s}{\sqrt{s^2+1}}}$$

$$L\{J_1(ax)\} = \left(1 - \frac{s/a}{\sqrt{\frac{s^2}{a^2}+1}}\right)\cdot\frac{1}{a}$$

$$= \frac{1}{a}\left(1 - \frac{s}{\sqrt{s^2+a^2}}\right)$$

Convolution Theorem :

If $L\{f_1(t)\} = F_1(s)$

and $L\{f_2(t)\} = F_2(s)$

then, $L\left\{\int_0^t f_1(x) \cdot f_2(t-x) \cdot dx\right\} = F_1(s) \cdot F_2(s)$.

Mostly used to find Laplace Inverse. Transformation.

for, Example =

Find Laplace of $\int_0^t e^x \sin(t-x)\, dx$.

Let by Convolution theorem –

$$f_1(x) = e^x$$
$$f_2(x) = \sin x.$$

then, $L\left\{\int_0^t f_1(x) \cdot f_2(t-x)\, dx\right\} = F_1(s) \cdot F_2(s)$

$$\therefore L\left\{\int_0^t e^x \cdot \sin(t-x)\, dx\right\} = L\{e^t\} \cdot L\{\sin x\}$$

$$= \frac{1}{s-1} \cdot \frac{1}{s^2-1}$$

$$= \frac{1}{(s-1)^2(s+1)}$$

Laplace Transform of Derivatives and Integrals. Periodic function, Impulse function.

Laplace Transform of the derivative of f(t):

Let $L\{f(t)\} = F(s)$

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

Proof:- $$L\{f'(t)\} = \int_0^\infty e^{-st}.\,f'(t)\,dt$$

by integration by Part

or $$L\{f'(t)\} = \left[e^{-st}.\,f(t)\right]_0^\infty - \int_0^\infty (-se^{-st}).\,f(t)\,dt$$

or $$L\{f'(t)\} = \left[0 - f(0)\right] + s\int_0^\infty e^{-st}.\,f(t)\,dt$$

$$\boxed{L\{f'(t)\} = s\,L\{f(t)\} - f(0)}$$

In General form:-

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - f^{n-1}(0)$$

i.e. For, n=3.

$$\boxed{L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f(0).}$$

Laplace Transform of Integral of f(t). (15).

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} F(s) \qquad \text{where, } L\{f(t)\} = F(s)$$

Proof:. Let $\phi(t) = \int_0^t f(t)\,dt$ and, $\phi(0) = 0$.

$\phi'(t) = f(t)$

Now,

$$L\{\phi'(t)\} = s L\{\phi(t)\} - \phi(0)$$

or, $$L\{f(t)\} = s L\{\phi(t)\}.$$

or, $$L\{\phi(t)\} = \frac{1}{s} L\{f(t)\}$$

or, $$\boxed{L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s).}$$

Laplace transform of $t^n f(t)$.

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [F(s)].$$

Proof:.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

∴ Differenitiating w.r.to 's'

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\infty \left(\frac{\partial}{\partial s} e^{-st}\right) \cdot f(t)\, dt$$

$$= \int_0^\infty -t e^{-st} f(t)\, dt$$

$$= -\int_0^\infty t \cdot e^{-st} f(t)\, dt$$

$$= -L\{t \cdot f(t)\}.$$

$$\Rightarrow \quad L\{t \cdot f(t)\} = (-1)\frac{d}{ds} F(s)$$

Similarly -

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$L\{t^3 f(t)\} = (-1)^3 \frac{d^3}{ds^3} F(s)$$

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

Laplace Transform of $\frac{1}{t} f(t)$.

$L\{f(t)\} = F(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} F(s)\, ds$.

Proof:-

$$F(s) = \int_0^{\infty} e^{-st} . f(t)\, dt$$

on integrating with respect to 's'

$$\int_s^{\infty} F(s)\, ds = \int_s^{\infty} \left[\int_0^{\infty} e^{-st} f(t)\, dt \right] ds$$

or $$= \int_0^{\infty} \left[\int_s^{\infty} e^{-st}\, ds \right] . f(t)\, dt$$

or $$= \int_0^{\infty} \left[\frac{e^{-st}}{-t} \right]_s^{\infty} . f(t)\, dt$$

or $$= \int_0^{\infty} \frac{-1}{t} \left[e^{-\infty} - e^{-st} \right] f(t)\, dt$$

or, $$= \int_0^{\infty} \frac{1}{t} . e^{-st} . f(t)\, dt = L\left\{\frac{1}{t} f(t)\right\}$$

so, $$\boxed{L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} F(s)\, ds}$$

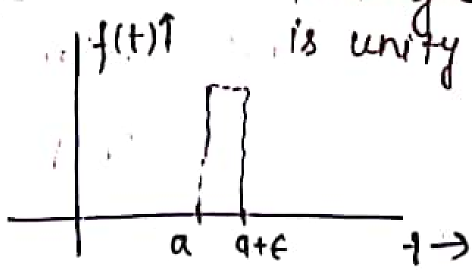
Impulse Function:-

When a large force acts for a short time, then the product of the force and the time is called impulse in applied Mechanics. The unit impulse function is the limiting function -

as $\epsilon \to 0$

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} & a < t < a+\epsilon \\ 0 & \text{otherwise -} \end{cases}$$

• area of rectangle is unity.



Laplace Transform of Unit Impulse function.

$$\int_a^b f(t)\,dt = (b-a)\cdot f(\eta)$$

$, a < \eta < a+\epsilon$
(Mean Value Theorem)

$$\int_0^\infty f(t)\cdot\delta(t-a)\,dt = \int_a^{a+\epsilon} f(t)\cdot\frac{1}{\epsilon}\,dt$$

$$= (a+\epsilon-a)\cdot\frac{1}{\epsilon}f(\eta) = f(\eta).$$

Now, Property =

$$\boxed{\int_0^\infty f(t)\cdot\delta(t-a)\,dt = f(a).}$$

as - $\epsilon \to 0$

then, $\eta \to a$.

$$\text{Sly,}\quad L\{\delta(t-a)\} = \int_0^\infty e^{-st}\cdot\delta(t-a)\,dt = e^{-as}.$$

$f(t) = e^{-st}$.

② Exp:. $L\{t^3\delta(t-2)\}$

$$= \int_0^\infty e^{-st}\cdot t^3 \cdot \delta(t-2)\,dt$$

here, $f(t) = e^{-st}\cdot t^3$

so, $$\int_0^\infty e^{-st}\cdot t^3\,\delta(t-2) = f(2) = e^{-2s}\cdot 2^3 = 8e^{-2s}.$$

Thus, $$\boxed{L\{t^3\delta(t-2)\} = 8e^{-2s}.}$$

Property (2)

$$\boxed{\int_{-\infty}^{\infty} f(t)\cdot\delta'(t-a)\,dt = -f'(a).}$$

The Unit Impulse Function is defined as -

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a. \end{cases}$$

thus, $$\boxed{\int_0^\infty \delta(t-a)\,dt = 1.}$$

Laplace Transform of Periodic function:

Let $f(t)$ be periodic function with period T, then -

$$\frac{\int_0^T e^{-st}.f(t)\,dt}{1-e^{-sT}} = L\{f(t)\} = F(s)$$

Exp: Find Laplace Transform.

$$f(t) = 2t \quad ; \quad \text{Period of } t = 3,$$

Sol: Thus, $T = 3$.

so,
$$L\{f(t)\} = \frac{1}{1-e^{sT}} \cdot \int_0^T e^{-st}.f(t)\,dt$$

or
$$L\{f(t)\} = \frac{1}{1-e^{3s}} \cdot \int_0^3 e^{-st}.2t\,dt$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}}\left[\frac{t e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\right]_0^3$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}}\left[\frac{t e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_0^3.$$

$$L\{f(t)\} = \frac{2}{1-e^{3s}}\left[\frac{3.e^{-3s}}{-s} + \frac{e^{-3s}}{s^2} - 0 - \frac{1}{s^2}\right]$$

[redacted]:

Inverse Laplace Transform

If $F(s)$ is the Laplace Transform of a function $f(t)$.

then $f(t)$ is known as Laplace inverse or Inverse Laplace transform

If $L\{f(t)\} = F(s)$

then, $L^{-1}\{F(s)\} = f(t)$.

Where, L^{-1} is called the inverse Laplace transform operator.

Important formulae:-

1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

3) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

4) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

5) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

6) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

7) $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

8) $L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$
$= e^{at}.f(t)$.

9) $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(t)\,dt$.

10) $L^{-1}\{1\} = \delta(t)$.

11) $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t f(t)$

(12) $L^{-1}\{e^{-as}F(s)\} = f(t-a)\cdot U(t-a)$. (Second shifting property)

(13) $L^{-1}\{e^{-as}\} = \delta(t-a)$

14) $L^{-1}\{sF(s)\} = \frac{d}{dt}f(t) + f(0)\cdot\delta(t)$.

Derivative.

(15) $L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -t\,f(t)$

or, $L^{-1}\{F(s)\} = \frac{-1}{t}L^{-1}\left\{\frac{d}{ds}F(s)\right\}$

Integral.

(16) $L^{-1}\left[\int_s^{\infty}F(s)\,ds\right] = \frac{f(t)}{t}$

$\Rightarrow L^{-1}[F(s)] = t\,L^{-1}\left[\int_s^{\infty}F(s)\,ds\right]$

Inverse Laplace Transform by Convolution.

Since $L\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1(s)\cdot F_2(s)$

or, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(x)\cdot f_2(t-x)\,dx$.

(1) Find Laplace Inverse of—

$$F(s) = \frac{7}{s^2+4}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{7}{s^2+4}\right\}$$

∵ $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$

or, $L\{\sin at\} = \frac{a}{s^2+a^2}$

or, $f(t) = 7L^{-1}\left\{\frac{1}{s^2+4}\right\}$

or, $f(t) = \frac{7}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

or, $\boxed{f(t) = \frac{7}{2}\sin(2t)}$

(2) $F(s) = \frac{5}{s-4} + \frac{7s}{s^2+16}$

$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{5}{s-4} + \frac{7s}{s^2+16}\right\}$$

or $f(t) = 5L^{-1}\left\{\frac{1}{s-4}\right\} + 7L^{-1}\left\{\frac{s}{s^2+16}\right\}$

or $\boxed{f(t) = 5e^{4t} + 7\cos(4t)}$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

(3)

$$L^{-1}\left\{\frac{S+1}{S^2+4s-5}\right\}$$

Partial fraction.

$$\frac{s+1}{s^2+4s-5} = \frac{(s+1)}{(s^2+4s+4-9)}$$

$$= \frac{(s+1)}{(s+2)^2-9}$$

$$= \frac{(s+2)-1}{(s+2)^2-9}$$

$$= L^{-1}\left\{\frac{(S+2)}{(s+2)^2-9} - \frac{1}{(s+2)^2-9}\right\}$$

$$= L^{-1}\left\{\frac{(s+2)}{(s+2)^2-3^2}\right\} - L^{-1}\left\{\frac{1}{(s+2)^2-9}\right\}$$

By shifting Property -

If

$L\{f(t)\} = F(s)$

$L\{e^{at} f(t)\} = F(s-a)$

i.e.

$L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$

$= e^{at} f(t)$.

$$= e^{-2t} L^{-1}\left\{\frac{s}{s^2-3^2}\right\} - e^{-2t} L^{-1}\left\{\frac{1}{s^2-9}\right\}$$

$$= e^{-2t}\cosh(3t) - \frac{e^{-2t}}{3} L^{-1}\left\{\frac{3}{s^2-9}\right\}$$

$$= e^{-2t}\cosh(3t) - \frac{e^{-2t}}{3}\sinh(3t)$$

$$= \frac{e^{-2t}}{3}\left[3\cosh(3t) - \sinh(3t)\right]$$

and,

$L\{\cosh(at)\} = \frac{s}{s^2-a^2}$

$L\{\sinh(at)\} = \frac{a}{s^2-a^2}$.

so,

$$\boxed{L^{-1}\left\{\frac{s+1}{s^2+4s-5}\right\} = \frac{e^{-2t}}{3}\left[3\cosh(3t) - \sinh(3t)\right]}$$

(4) $F(s) = \dfrac{s}{(s-4)^5}$

or, $F(s) = \dfrac{(s-4+4)}{(s-4)^5}$

or $F(s) = \dfrac{(s-4)}{(s-4)^5} + \dfrac{4}{(s-4)^5}$

or $F(s) = \dfrac{1}{(s-4)^4} + \dfrac{4}{(s-4)^5}$

∵ By shifting Property:
If $L^{-1}\{F(s)\} = f(t)$
then, $L^{-1}\{F(s-a)\} = e^{at} \cdot f(t)$.

Taking Laplace inverse -

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s-4)^4}\right\} + 4L^{-1}\left\{\frac{1}{(s-4)^5}\right\}$$

or $F(t) = e^{4t} L^{-1}\left\{\dfrac{1}{s^4}\right\} + 4e^{4t} L^{-1}\left\{\dfrac{1}{s^5}\right\}$

$L^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}$

or $F(t) = e^{4t} \cdot \dfrac{t^3}{3!} + 4e^{4t} \cdot \dfrac{t^4}{4!}$

or, $F(t) = \dfrac{e^{4t}}{6}\left[t^3 + t^4\right]$

$$\boxed{\text{or, } F(t) = \frac{t^3 e^{4t}}{6}(t+1)}$$

(5)

$$F(s) = \frac{e^{-3s} - e^{-6s}}{s^8}$$

or, $$F(s) = \frac{e^{-3s}}{s^8} - \frac{e^{-6s}}{s^8}$$

Taking Laplace Inverse-

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{e^{-3s}}{s^8}\right\} - L^{-1}\left\{\frac{e^{-6s}}{s^8}\right\}.$$

Now,

$$L^{-1}\left\{\frac{1}{s^8}\right\} = \frac{t^7}{7!}$$

Thus,

$$L^{-1}\left\{\frac{e^{-3s}}{s^8}\right\} = \frac{(t-3)^7}{7!} U(t-3)$$

and,

$$L^{-1}\left\{\frac{e^{-6s}}{s^8}\right\} = \frac{(t-6)^7}{7!} U(t-6)$$

$L\{f(t)\} = F(s)$

$L\{f(t) \cdot U(t-a)\}$

$= F(s-a) \cdot e^{-as}$.

So, inverse -

$L^{-1}\{F(s)\} = f(t)$

$L^{-1}\{e^{-as} F(s)\} = f(t-a) \cdot U(t-a)$

$$\boxed{U(t-a) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a. \end{cases} \quad \text{Unit step function.}}$$

and,

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

so,

$$\boxed{f(t) = L^{-1}\{F(s)\} = \frac{1}{7!}\left[(t-3)^7 \cdot U(t-3) - (t-6)^7 U(t-6)\right]}$$

6) $F(s) = -\dfrac{e^{-4s}(s+7)}{s^2+25}$

Let $f(s) = \dfrac{s+7}{s^2+25} = \dfrac{s}{s^2+25} + \dfrac{7}{s^2+25}$

$$L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{7}{5}L^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$L^{-1}\{f(s)\} = \cos(5t) + \frac{7}{5}\sin(5t)$$

Now, using Property

Unit step function: $U(t-a) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a \end{cases}$

if $L^{-1}\{f(s)\} = f(t)$

then, $L^{-1}\{e^{-as}f(s)\} = f(t-a) \cdot U(t-a)$.

Since, $L^{-1}\{f(s)\} = \cos(5t) + \frac{7}{5}\sin(5t)$

then, $L^{-1}\{e^{-4s}f(s)\} = \left[\cos[5(t-4)] + \frac{7}{5}\sin[5(t-4)]\right] U(t-4)$

Now, Given that

$$F(s) = -e^{-4s}f(s) = -e^{-4s}\frac{(s+7)}{s^2+25}$$

so, $\boxed{L^{-1}\{F(s)\} = -\left[\cos(5t-20) + \frac{7}{5}\sin(5t-20)\right] U(t-4)}$

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Find Laplace Inverse.

$$L^{-1}\left\{\ln\left(\frac{s+2}{s+4}\right)\right\} = f(t)$$

$$\Rightarrow L^{-1}\{\ln(s+2) - \ln(s+4)\} = f(t)$$

$$\Rightarrow L^{-1}\{\ln(s+2) - \ln(s+4)\} = f(t)$$

From Differential theorem -

$$\Rightarrow \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}[\ln(s+2) - \ln(s+4)]\right\} = f(t)$$

$$\Rightarrow F(t) = \frac{-1}{t} L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+4}\right\}$$

$$F(t) = \frac{-1}{t} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{t} L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$F(t) = \frac{-1}{t} e^{-2t} + \frac{1}{t} e^{-4t}$$

$$F(t) = \frac{1}{t}\left[e^{-4t} - e^{-2t}\right]$$

$$\Rightarrow \boxed{L^{-1}\left\{\ln\left(\frac{s+2}{s+4}\right)\right\} = \frac{1}{t}\left[e^{-4t} - e^{-2t}\right]}$$

If

$$L\{f(t)\} = F(s)$$

then

$$L\{t \cdot f(t)\} = \frac{-d}{ds} F(s)$$

$\Downarrow$

$$L^{-1}\{F(s)\} = f(t)$$

$$L^{-1}\left\{\frac{-d}{ds} f(s)\right\} = t \cdot f(t)$$

$$f(t) = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds} F(s)\right\}$$

$$f(t) = 5+3t^2+4\int_0^t f(u)\sin(4(t-u))du \quad \text{——(1)}$$

Using Convolution Theorem

using Laplace Transform.

Since, Convolution Theorem -

$$\int_0^t f(t-u)\cdot g(u)\,du = F(s)\cdot G(s)$$

Let $L\{f(t)\} = F(s)$.

Apply Laplace in -

$$f(t) = 5+3t^2+4\int_0^t f(u)\sin(4(t-u))du$$

$$\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

or $f(t) = 5+3t^2+4\int_0^t f(t-u)\sin 4u\cdot du$

Apply Laplace -

$$L\{f(t)\} = L\{5\}+3L\{t^2\}+4L\left\{\int_0^t f(t-u)\cdot\sin 4u\cdot du\right\}$$

or $F(s) = L\{5\}+3L\{t^2\}+4\cdot L\{f(t)\}\cdot L\{\sin 4t\}$

or $F(s) = \dfrac{5}{s}+3\cdot\dfrac{2!}{s^3}+4F(s)\cdot\dfrac{4}{s^2+4^2}$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

or, $F(s) = \dfrac{5}{s}+\dfrac{6}{s^3}+\dfrac{16}{s^2+16}\cdot F(s)$

or $F(s)\left[1-\frac{16}{s^2+16}\right] = \frac{5s^2+6}{s^3}$

or, $F(s)\left[\frac{s^2+16-16}{s^2+16}\right] = \frac{5s^2+6}{s^3}$

or, $F(s) = \frac{(5s^2+6)(s^2+16)}{s^5}$

or, $F(s) = \frac{(5s^4+80s^2+6s^2+96)}{s^5}$

or, $F(s) = \frac{5}{s}+86\cdot\frac{1}{s^3}+\frac{96}{s^5}$

Taking Laplace inverse -

$L^{-1}\{F(s)\} = 5L^{-1}\left\{\frac{1}{s}\right\}+86L^{-1}\left\{\frac{1}{s^3}\right\}+96\left\{\frac{1}{s^5}\right\}$

or $f(t) = 5+86\cdot\frac{1}{2}\cdot t^2+96\cdot\frac{1}{4!}t^4$

or, $\boxed{f(t) = 5+43t^2+4t^4}$

$\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}t^n$

Application to solve Simple linear and Simultaneous differential equation.

Ordinary differential equation with Constant coefficient can be easily solved by Laplace transform Method.

In this Method we use-

$$L\{y''(t)\} = s^2 Y(s) - s y(0) - y'(0)$$

$$L\{y'(t)\} = s Y(s) - y(0)$$

$$L\{y(t)\} = Y(s)$$

where, $y(0)$ and $y'(0)$ is Given.

* Same use in Simultaneous Differential equation

$$L\{y^n(t)\} = s^n y(s) - s^{n-1} y(0) - s^{n-2} y'(0) \;-\!- \; - y(0).$$

For this, we solve Some example:

1) Use the Laplace Transform to slove the following IVP

$$y' + y = \sin(t) \qquad y(0) = 0$$

Sol^n:

Taking Laplace -

Let $L\{y(t)\} = y(s)$

$$L\{y'(t) + y(t)\} = L\{\sin t\}$$

or $\quad L\{y'(t)\} + L\{y(t)\} = L\{\sin t\}$

$L\{y'(t)\} = sy(s) - y(0)$

or, $\quad sy(s) - y(0) + y(s) = \dfrac{1}{s^2+1}$

or $\quad (s+1)\,y(s) - 0 = \dfrac{1}{s^2+1}$

or, $\quad y(s) = \dfrac{1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{(s^2+1)}$

Decompose by Partial Fraction.

or $\quad y(s) = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+1)}$$

or $\quad 1 = A(s^2+1) + (Bs+C)(s+1)$

or $\quad 1 = A(s^2+1) + Bs^2 + Bs + Cs + C$

or $\quad 1 = (A+B)s^2 + (B+C)s + (A+C)$

$\Rightarrow \quad A+B = 0 \Rightarrow B = -A$

$B+C = 0 \Rightarrow B = -C$

$A+C = -1 \Rightarrow -B-B = -1 \Rightarrow 2B = 1 \Rightarrow \boxed{B = 1/2}$

so, $\boxed{A = -B = -1/2}$, $\boxed{C = -B = -1/2}$

so,

$$y(s) = \frac{-1/2}{(s+1)} + \frac{-\frac{s}{2} + \frac{1}{2}}{(s^2+1)}$$

or, $\quad y(s) = -\frac{1}{2} \cdot \frac{1}{(s+1)} - \frac{1}{2} \frac{(s-1)}{(s^2+1)}$

or, $\quad y(s) = -\frac{1}{2} \cdot \frac{1}{(s+1)} - \frac{1}{2} \frac{s}{(s^2+1)} + \frac{1}{2} \frac{1}{(s^2+1)}$

Now, Taking Laplace Inverse:

$$L^{-1}\{y(s)\} = \frac{-1}{2} L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2} L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or, $y(t) = \frac{-1}{2} e^{-t} - \frac{1}{2} \cos t + \frac{1}{2} \sin t$

or, $\boxed{y(t) = \frac{1}{2}\left[\sin t - \cos t - e^{-t}\right]}$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$$

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$$

Solving using Laplace Transform -

(2) $y''+2y'+4y = e^{2t}$ $\quad y(0)=1 \quad y'(0)=0$

Taking Laplace -

$$\mathcal{L}\{y''+2y'+4y\} = \mathcal{L}\{e^{2t}\}$$

or, $[s^2y(s) - sy(0) - y'(0)] + 2[sy(s) - y(0)] + 4y(s) = \frac{1}{(s-2)}$

or $(s^2+2s+4)\,y(s) - s - 0 + 2(-1) = \frac{1}{(s-2)}$

or $(s^2+2s+4)\,y(s) = \frac{1}{(s-2)} + s+2$

or $(s^2+2s+4)\,y(s) = \frac{1+(s+2)(s-2)}{(s-2)}$

or $y(s) = \frac{1+s^2-4}{(s-2)(s^2+2s+4)} = \frac{s^2-3}{(s-2)(s^2+2s+4)}$

Partial fraction decomposition -

$$\frac{s^2-3}{(s-2)(s^2+2s+4)} = \frac{A}{(s-2)} + \frac{Bs+C}{(s^2+2s+4)}$$

or, $s^2-3 = A(s^2+2s+4) + (Bs-C)(s-2)$

or $s^2-3 = A(s^2+2s+4) + (Bs^2-2Bs-Cs+2C)$

or $s^2-3 = (A+B)s^2 + (2A-2B-C)s + (4A+2C)$

$A+B = 1 \Rightarrow 4A+4B = 4$ —— (1)

$2A-2B-C=0 \Rightarrow \underline{C = 2A-2B}$ Put in

$4A+2C=-3$

$4A+2(2A-2B)=-3$

$8A-4B=-3$ —— (2)

on adding (1) & (2)

$$12A = 1 \Rightarrow \boxed{A = 1/12}$$

$$B = 1-A = 1-\frac{1}{12} = \frac{11}{12} \Rightarrow \boxed{B = 11/12}$$

or $$C = 2(A-B) = 2\left(\frac{11}{12}-\frac{1}{12}\right) = \frac{2\times 10}{12} = \frac{20}{12}$$

$$y(s) = \frac{1/12}{(s-2)} + \frac{\left(\frac{11}{12}s+\frac{20}{12}\right)}{(s^2+2s+4)}$$

$$y(s) = \frac{1}{12}\cdot\frac{1}{(s-2)} + \frac{1}{12}\cdot\frac{(11s+20)}{(s^2+2s+4)}$$

or $$y(s) = \frac{1}{12}\cdot\frac{1}{(s-2)} + \frac{11}{12}\cdot\frac{s}{(s^2+2s+4)} + \frac{20}{12}\cdot\frac{1}{(s^2+2s+4)}$$

or $$y(s) = \frac{1}{12}\left[\frac{1}{(s-2)} + \frac{11s}{(s+1)^2+(\sqrt{3})^2} + \frac{20}{(s+1)^2+(\sqrt{3})^2}\right]$$

Taking Laplace inverse-

$$L^{-1}\{y(s)\} = \frac{1}{12} L^{-1}\left[\frac{1}{s-2}\right] + \frac{11}{12} L^{-1}\left\{\frac{s}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{20}{12} L^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

$$= \frac{1}{12} L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{11}{12} L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{20}{12} L^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

$$= \frac{1}{12} e^{2t} + \frac{11}{12} L^{-1}\left\{\frac{(s+1)}{(s+1)^2+(\sqrt{3})^2}\right\} + \frac{(20-11)}{12} L^{-1}\left\{\frac{1}{(s+1)^2+(\sqrt{3})^2}\right\}$$

or, $$y(t) = \frac{1}{12} e^{2t} + \frac{11}{12} e^{-t}\cos(\sqrt{3}t) + \frac{9}{12\sqrt{3}} e^{-t}\sin(\sqrt{3}t)$$

or, $$\boxed{y(t) = \frac{1}{12}\left[e^{2t} + 11 e^{-t}\cos(\sqrt{3}t) + 3\sqrt{3}\, e^{-t}\sin(\sqrt{3}t)\right]}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$L\{f(t)\} = F(s)$$

$$L\{e^{at} f(t)\} = F(s-a)$$

(3) Use Laplace to solve-

$$\frac{dx}{dt} = -x + y \qquad x(0) = 0, \quad y(0) = 7$$

$$\frac{dy}{dt} = 2x$$

Taking Laplace

$$L\{x'(t)\} = L^{-1}\{-x(t) + y(t)\}$$

$$L\{y'(t)\} = L^{-1}\{2x(t)\}$$

or, $sX(s) - X(0) = -X(s) + Y(s)$

$sY(s) - Y(0) = 2X(s)$

or, $sX(s) + X(s) - Y(s) = 0$

$sY(s) - 7 - 2X(s) = 0$

or $\left.\begin{aligned} (s+1)X(s) - Y(s) &= 0 \quad \longrightarrow \text{multiply by } s \\ -2X(s) + sY(s) &= 7 \end{aligned}\right\}$ —(1)

we get

$$s(s+1)X(s) - sY(s) = 0$$

$$-2X(s) + sY(s) = 7$$

on adding

$$(s(s+1) - 2)X(s) = 7$$

$$X(s) = \frac{7}{s^2+s-2} = \frac{7}{s^2+2s-s-2}$$

or $$X(s) = \frac{7}{(s+2)(s-1)}$$

Now, Partial Fraction -

$$\frac{7}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

or $$7 = (A+B)s + (2B-A)$$

$$\left.\begin{aligned} A+B &= 0 \\ -A+2B &= 7 \end{aligned}\right\} \Rightarrow 3B = 7 \Rightarrow B = \frac{7}{3}, \quad A = \frac{-7}{3}.$$

$$X(s) = \frac{-7/3}{s+2} + \frac{7/3}{s-1}$$

Taking Laplace Inverse -

$$L^{-1}\{X(s)\} = \frac{-7}{3} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{7}{3} L^{-1}\left\{\frac{1}{s-1}\right\}$$

or, $$x(t) = \frac{-7}{3} e^{-2t} + \frac{7}{3} e^{t}$$

or, $$\boxed{x(t) = \frac{7}{3}\left[e^{t} - e^{-2t}\right]}$$

Now From (1)'

$$Y(s) = (s+1)X(s) = \frac{(s+1)\cdot 7}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

Now, Partial Fraction-

$$7s + 7 = (A+B)s + (2B-A)$$

$$\left.\begin{array}{r} A+B=7 \\ -A+2B=7 \end{array}\right\} \Rightarrow 3B = 14 \Rightarrow B = \frac{14}{3}$$

$$A = 7-B = 7-\frac{14}{3} = \frac{21-14}{3} = \frac{7}{3}$$

so $$Y(s) = \frac{7/3}{(s+2)} + \frac{14/3}{(s-1)}$$

Taking Laplace inverse -

$$L^{-1}\{Y(s)\} = \frac{7}{3} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{14}{3} L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$y(t) = \frac{7}{3}e^{-2t} + \frac{14}{3}e^{t}$$ $\because L\{e^{at}\} = \frac{1}{s-a}$

or $$\boxed{y(t) = \frac{7}{3}\left[e^{-2t} + 2e^{t}\right]}$$

(4)

Use the Laplace. Solve. Simultaneous. Eqn.

$$\frac{dx}{dt} = -y + t \quad \Rightarrow \quad x'(t) + y(t) = t$$

$$\frac{dy}{dt} = -4x \quad \Rightarrow \quad y'(t) + 4x(t) = 0$$

$$x(0) = 1, \quad y(0) = 2$$

Taking Laplace both side-

$$L\{x'(t) + y(t)\} = L\{t\}$$

$$L\{y'(t) + 4x(t)\} = 0$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $L\{x(t)\} = X(s)$

$L\{y(t)\} = Y(s)$

or $$sX(s) - X(0) + Y(s) = \frac{1}{s^2}$$

$$sY(s) - Y(0) + X(s) = 0$$

or $$sX(s) - 1 + Y(s) = \frac{1}{s^2}$$

$$X(s) - 2 + sY(s) = 0$$

or $$sX(s) + Y(s) = \frac{1}{s^2} + 1 = \frac{s^2+1}{s^2} \longrightarrow \times 1$$

$$X(s) + sY(s) = 2 \longrightarrow \text{multiply by } s$$

$\Big\}$ — Eqn (1)

we get-

$$sX(s) + Y(s) = \frac{s^2+1}{s^2}$$

$$sX(s) + s^2Y(s) = 2s$$

On subtracting

$$Y(s)[1-s^2] = \frac{s^2+1}{s^2} - 2s$$

or, $$(s^2-1)\,Y(s) = 2s - \frac{s^2+1}{s^2} = \frac{2s^3-s^2-1}{s^2}$$

or, $$Y(s) = \frac{2s^3-s^2-1}{s^2(s^2-1)}$$

Now, $$\frac{2s^3-s^2-1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s+1)}$$

or, $$2s^3-s^2-1 = As(s^2-1) + B(s^2-1) + Cs^2(s+1) + Ds^2(s-1)$$

or $$2s^3-s^2-1 = A(s^3-s) + B(s^2-1) + C(s^3+s^2) + D(s^3-s^2)$$

or $$2s^3-s^2-1 = (A+C+D)s^3 + (B+C-D)s^2 - As - B$$

on comparing.

$$A+C+D=2 \Rightarrow C+D=2$$
$$B+C-D=-1 \Rightarrow C-D=-2$$
$$\Bigg\} \Rightarrow C=0,\ D=2$$
$$-A=0 \Rightarrow A=0$$
$$-B=-1 \Rightarrow B=1$$

so, $$Y(s) = \frac{1}{s^2} + \frac{2}{(s+1)}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + 2L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = t + 2e^{-t}}$$

Now,

$$X(s) = 2 - s\,Y(s) \longrightarrow \underline{\text{from eq}^n (1)}$$

or $$X(s) = 2 - s\left(\frac{2s^3 - s^2 - 1}{s^2(s-1)(s+1)}\right)$$

or $$X(s) = \frac{2s(s^2-1) - (2s^3 - s^2 - 1)}{s(s-1)(s+1)}$$

or, $$X(s) = \frac{2s^3 - 2s - 2s^3 + s^2 + 1}{s(s-1)(s+1)} = \frac{s^2 - 2s + 1}{s(s-1)(s+1)}$$

or, $$\frac{s^2 - 2s + 1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

or $$s^2 - 2s + 1 = A(s^2-1) + B(s^2+s) + C(s^2-s)$$

or $$s^2 - 2s + 1 = (A+B+C)s^2 + (B-C)s - A$$

$$\left.\begin{aligned} A+B+C &= 1 \\ B-C &= -2 \\ \underline{A} &\underline{= -1} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} B+C &= 2 \\ B-C &= -2 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= 0 \\ C &= +2 \end{aligned}$$

$$X(s) = \frac{-1}{s} + \frac{2}{s+1}$$

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{-1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{x(t) = -1 + 2e^{-t}}$$

So, solution is-

$$\boxed{\left.\begin{aligned} x(t) &= 2e^{-t} - 1 \\ y(t) &= 2e^{-t} + t \end{aligned}\right\}}$$