Heat & Mass Transfer (BME-27)

Steady State Heat Conduction: Fins

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HEAT TRANSFER FROM FINNED SURFACES

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

Newton's law of cooling: The rate of heat transfer from a surface to the surrounding medium

When T_s and T_∞ are fixed, *two ways* to increase the rate of heat transfer are

- To increase the *convection heat transfer coefficient h.* This may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate.
- To increase the *surface area A_s* by attaching to the surface *extended surfaces* called ***fins*** made of highly conductive materials such as aluminum.

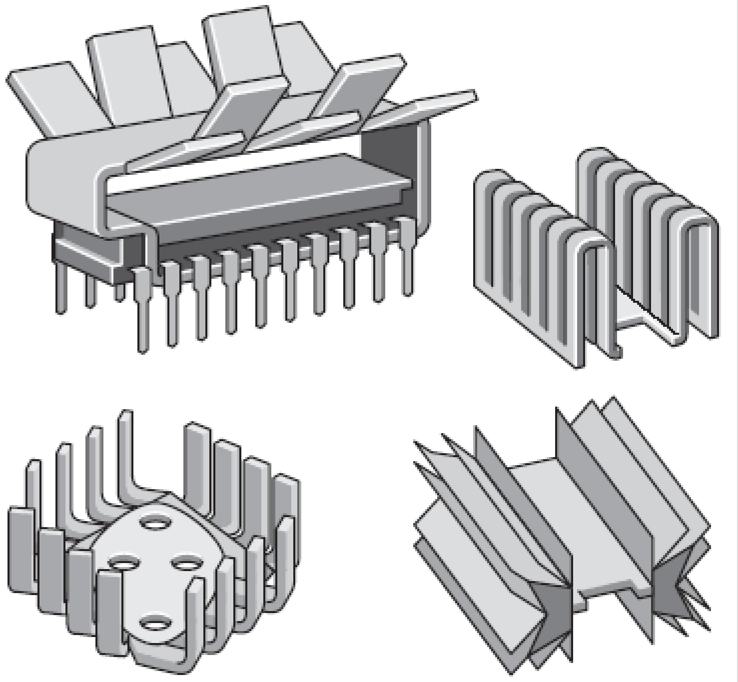
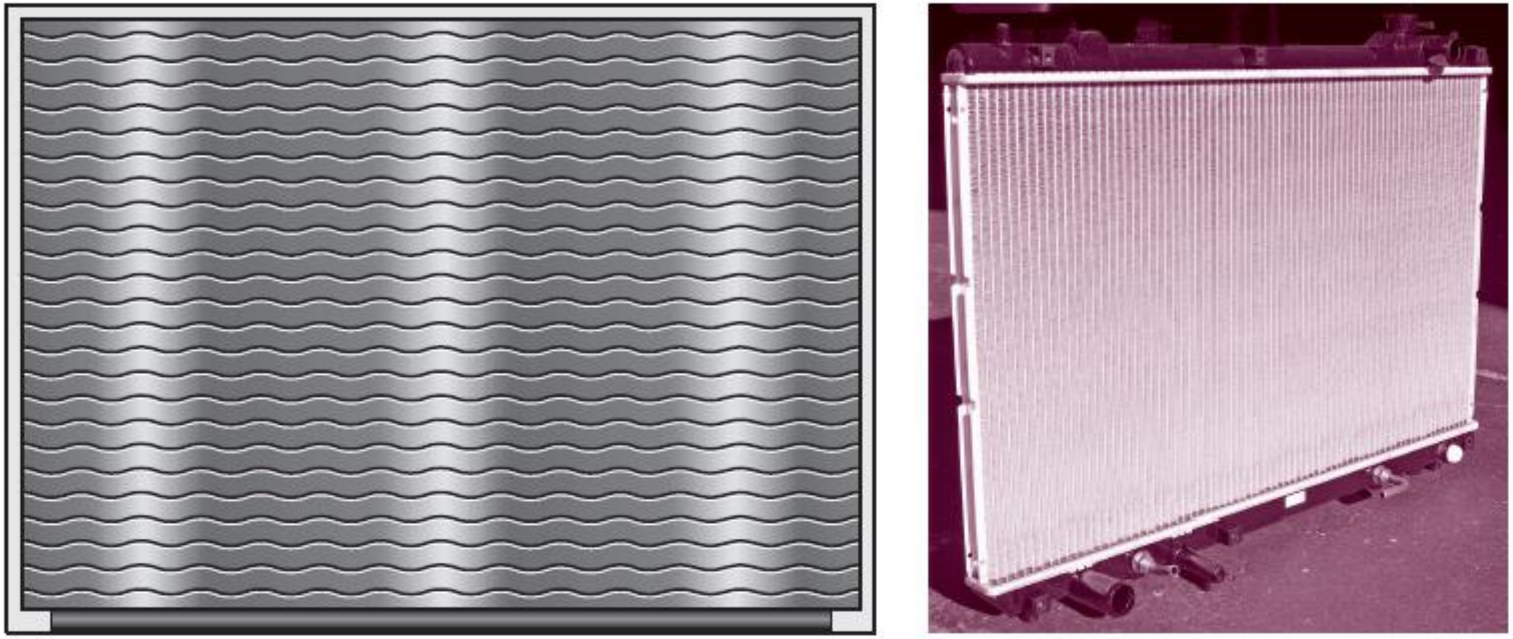


FIGURE 3–35
Some innovative fin designs.

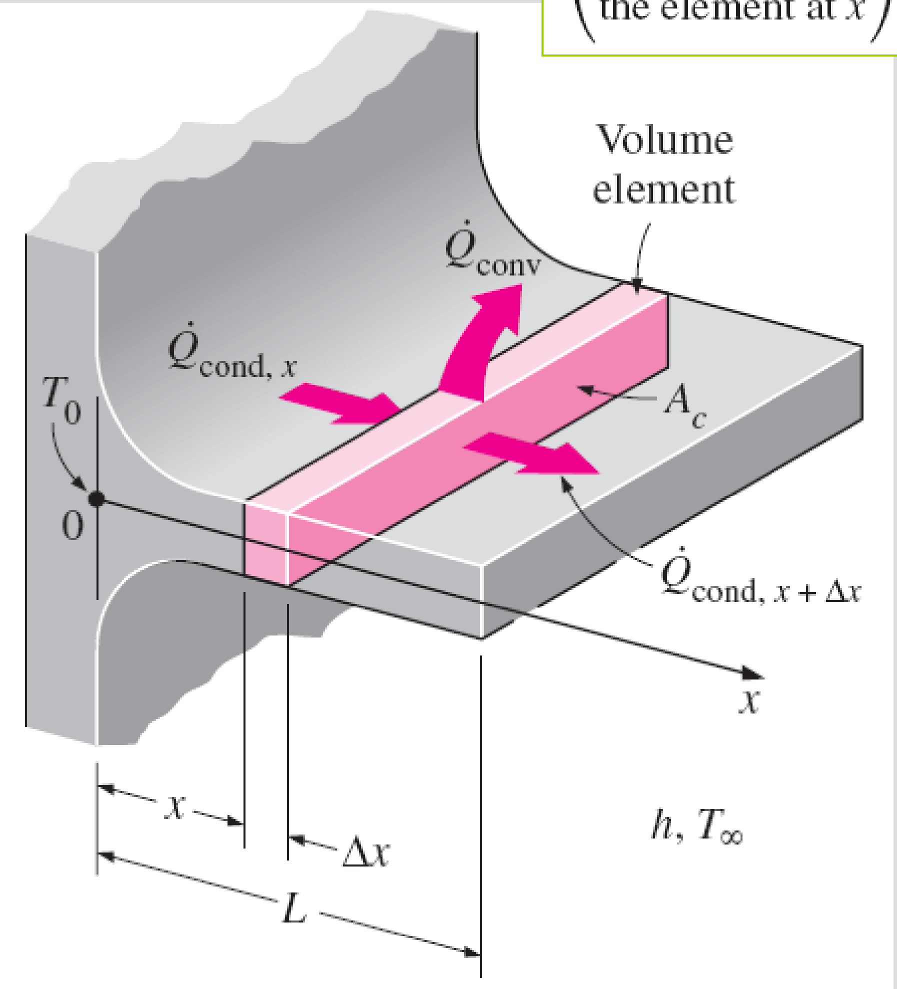
FIGURE 3–33
Presumed cooling fins on dinosaur stegosaurus. (© Alamy RF.)

The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air.



Fin Equation

$$\begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element}\end{pmatrix}$$



Volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and perimeter of p.

$$\dot{Q}_{cond,\,x} = \dot{Q}_{cond,\,x+\Delta x} + \dot{Q}_{conv}$$

$$\dot{Q}_{conv} = h(p\,\Delta x)(T - T_\infty)$$

$$\frac{\dot{Q}_{cond,\,x+\Delta x} - \dot{Q}_{cond,\,x}}{\Delta x} + hp(T - T_\infty) = 0$$

$$\Delta x \to 0$$

$$\frac{d\dot{Q}_{cond}}{dx} + hp(T - T_\infty) = 0$$

$$\dot{Q}_{cond} = -kA_c\frac{dT}{dx}$$

$$\frac{d}{dx}\left(kA_c\frac{dT}{dx}\right) - hp(T - T_\infty) = 0$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

Differential equation

$$m^2 = \frac{hp}{kA_c}$$

$$\theta = T - T_\infty$$

Temperature excess

The general solution of the differential equation

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

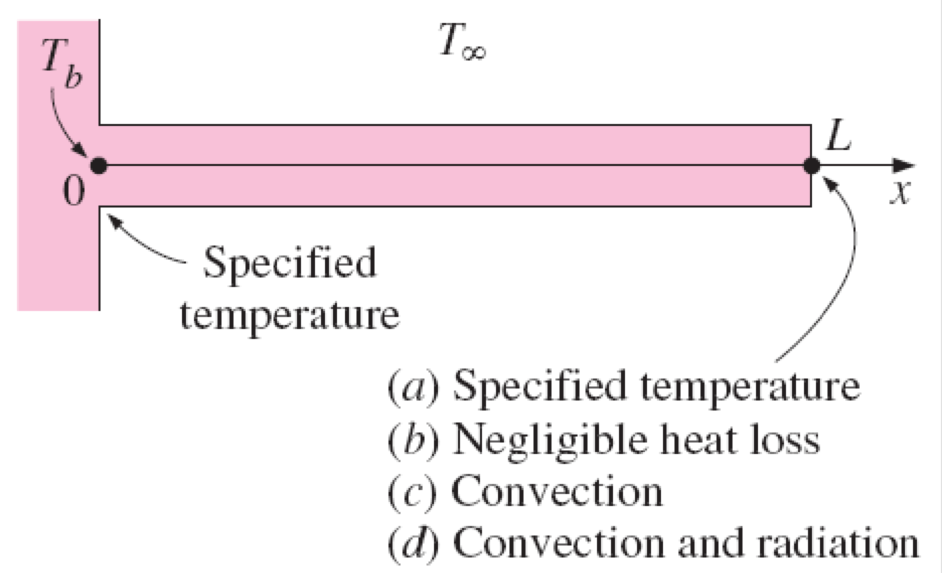
Boundary condition at fin base

$$\theta(0) = \theta_b = T_b - T_\infty$$

1 Infinitely Long Fin ($T_{\text{fin tip}} = T_\infty$)

Boundary condition at fin tip

$$\theta(L) = T(L) - T_\infty = 0 \qquad L \rightarrow \infty$$



Boundary conditions at the fin base and the fin tip.

The variation of temperature along the fin

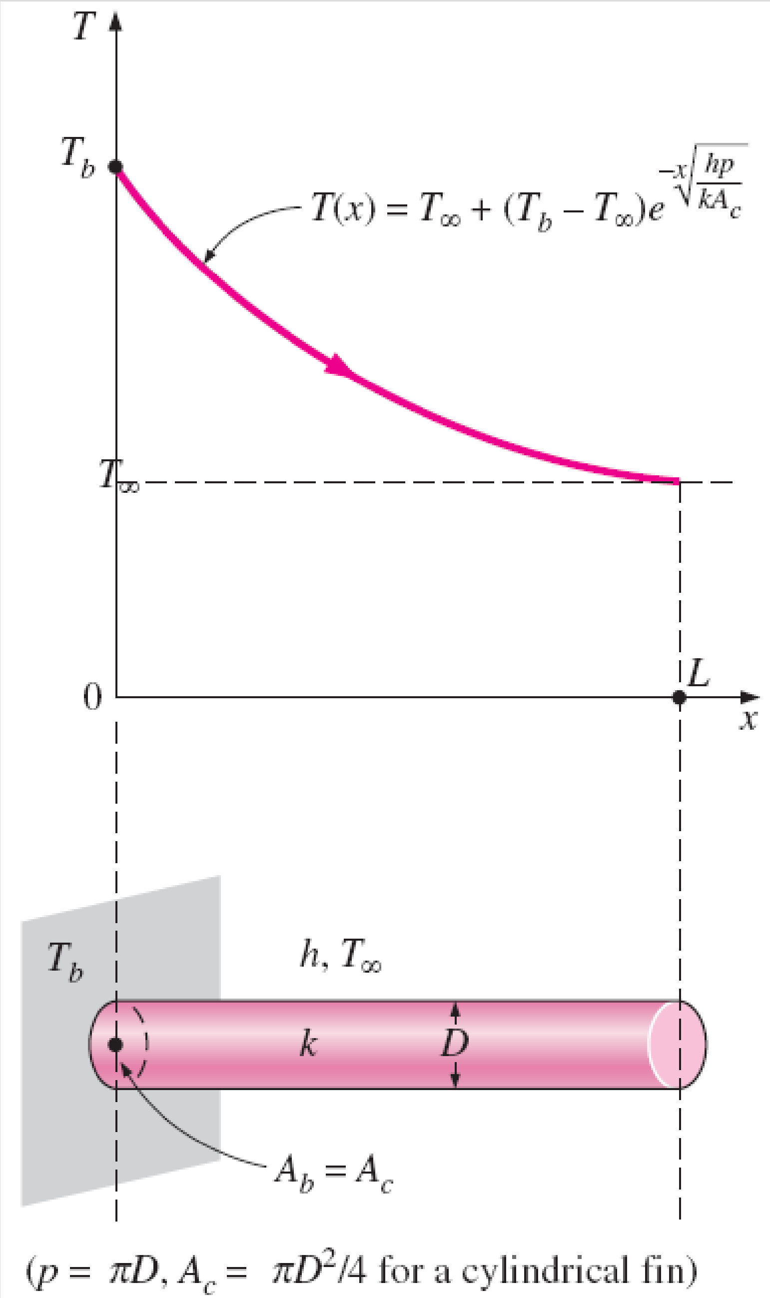
$$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}}$$

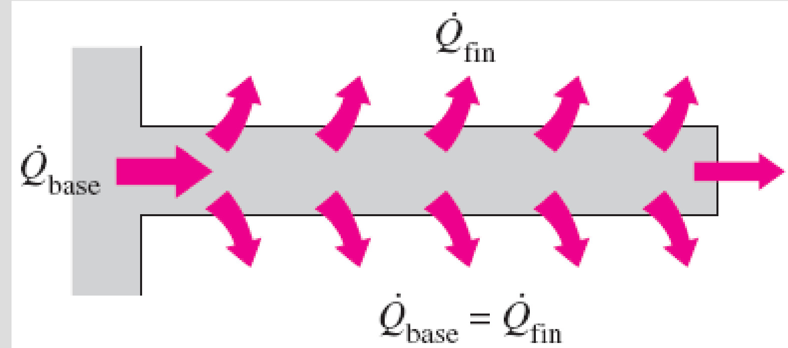
$$\theta = T - T_\infty$$

$$m = \sqrt{hp/kA_c}$$

The steady rate of *heat transfer* from the entire fin

$$\dot{Q}_{\text{long fin}} = -kA_c \left. \frac{dT}{dx} \right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$





Under steady conditions, heat transfer from the exposed surfaces of the fin is equal to heat conduction to the fin at the base.

The rate of heat transfer from the fin could also be determined by considering heat transfer from a differential volume element of the fin and integrating it over the entire surface of the fin:

$$\dot{Q}_{fin} = \int_{A_{fin}} h[T(x) - T_\infty]\, dA_{fin} = \int_{A_{fin}} h\theta(x)\, dA_{fin}$$

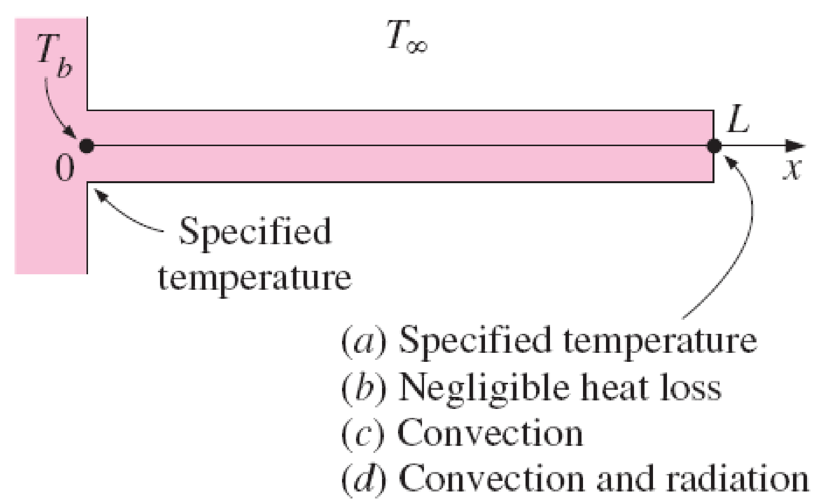
A long circular fin of uniform cross section and the variation of temperature along it.

2 Negligible Heat Loss from the Fin Tip (Adiabatic fin tip, $Q_{\text{fin tip}} = 0$)

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic assumption is for heat transfer from the fin tip to be negligible since the surface area of the fin tip is usually a negligible fraction of the total fin area.

Boundary condition at fin tip

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$



The variation of temperature along the fin

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

Heat transfer from the entire fin

$$\dot{Q}_{\text{adiabatic tip}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL$$

3 Specified Temperature ($T_{fin,tip} = T_L$)

In this case the temperature at the end of the fin (the fin tip) is fixed at a specified temperature T_L.

This case could be considered as a generalization of the case of *Infinitely Long Fin* where the fin tip temperature was fixed at T_∞.

Boundary condition at fin tip: $\theta(L) = \theta_L = T_L - T_\infty$

Specified fin tip temperature:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{[(T_L - T_\infty)/(T_b - T_\infty)]\sinh mx + \sinh m(L-x)}{\sinh mL}$$

Specified fin tip temperature:

$$\dot{Q}_{\text{specified temp.}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}$$

$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\cosh mL - [(T_L - T_\infty)/(T_b - T_\infty)]}{\sinh mL}$$

4 Convection from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that may also include the effects of radiation. Consider the case of convection only at the tip. The condition at the fin tip can be obtained from an energy balance at the fin tip.

$(\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}})$

Boundary condition at fin tip: $$-kA_c \left.\frac{dT}{dx}\right|_{x=L} = hA_c[T(L) - T_\infty]$$

Convection from fin tip: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x) + (h/mk) \sinh m(L - x)}{\cosh mL + (h/mk) \sinh mL}$$

Convection from fin tip:

$$\dot{Q}_{\text{convection}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}$$

$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\sinh mL + (h/mk) \cosh mL}{\cosh mL + (h/mk) \sinh mL}$$

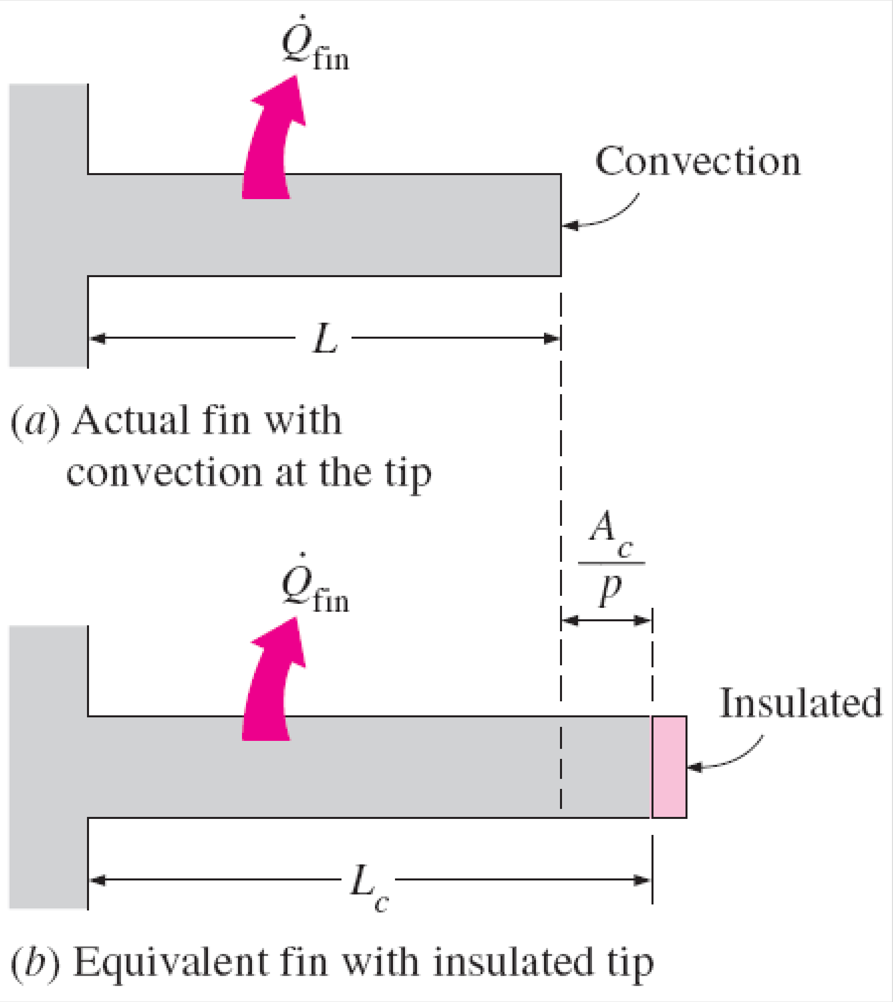
A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as

$$L_c = L + \frac{A_c}{p}$$

$$L_{c,\,\text{rectangular fin}} = L + \frac{t}{2}$$

$$L_{c,\,\text{cylindrical fin}} = L + \frac{D}{4}$$

t the thickness of the rectangular fins
D the diameter of the cylindrical fins



Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

Fin Efficiency

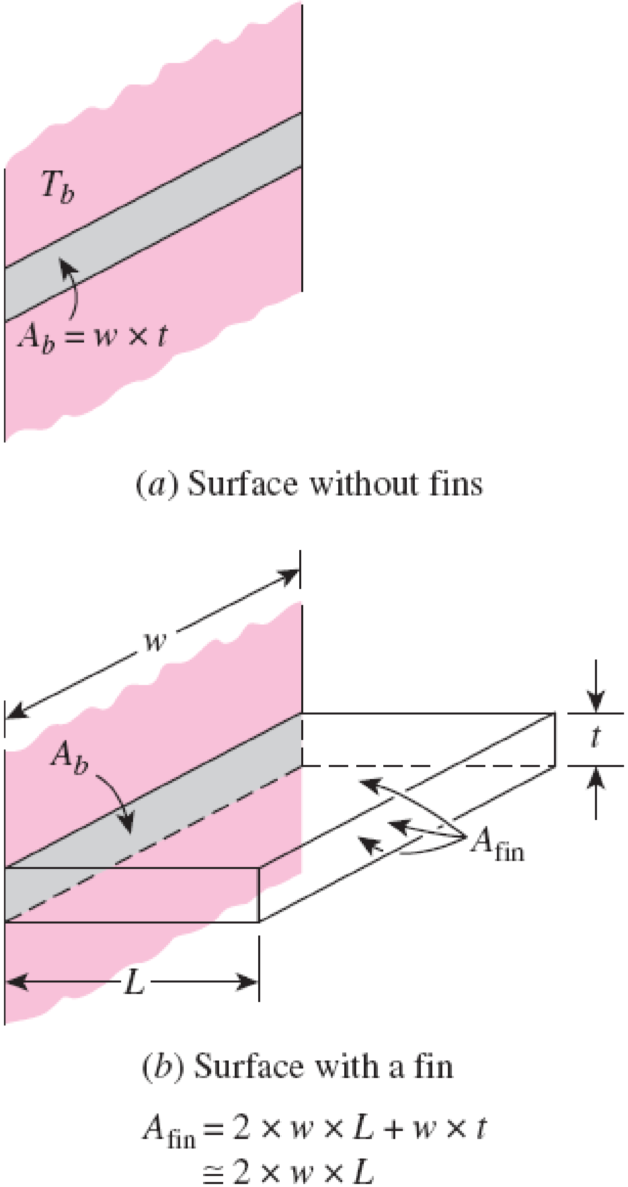


FIGURE 3–41
Fins enhance heat transfer from a surface by enhancing surface area.

$$\dot{Q}_{fin,\ max} = hA_{fin}\,(T_b - T_\infty)$$

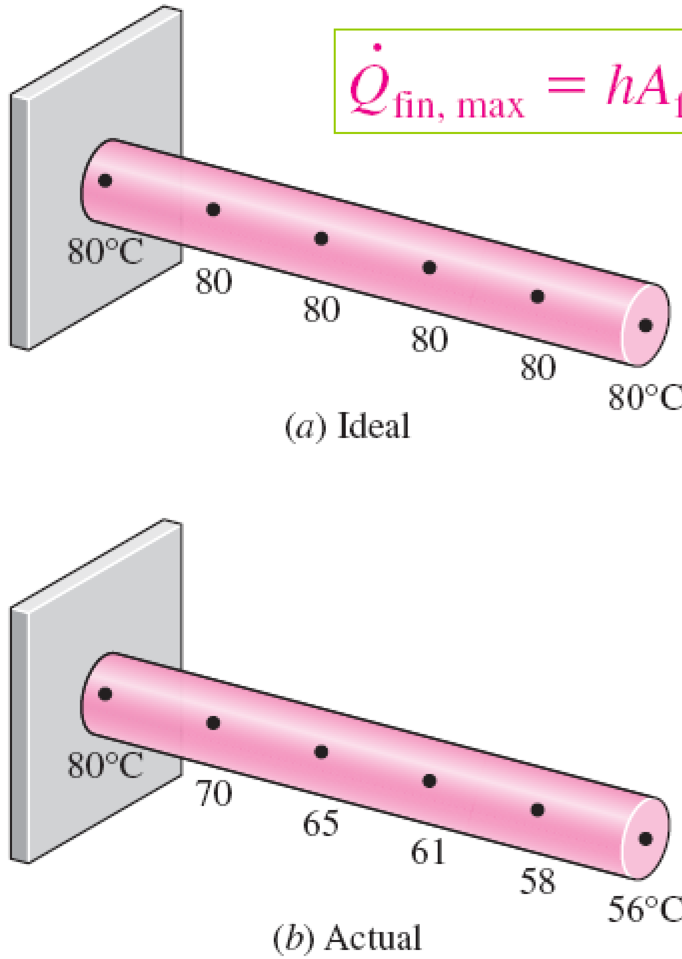


FIGURE 3–42
Ideal and actual temperature distribution along a fin.

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}(T_b - T_\infty)$$

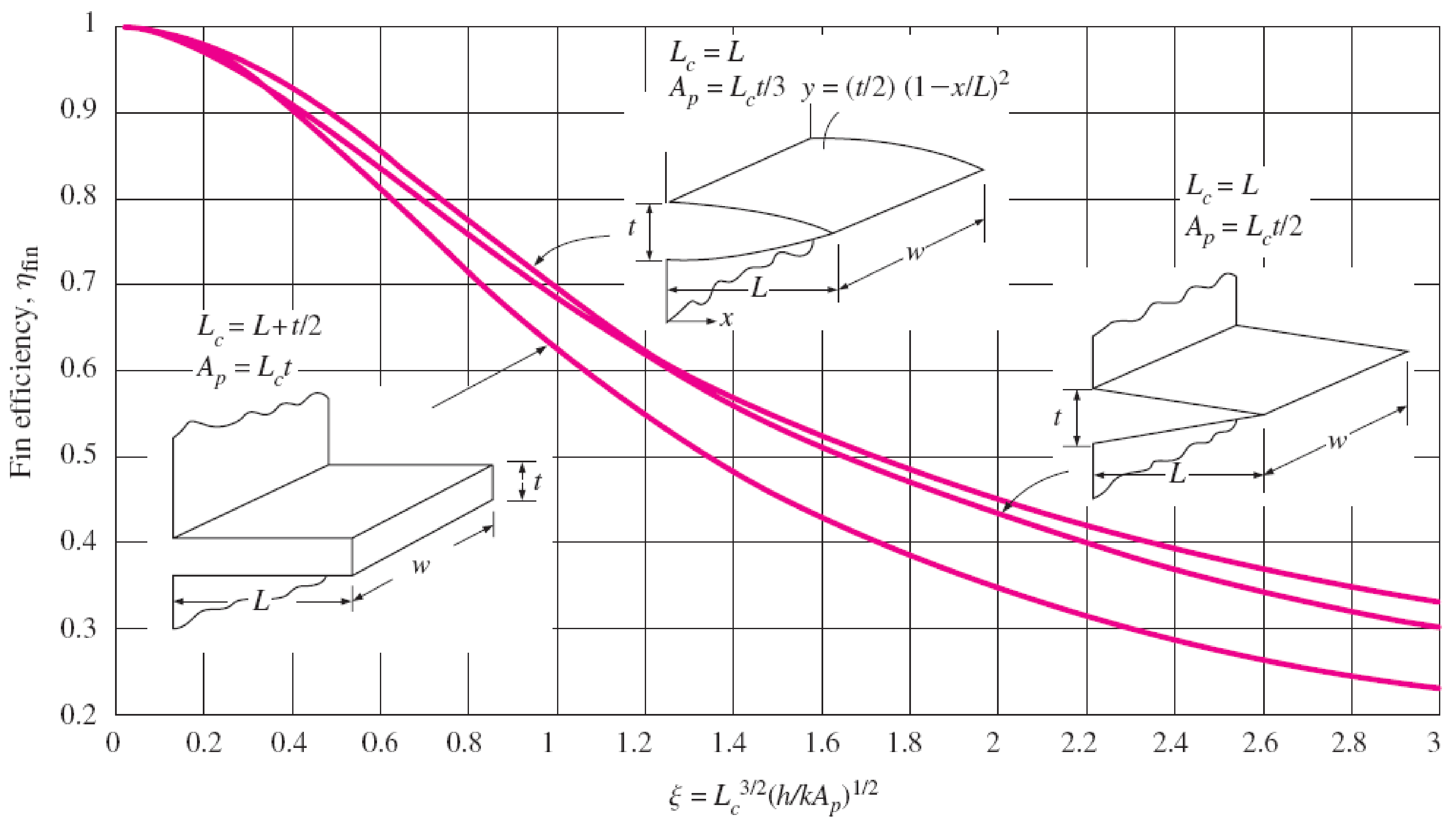
Zero thermal resistance or *infinite thermal conductivity* ($T_{\text{fin}} = T_b$)

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}}$$

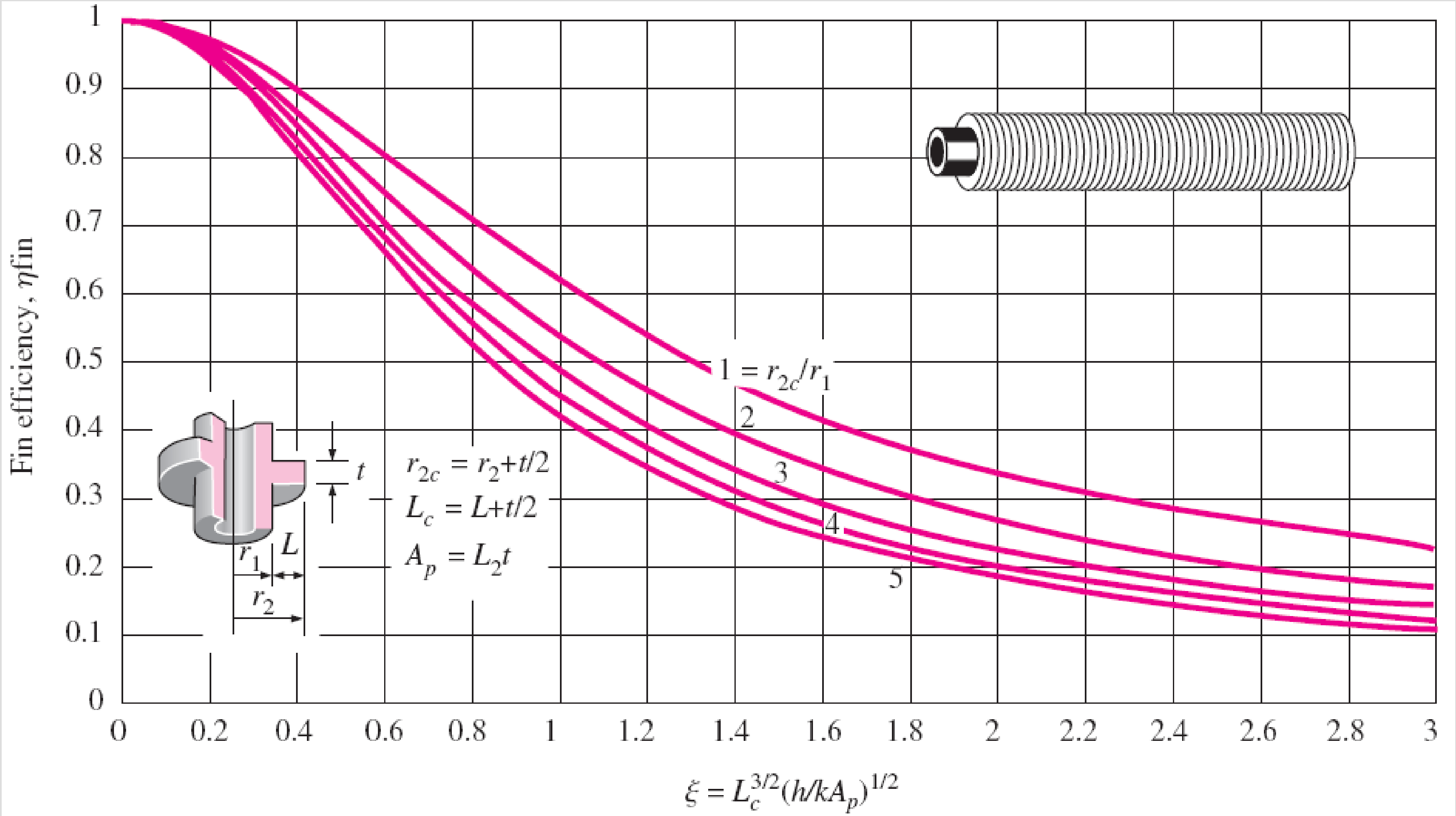
$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\,\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}\,hA_{\text{fin}}(T_b - T_\infty)$$

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{mL}$$

$$\eta_{\text{adiabatic tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{\tanh mL}{mL}$$



Efficiency of straight fins of rectangular, triangular, and parabolic profiles.



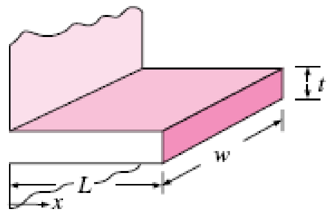
Efficiency of annular fins of constant thickness *t*.

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$$m = \sqrt{2h/kt}$$
$$L_c = L + t/2$$
$$A_{fin} = 2wL_c$$

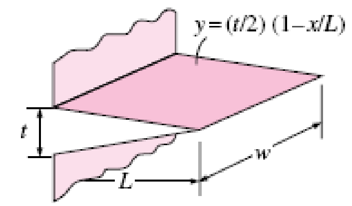
$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$



Straight triangular fins

$$m = \sqrt{2h/kt}$$
$$A_{fin} = 2w\sqrt{L^2 + (t/2)^2}$$

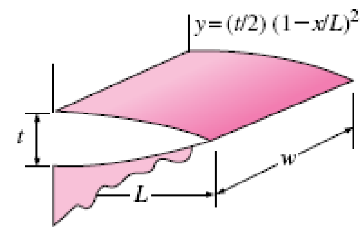
$$\eta_{fin} = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$



Straight parabolic fins

$$m = \sqrt{2h/kt}$$
$$A_{fin} = wL[C_1 + (L/t)\ln(t/L + C_1)]$$
$$C_1 = \sqrt{1 + (t/L)^2}$$

$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$

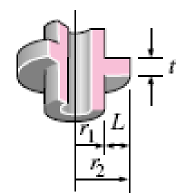


Circular fins of rectangular profile

$$m = \sqrt{2h/kt}$$
$$r_{2c} = r_2 + t/2$$
$$A_{fin} = 2\pi(r_{2c}^2 - r_1^2)$$

$$\eta_{fin} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

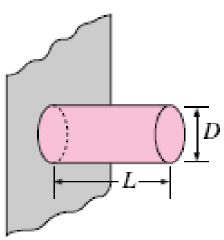
$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$



Pin fins of rectangular profile

$$m = \sqrt{4h/kD}$$
$$L_c = L + D/4$$
$$A_{fin} = \pi DL_c$$

$$\eta_{fin} = \frac{\tanh mL_c}{mL_c}$$

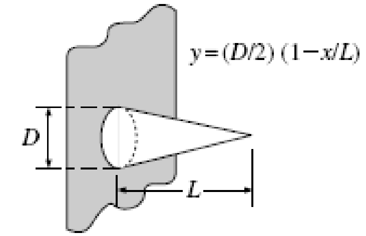


Pin fins of triangular profile

$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$$

$$\eta_{fin} = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$



Pin fins of parabolic profile

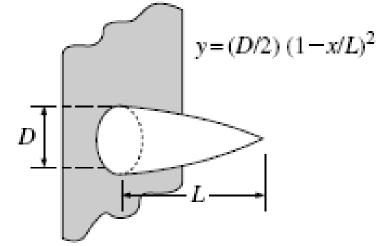
$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi L^3}{8D}\left[C_3C_4 - \frac{L}{2D}\ln(2DC_4/L + C_3)\right]$$

$$C_3 = 1 + 2(D/L)^2$$

$$C_4 = \sqrt{1 + (D/L)^2}$$

$$\eta_{fin} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$

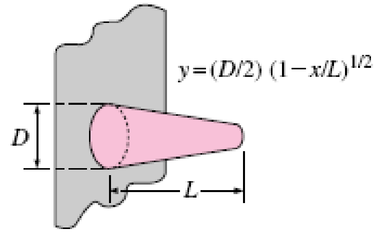


Pin fins of parabolic profile (blunt tip)

$$m = \sqrt{4h/kD}$$

$$A_{fin} = \frac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$$

$$\eta_{fin} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$



- Fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles.
- The fin efficiency decreases with increasing fin length. Why?
- How to choose fin length? Increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.
- Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically.
- The efficiency of most fins used in practice is above 90 percent.

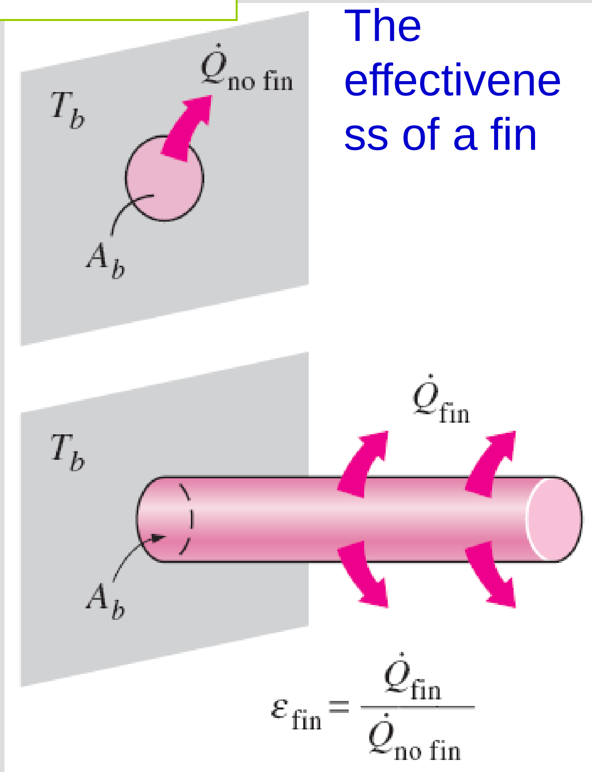
Fin Effectiveness

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{\dot{Q}_{fin}}{hA_b(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b}$$

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{\dot{Q}_{fin}}{hA_b(T_b - T_\infty)} = \frac{\eta_{fin} hA_{fin}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{fin}}{A_b}\eta_{fin}$$

$$\varepsilon_{long\ fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{\sqrt{hpkA_c}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}}$$

- The *thermal conductivity* ***k*** of the fin should be as high as possible. Use aluminum, copper, iron.
- The ratio of the *perimeter* to the *cross-sectional area* of the fin ***p/A_c*** should be as high as possible. Use slender pin fins.
- *Low convection heat transfer coefficient* ***h***. Place fins on gas (air) side.



The effectiveness of a fin

The total rate of heat transfer from a finned surface

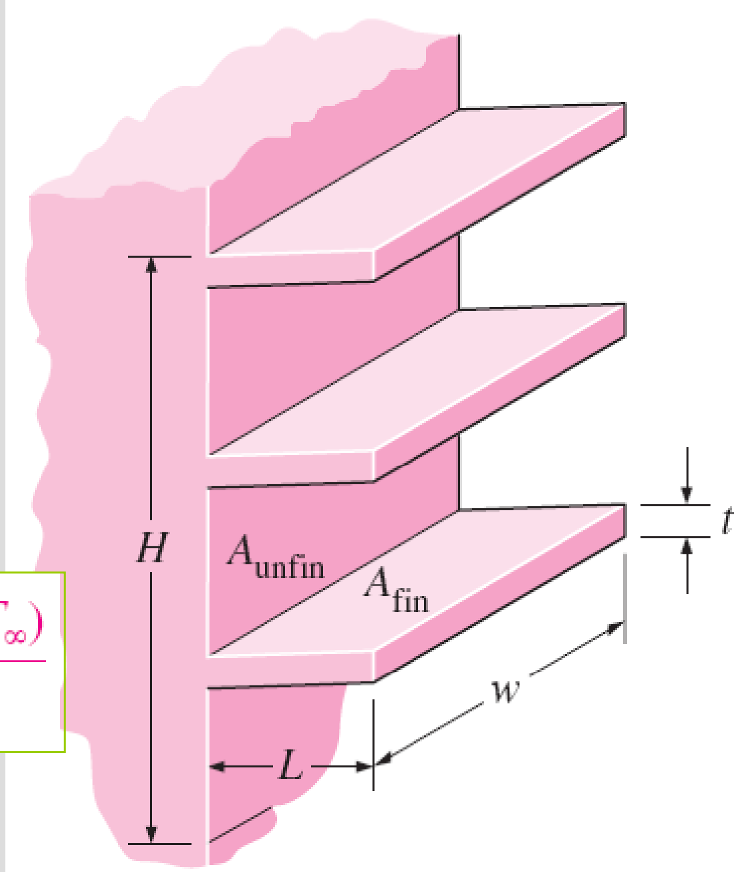
$$\dot{Q}_{\text{total, fin}} = \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}}$$
$$= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)$$
$$= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)$$

Overall effectiveness for a finned surface

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)}$$

The overall fin effectiveness depends on the fin density (number of fins per unit length) as well as the effectiveness of the individual fins.

The overall effectiveness is a better measure of the performance of a finned surface than the effectiveness of the individual fins.

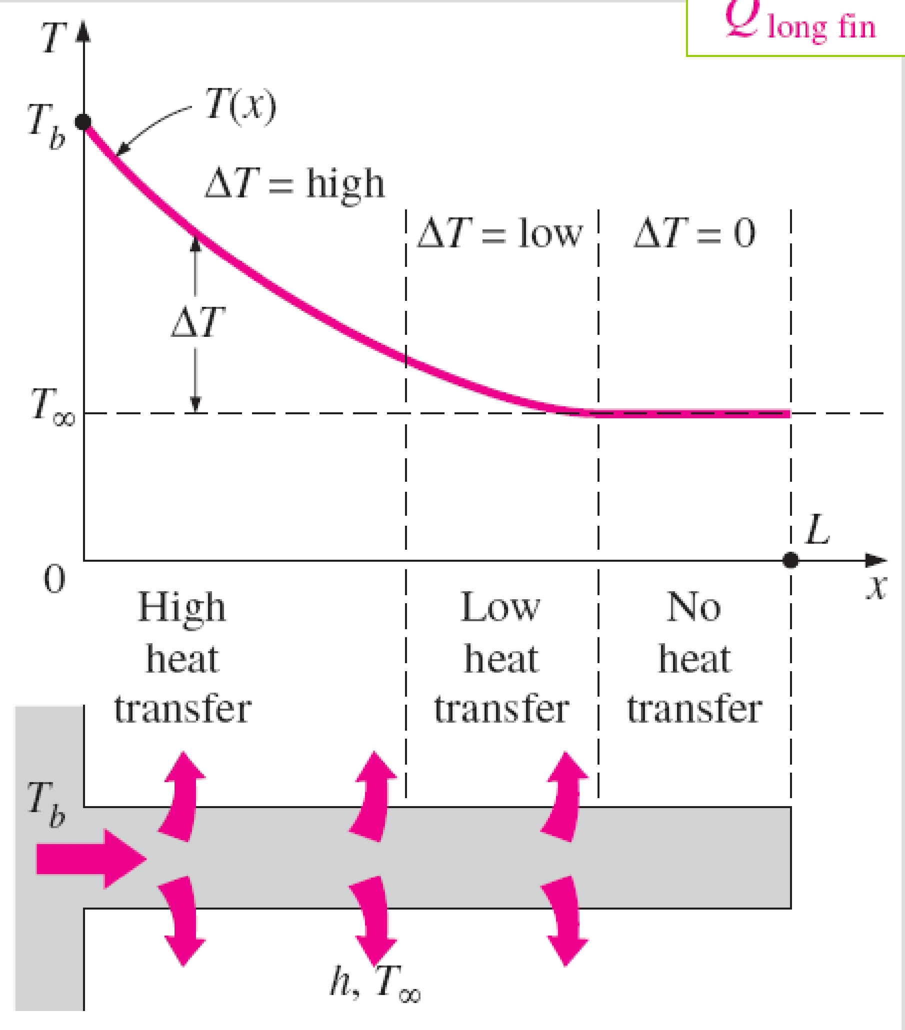


$$A_{\text{no fin}} = w \times H$$
$$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$$
$$A_{\text{fin}} = 2 \times L \times w + t \times w$$
$$\cong 2 \times L \times w \text{ (one fin)}$$

Various surface areas associated with a rectangular surface with three fins.

Proper Length of a Fin

$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh mL$$



The variation of heat transfer from a fin relative to that from an infinitely long fin

mL	$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

$mL = 5 \rightarrow$ an infinitely long fin

$mL = 1$ offer a good compromise between heat transfer performance and the fin size.

Because of the gradual temperature drop along the fin, the region near the fin tip makes little or no contribution to heat transfer.

A common approximation used in the analysis of fins is to assume the fin temperature to vary in one direction only (along the fin length) and the temperature variation along other directions is negligible.

Perhaps you are wondering if this one-dimensional approximation is a reasonable one.

This is certainly the case for fins made of thin metal sheets such as the fins on a car radiator, but we wouldn't be so sure for fins made of thick materials.

Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1 percent) when

$$\frac{h\delta}{k} < 0.2$$

where δ is the characteristic thickness of the fin, which is taken to be the plate thickness *t* for rectangular fins and the diameter *D* for cylindrical ones.

HEAT TRANSFER IN COMMON CONFIGURATIONS

So far, we have considered heat transfer in *simple* geometries such as large plane walls, long cylinders, and spheres.

This is because heat transfer in such geometries can be approximated as *one-dimensional*.

But many problems encountered in practice are two- or three-dimensional and involve rather complicated geometries for which no simple solutions are available.

An important class of heat transfer problems for which simple solutions are obtained encompasses those involving two surfaces maintained at *constant* temperatures T_1 and T_2.

The steady rate of heat transfer between these two surfaces is expressed as

$$Q = Sk(T_1 - T_2)$$

***S*: conduction shape factor**

k: the thermal conductivity of the medium between the surfaces

The conduction shape factor depends on the *geometry* of the system only.

Conduction shape factors are applicable only when heat transfer between the two surfaces is by ***conduction***.

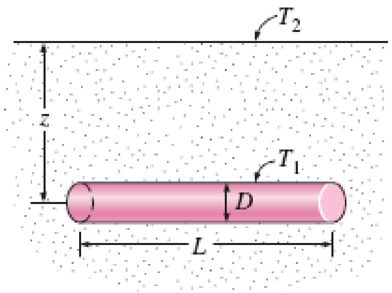
$$S = 1/kR$$

Relationship between the conduction shape factor and the thermal resistance

Conduction shape factors S for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity k between the surfaces at temperatures T_1 and T_2

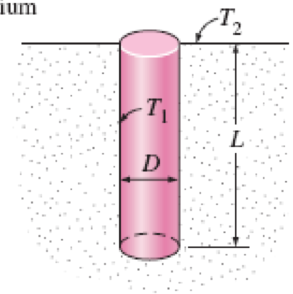
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



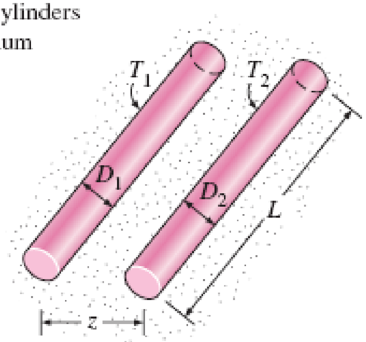
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

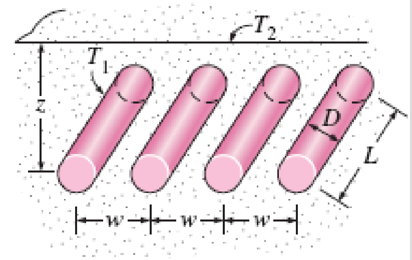
$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D, z,$ and $w > 1.5D$)

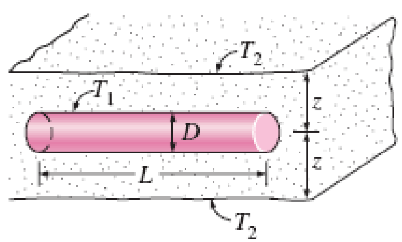
$$S = \frac{2\pi L}{\ln\left(\frac{2w}{\pi D}\sinh\frac{2\pi z}{w}\right)}$$

(per cylinder)



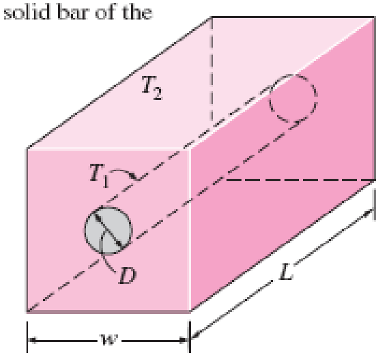
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



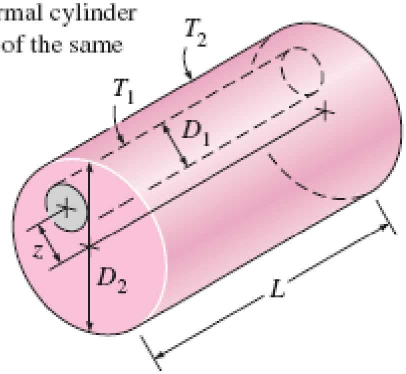
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$



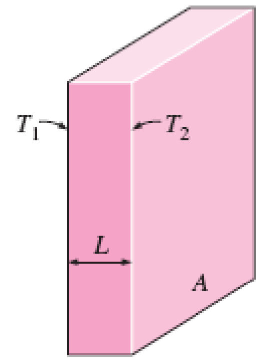
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length $(L > D_2)$

$$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



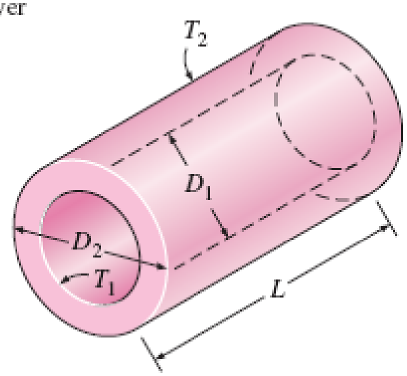
(8) Large plane wall

$$S = \frac{A}{L}$$



(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



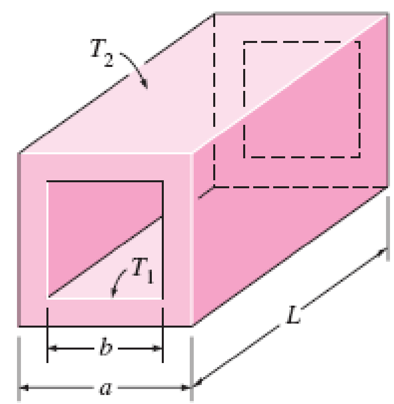
(10) A square flow passage

(*a*) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948a/b)}$$

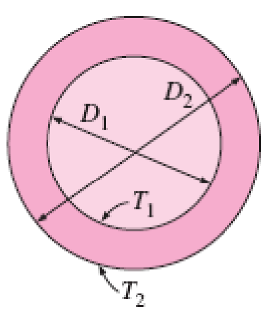
(*b*) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

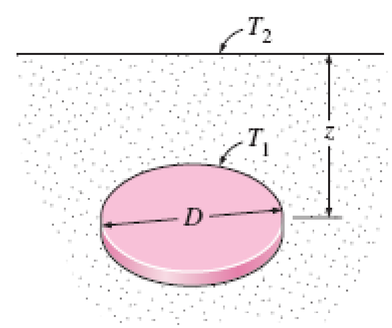
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$

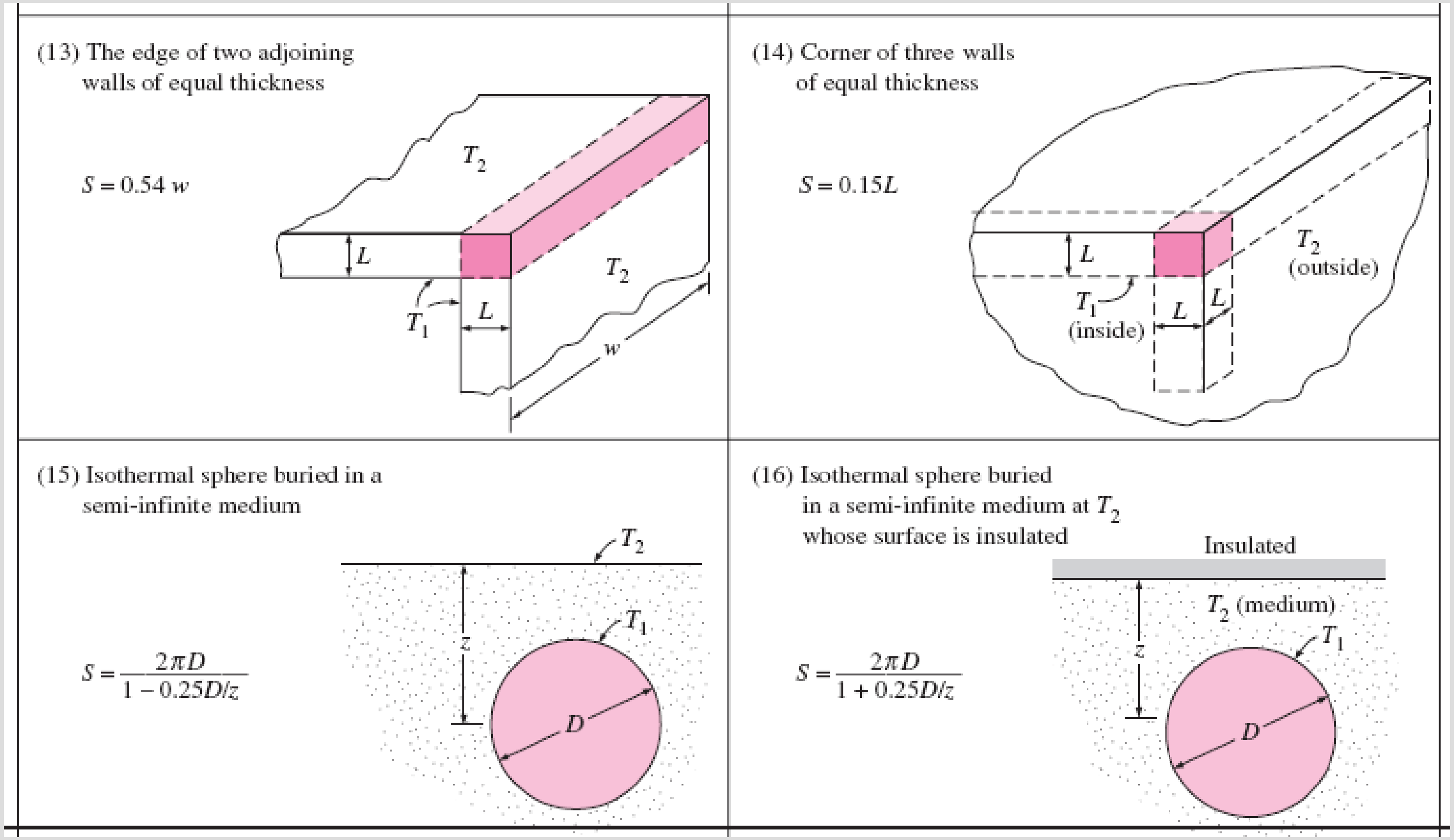


(12) Disk buried parallel to the surface in a semi-infinite medium $(z >> D)$

$$S = 4D$$

$(S = 2D$ when $z = 0)$





Once the value of the shape factor is known for a specific geometry, the total steady heat transfer rate can be determined from the following equation using the specified two constant temperatures of the two surfaces and the thermal conductivity of the medium between them.

$$Q = Sk(T_1 - T_2)$$

Summary

- Heat Transfer from Finned Surfaces
  - ✓ Fin Equation
  - ✓ Fin Efficiency
  - ✓ Fin Effectiveness
  - ✓ Proper Length of a Fin
- Heat Transfer in Common Configurations

Heat and Mass Transfer

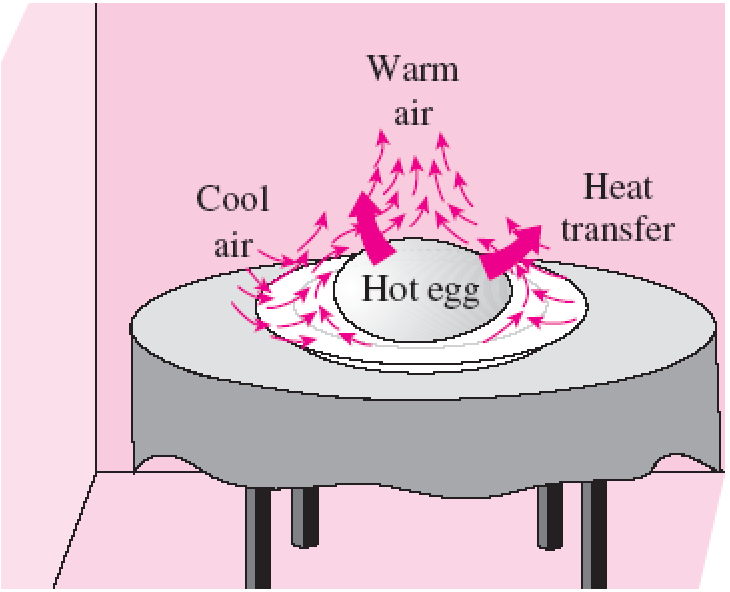
Topic: NATURAL CONVECTION

Objectives

- Understand the physical mechanism of natural convection
- Derive the governing equations of natural convection, and obtain the dimensionless Grashof’s number by nondimensionalizing them
- Evaluate the Nusselt number for natural convection associated with vertical, horizontal, and inclined plates as well as cylinders and spheres
- Analyze natural convection inside enclosures such as double-pane windows

PHYSICAL MECHANISM OF NATURAL CONVECTION

- Many familiar heat transfer applications involve natural convection as the primary mechanism of heat transfer. Examples?
  - Ex- As soon as the hot egg is exposed to cooler air, the temperature of the outer surface of the egg shall drop somewhat, and the temperature of air adjacent to the shell rises as a result of HEAT CONDUCTION from the shell to air

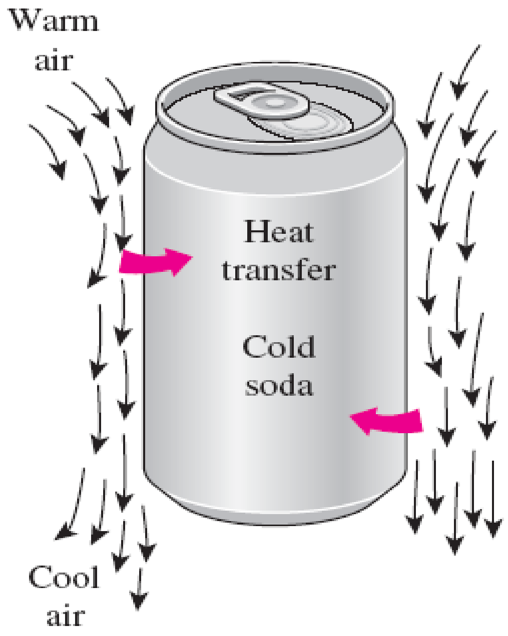


The cooling of a boiled egg in a cooler environment by natural convection.

- Consequently, the egg is surrounded by a thin layer of warmer air and the heat is then transferred from warmer layer to the outer layers of air.
- The temperature of air adjacent to the egg is higher and thus its density is lower, since at constant pressure the density of a gas is inversely proportional to its temperature.
- Natural laws dictates that the light gas rise. The space vacated by the warmer air in the vicinity of egg is replaced by the cooler air nearby and the presence of cooler air in the vicinity of eggs speed up the cooling process.
- The rise of warmer air and the flow of cooler air into its place continues until the egg is cooled to the temperature of surrounding air.

- The motion that results from the continual replacement of the heated air in the vicinity of the hot abject by the cooler air nearby is called a **natural convection current,** and the heat transfer that is enhanced as a result of this current is called **natural convection heat transfer.**

The warming up of a cold drink in a warmer environment by natural convection.



General Considerations

- Free convection refers to fluid motion induced by **buoyancy forces**
- Buoyancy forces may arise in a fluid for which there are **density gradient** and a **body force** that is proportional to **density**
- In heat transfer, **density gradients** are due to **temperature gradients** and the **body force is gravitational**
- **Buoyancy force:** The upward force exerted by a fluid on a body completely or partially immersed in it in a gravitational field
  - The magnitude of the buoyancy force is equal to the weight of the fluid displaced by the body **(Archimedes' Principle)**

$$F_{\text{buoyancy}} = \rho_{\text{fluid}}\, g V_{\text{body}}$$

- The net vertical force acting on a body

$$\begin{aligned} F_{\text{net}} &= W - F_{\text{buoyancy}} \\ &= \rho_{\text{body}}\, g V_{\text{body}} - \rho_{\text{fluid}}\, g V_{\text{body}} \\ &= (\rho_{\text{body}} - \rho_{\text{fluid}})\, g V_{\text{body}} \end{aligned}$$

- **Archimedes' principle:** A body immersed in a fluid will experience a "weight loss" in an amount equal to the weight of the fluid it displaces.
- The **"chimney effect"** that induces the upward flow of hot combustion gases through a chimney is due to the buoyancy effect.
- **NOTE:** Without buoyancy, heat transfer would be by **conduction instead of natural convection** (i.e. if there is no noticeable gravity in space, there can be no natural convection in space-craft even if the spacecraft is filled with atmospheric air)
- Since, in heat transfer the primary variable is temperature, so it is desirable to express net buoyancy force in terms of temperature difference

- A property that represents the variation of density of fluid with temperature at constant pressure is **“volume expansion coefficient”** of **“coefficient of thermal expansion of fluid”** (β)

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \qquad (1/\mathrm{K})$$

$$\beta \approx -\frac{1}{\rho}\frac{\Delta \rho}{\Delta T} = -\frac{1}{\rho}\frac{\rho_\infty - \rho}{T_\infty - T} \qquad (\text{at constant } P)$$

$$\rho_\infty - \rho = \rho\beta(T - T_\infty) \qquad (\text{at constant } P)$$

$$\beta_{\text{ideal gas}} = \frac{1}{T} \qquad (1/\mathrm{K}) \qquad (P = \rho R T)$$

ideal gas

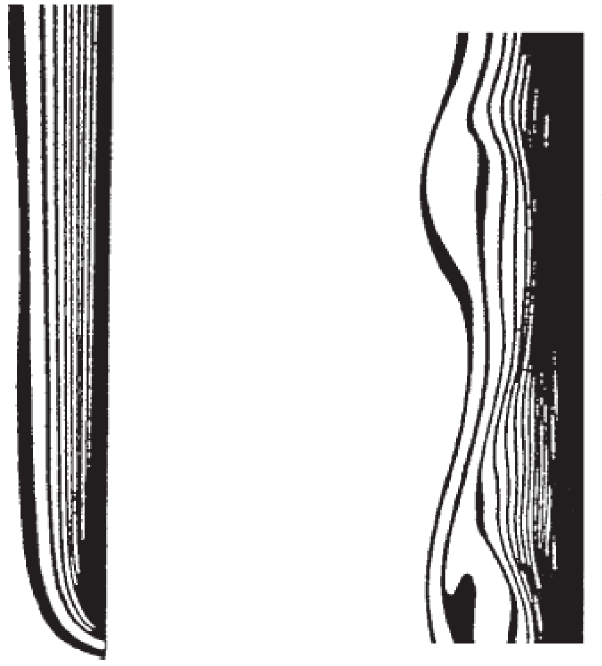
- **Significance:** The larger the temperature difference between the fluid adjacent to a hot (or cold) surface and the fluid away from it, the ***larger*** the buoyancy force and the ***stronger*** the natural convection currents, and thus the ***higher*** the heat transfer rate

- In natural convection, no blowers are used, and therefore the flow rate cannot be controlled externally.
  - The flow rate in this case is established by the dynamic balance of ***buoyancy*** and ***friction.***
- An interferometer produces a map of interference fringes, which can be interpreted as lines of *constant temperature.*

The smooth and parallel lines in (*a*) indicate that the flow is *laminar,* whereas the eddies and irregularities in (*b*) indicate that the flow is *turbulent.*

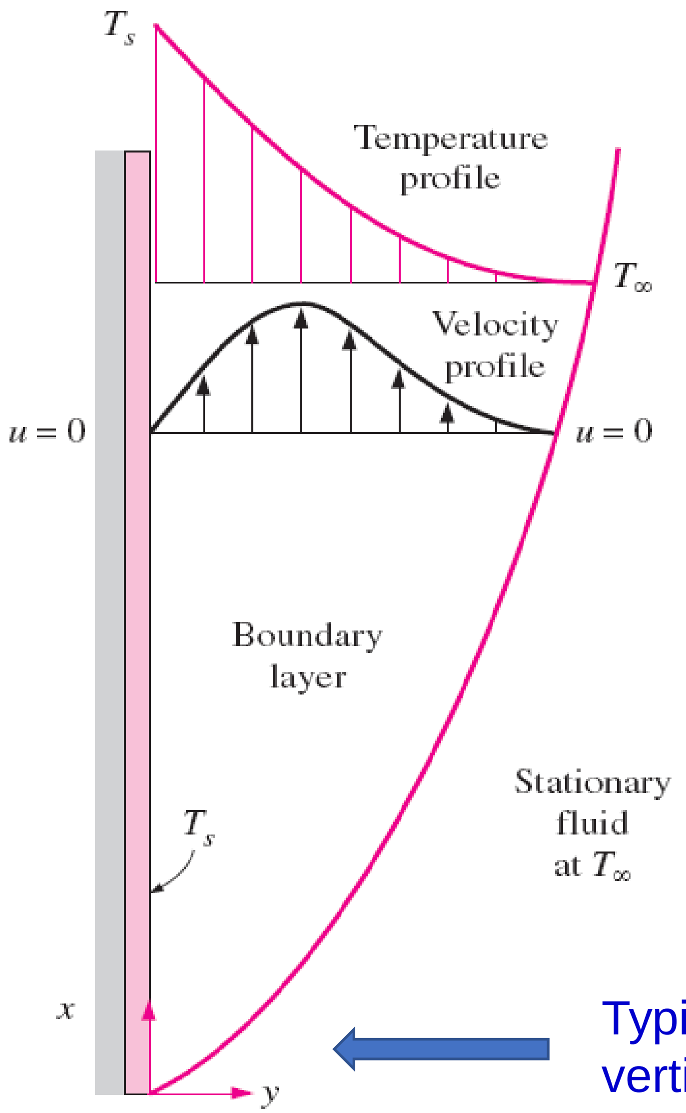
The lines are closest near the surface, indicating a *higher temperature gradient.*

Isotherms in natural convection over a hot plate in air.



(*a*) Laminar flow (*b*) Turbulent flow

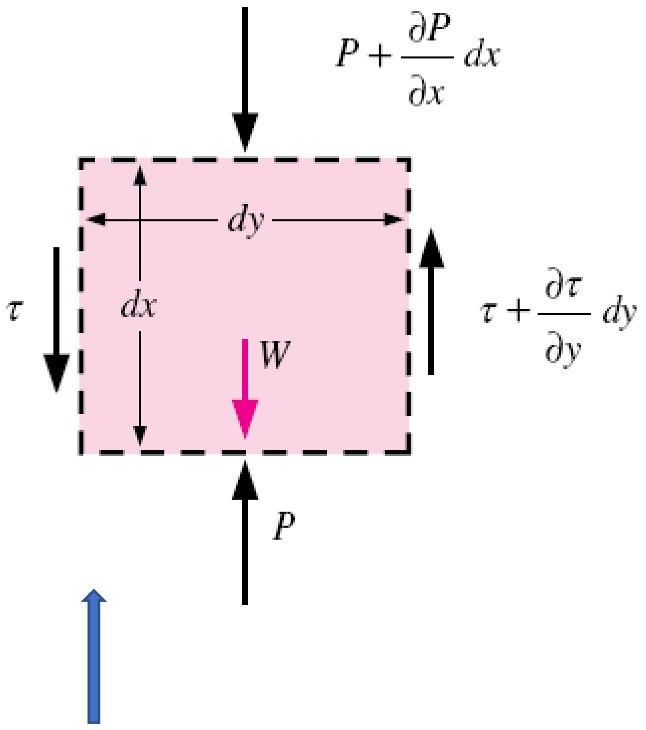
EQUATION OF MOTION AND THE GRASHOF NUMBER



- The thickness of the boundary layer increases in the flow direction
- Unlike forced convection, the **fluid velocity is *zero* at the outer edge of the velocity boundary layer** as well as at the surface of the plate.
- At the surface, the fluid temperature is equal to the plate temperature, and gradually decreases to the temperature of the surrounding fluid at a distance sufficiently far from the surface.
- **In the case of *cold surfaces*, the shape of the velocity and temperature profiles remains the same but their direction is reversed.**

Typical velocity and temperature profiles for natural convection flow over a hot vertical plate at temperature T_s inserted in a fluid at temperature T_∞.

Derivation of the equation of motion that governs the natural convection flow in laminar boundary layer



Forces acting on a differential volume element in the natural convection boundary layer over a vertical flat plate.

Newton's 2nd law of motion →

$$\delta m \cdot a_x = F_x$$

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$F_x = \left(\frac{\partial \tau}{\partial y} dy\right)(dx \cdot 1) - \left(\frac{\partial P}{\partial x} dx\right)(dy \cdot 1) - \rho g(dx \cdot dy \cdot 1)$$

$$= \left(\mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g\right)(dx \cdot dy \cdot 1)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} - \rho g$$

$$\frac{\partial P_\infty}{\partial x} = -\rho_\infty g \qquad \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} + (\rho_\infty - \rho)g$$

$$\rho_\infty - \rho = \rho\beta(T - T_\infty)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty)$$

This is the equation that governs the fluid motion in the boundary layer due to the effect of buoyancy. The momentum equation involves the temperature, and thus the momentum and energy equations must be solved simultaneously.

The complete set of conservation equations, continuity, momentum, and energy that govern natural convection flow over vertical isothermal plates are:

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty)$$

Energy: $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

with the following boundary conditions (see Fig.

At $y = 0$: $u(x, 0) = 0,\ v(x, 0) = 0,\quad T(x, 0) = T_s$

At $y \to \infty$: $u(x, \infty) \to 0,\ v(x, \infty) \to 0,\quad T(x, \infty) \to T_\infty$

The above set of three partial differential equations can be reduced to a set of two ordinary nonlinear differential equations by the introduction of a similarity variable. But the resulting equations must still be solved along with their transformed boundary conditions numerically.

The Grashof Number

The governing equations of natural convection and the boundary conditions can be nondimensionalized by dividing all dependent and independent variables by suitable constant quantities:

$$x^* = \frac{x}{L_c} \qquad y^* = \frac{y}{L_c} \qquad u^* = \frac{u}{V} \qquad v^* = \frac{v}{V} \qquad \text{and} \qquad T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

Substituting them into the momentum equation and simplifying gives:-

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \left[\frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}\right] \frac{T^*}{\mathrm{Re}_L^2} + \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\mathrm{Gr}_L = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

Grashof number: Represents the natural convection effects in momentum equation

g = gravitational acceleration, m/s^2

β = coefficient of volume expansion, 1/K ($\beta = 1/T$ for ideal gases)

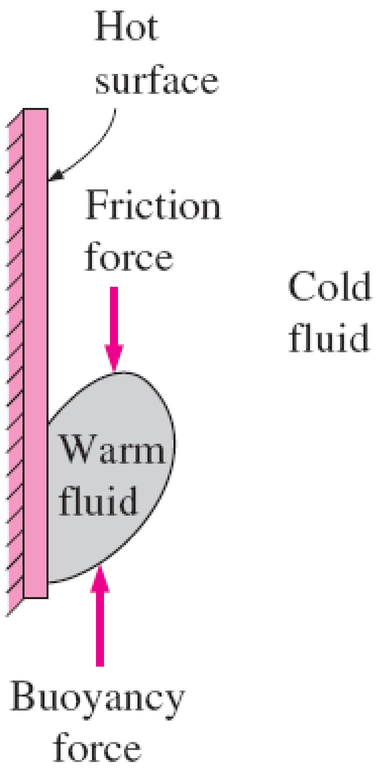
T_s = temperature of the surface, °C

T_∞ = temperature of the fluid sufficiently far from the surface, °C

L_c = characteristic length of the geometry, m

ν = kinematic viscosity of the fluid, m^2/s

- The Grashof number provides the main criterion in determining whether the fluid flow is **laminar or turbulent** in natural convection.
- For vertical plates, the critical Grashof number is observed to be about 10^9 for laminar
- **Conclusion:-** The **role** played by Reynold’s number in forced convection is played by Grashof’s number in natural convection
- **Reasons of Non-Dimensionalising**
  - Easier to recognize when to apply familiar mathematical techniques
  - It reduces the number of times we might have to solve the equation numerically
  - It gives us insight into what might be small parameter that could be ignored or treated approximately
  - It facilitates scale up of obtained results to real condition.



The Grashof number Gr is a measure of the relative magnitudes of the *buoyancy force* and the opposing *viscous force* acting on the fluid.

When a surface is subjected to external flow, the problem involves both natural and forced convection.

The relative importance of each mode of heat transfer is determined by the value of the coefficient Gr/Re^2:

Natural convection effects are negligible if $Gr/Re^2 \ll 1$.

Free convection dominates and the forced convection effects are negligible if $Gr/Re^2 \gg 1$.

Both effects are significant and must be considered if $Gr/Re^2 \approx 1$ (mixed convection).

- Pertinent Dimensionless Parameters
  - Grashof Number:

$$Gr_L = \frac{g\beta(T_s - T_\infty)L^3}{\nu^2} \sim \frac{\text{Buoyancy Force}}{\text{Viscous Force}}$$

$L \rightarrow$ characteristic length of surface

$\beta \rightarrow$ thermal expansion coefficient (a thermodynamic property of the fluid)

$$\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$$

Liquids: $\beta \rightarrow$ Tables A.5, A.6

Perfect Gas: $\beta = 1/T\,(\mathrm{K})$

  - Rayleigh Number:

$$Ra_L = Gr_L\,\mathrm{Pr} = \frac{g\beta(T_s - T_\infty)L^3}{\nu\alpha}$$

Rayleigh Number, Ra=Gr.Pr

- In fluid mechanics, the Rayleigh number for a fluid is a dimensionless number associated with buoyancy driven flow (also known as free convection or natural convection).
- When the Rayleigh number is below the critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection.
- The Rayleigh number is defined as the product of the Grashof number, which describes the relationship between buoyancy and viscosity within a fluid, and the Prandtl number, which describes the relationship between momentum diffusivity and thermal diffusivity.
- Hence the Rayleigh number itself may also be viewed as the ratio of buoyancy and viscosity forces times the ratio of momentum and thermal diffusivities.

NATURAL CONVECTION OVER SURFACES

- Natural convection heat transfer on a surface depends on the **geometry** of the surface as well as **its orientation,** the **variation of temperature** on the surface and the thermophysical properties of the fluid involved.
- With the exception of some simple cases, heat transfer relations in natural convection are based on experimental studies.

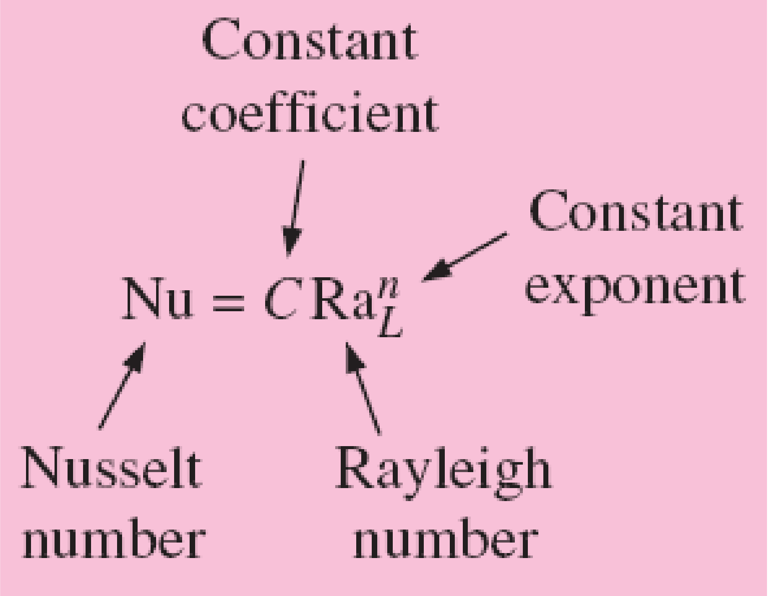
$$\text{Nu} = \frac{hL_c}{k} = C(\text{Gr}_L \, \text{Pr})^n = C \, \text{Ra}_L^n$$

$$\text{Ra}_L = \text{Gr}_L \, \text{Pr} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2} \text{Pr}$$

The constants C and n depend on the *geometry* of the surface and the *flow regime,* which is characterized by the range of the Rayleigh number.

The value of n is usually 1/4 for laminar flow and 1/3 for turbulent flow.

All fluid properties are to be evaluated at the film temperature $T_f = (T_s + T_\infty)/2$.



Natural convection heat transfer correlations are usually expressed in terms of the Rayleigh number raised to a constant *n* multiplied by another constant *C*, both of which are determined experimentally.

- General correlations for vertical plate

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = CRa_L^n$$

where,

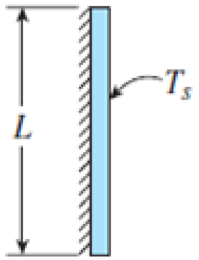
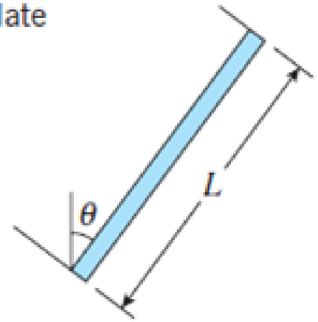
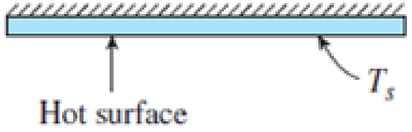
Laminar	$10^4 \leq Ra_L \leq 10^9$	C = 0.59	n = 1/4
Turbulent	$10^9 \leq Ra_L \leq 10^{13}$	C = 0.10	n = 1/3

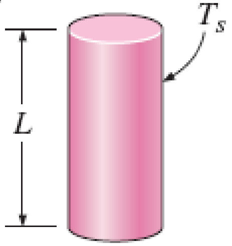
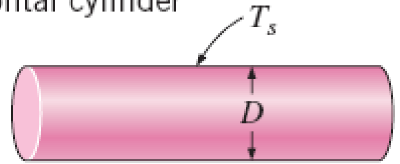
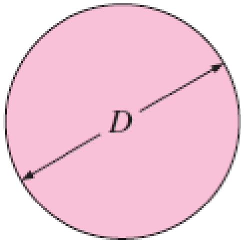
- For wide range and more accurate solution, use correlation Churchill and Chu

  - All Conditions:

$$\overline{Nu}_L = \left\{ 0.825 + \frac{0.387\, Ra_L^{1/6}}{\left[1 + (0.492/\mathrm{Pr})^{9/16}\right]^{8/27}} \right\}^2$$

Empirical correlations for the average Nusselt number for natural convection over surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical plate 	L	10^4–10^9 10^{10}–10^{13} Entire range	$Nu = 0.59Ra_L^{1/4}$ $Nu = 0.1Ra_L^{1/3}$ $Nu = \left\{0.825 + \frac{0.387Ra_L^{1/6}}{[1 + (0.492/Pr)^{9/16}]^{8/27}}\right\}^2$ (complex but more accurate)
Inclined plate 	L		Use vertical plate equations for the upper surface of a cold plate and the lower surface of a hot plate Replace g by $g\cos\theta$ for $0 < \theta < 60°$
Horizontal plate (Surface area A and perimeter p) (*a*) Upper surface of a hot plate (or lower surface of a cold plate)  (*b*) Lower surface of a hot plate (or upper surface of a cold plate) 	A_s/p	10^4–10^7 10^7–10^{11} 10^5–10^{11}	$Nu = 0.54Ra_L^{1/4}$ $Nu = 0.15Ra_L^{1/3}$ $Nu = 0.27Ra_L^{1/4}$

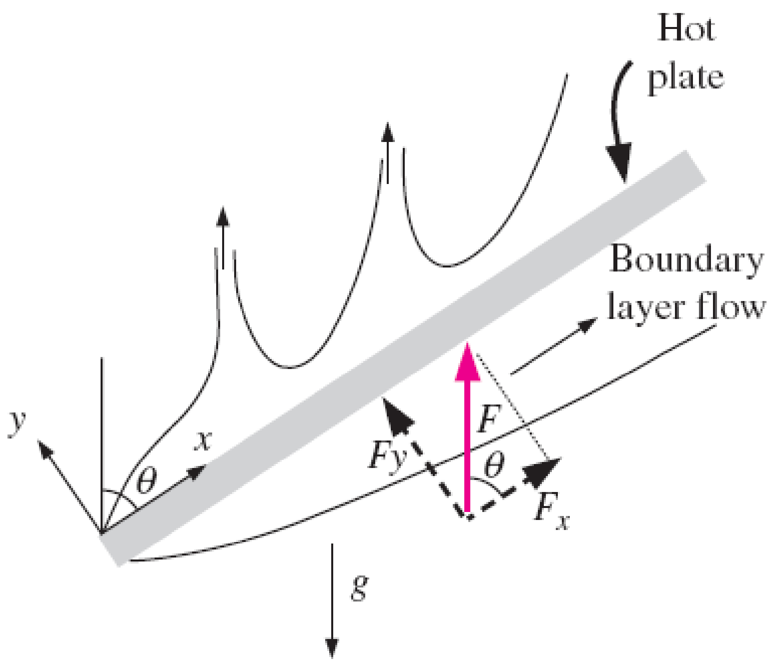
Vertical cylinder 	L		A vertical cylinder can be treated as a vertical plate when $D \geq \dfrac{35L}{Gr_L^{1/4}}$
Horizontal cylinder 	D	$Ra_D \leq 10^{12}$	$Nu = \left\{0.6 + \dfrac{0.387 Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}}\right\}^2$
Sphere 	D	$Ra_D \leq 10^{11}$ ($Pr \geq 0.7$)	$Nu = 2 + \dfrac{0.589 Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$

Vertical Plates (q_s = constant)

The relations for isothermal plates in the table can also be used for plates subjected to uniform heat flux, provided that the plate midpoint temperature $T_{L/2}$ is used for T_s in the evaluation of the film temperature, Rayleigh number, and the Nusselt number.

$$\mathrm{Nu} = \frac{hL}{k} = \frac{\dot{q}_s L}{k(T_{L/2} - T_\infty)} \qquad \dot{Q} = \dot{q}_s A_s$$

Inclined Plates



In a hot plate in a cooler environment for the lower surface of a hot plate, the convection currents are weaker, and the rate of heat transfer is lower relative to the vertical plate case.

On the upper surface of a hot plate, the thickness of the boundary layer and thus the resistance to heat transfer decreases, and the rate of heat transfer increases relative to the vertical orientation.

In the case of a cold plate in a warmer environment, the opposite occurs.

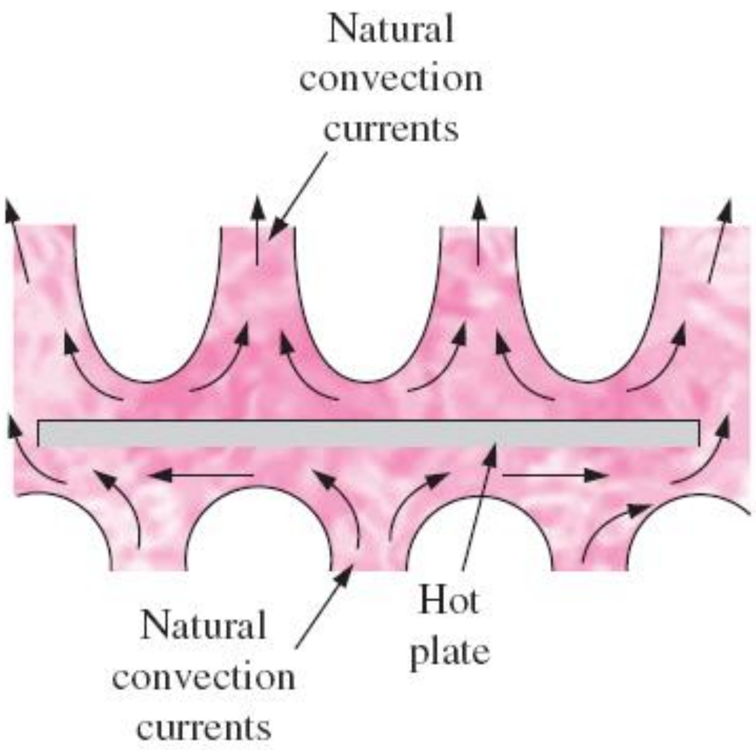
Natural convection flows on the upper and lower surfaces of an inclined hot plate.

Natural Convection

Example:

Consider a 0.6m x 0.6m thin square plate in a room at 30°C. One side of the plate is maintained at a temperature of 90°C, while the other side is insulated. Determine the rate of heat transfer from the plate by natural convection if the plate is vertical.

Horizontal Plates



Natural convection flows on the upper and lower surfaces of a horizontal hot plate.

- For a hot surface in a cooler environment, the net force acts upward, forcing the heated fluid to rise.
- If the hot surface is facing upward, the heated fluid rises freely, inducing strong natural convection currents and thus effective heat transfer.
- But if the hot surface is facing downward, the plate blocks the heated fluid that tends to rise, impeding heat transfer.
- The opposite is true for a cold plate in a warmer environment since the net force (weight minus buoyancy force) in this case acts downward, and the cooled fluid near the plate tends to descend.

$$L_c = \frac{A_s}{p}$$

$L_c = a/4$ for a horizontal square surface of length a

$L_c = D/4$ for a horizontal circular surface of diameter D