Subject Name-
ADVANCE QUANTUM MECHANICS

Subject Code- MPM-221

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Syllabus??

MPM-221: ADVANCE QUANTUM MECHANICS **Credit 04 (3-1-0)**

Unit I: Formulation of Relativistic Quantum Theory

Relativistic Notations, The Klein-Gordon equation, Physical interpretation, Probability current density & Inadequacy of Klein-Gordon equation, Dirac relativistic equation & Mathematical formulation, α and β matrices and related algebra, Properties of four matrices α and β, Matrix representation of $\alpha_i^{\prime s}$ and β, True continuity equation and interpretation.

Unit II: Covariance of Dirac Equation

Covariant form of Dirac equation, Dirac gamma (γ) matrices, Representation and properties, Trace identities, fifth gamma matrix γ^5, Solution of Dirac equation for free particle (Plane wave solution), Dirac spinor, Helicity operator, Explicit form, Negative energy states

Unit III: Field Quantization

Introduction to quantum field theory, Lagrangian field theory, Euler–Lagrange equations, Hamiltonian formalism, Quantized Lagrangian field theory, Canonical commutation relations, The Klein-Gordon field, Second quantization, Hamiltonian and Momentum, Normal ordering, Fock space, The complex Klein-Gordan field: complex scalar field

Unit IV: Approximate Methods

Time independent perturbation theory, The Variational method, Estimation of ground state energy, The Wentzel-Kramers-Brillouin (WKB) method, Validity of the WKB approximation, Time-Dependent Perturbation theory, Transition probability, Fermi-Golden Rule

Books & References:

1: Advance Quantum Mechanics by J. J. Sakurai (Pearson Education India)

2: Relativistic Quantum Mechanics by James D. Bjorken and Sidney D. Drell (McGraw-Hill Book Company; New York, 1964).

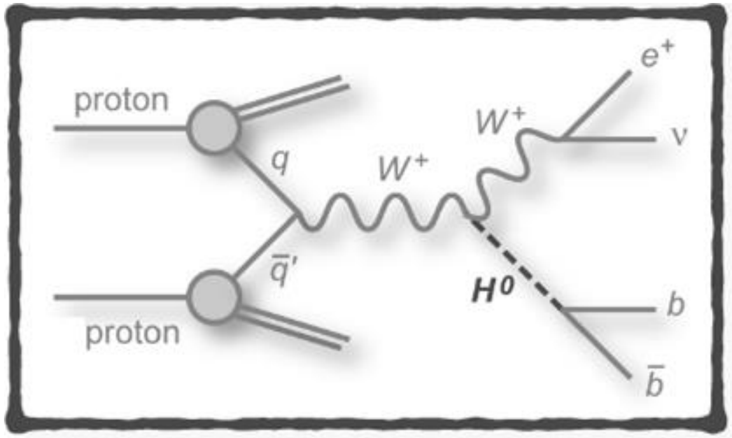
3: An Introduction to Relativistic Quantum Field Theory by S.S. Schweber (Harper & Row, New York, 1961).

4: Quantum Field Theory by F. Mandl & G. Shaw (John Wiley and Sons Ltd, 1984)

5: A First Book of Quantum Field Theory by A. Lahiri & P.B. Pal (Narosa Publishing House, New Delhi, 2000)

Session 2020-21

Lectures of Unit- III



Unit III: Field Quantization

Introduction to quantum field theory

Unit-III

1

Quantum Field Theory (QFT)

What is Quantum Field Theory? wiki

Quantum Field Theory :- as its name implies, it is the quantization of classical field, the most familiar example of which is the electromagnetic field. In standard quantum mechanics, we take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory is operator valued functions of space and time. This means that we are dealing with an infinite no. of degrees of freedom – at least one for every point in space.

For any relativistic system QFT is a necessary. But it is also a very useful tool in non-relativistic systems with many particles. QFT has had a major impact in condensed matter, high-energy physics, cosmology, quantum gravity and pure mathematics. It is literally the language in which the laws of Nature are

Unit III: Field Quantization

Classical field theory

Classical Field Theory :—

The dynamics of Fields :—

A field is a quantity defined at every point of space and time $(\vec{x}, t)$. While classical mechanics deals with a finite number of generalized coordinate $q_r(t)$, indexed by a label r, in field theory we are interested in the dynamics of fields

$$\phi_r(\vec{x}, t) \longrightarrow (1)$$

where both r and $\vec{x}$ are considered as labels. Thus we are dealing with a system with an infinite number of degrees of freedom — at least one for each point $\vec{x}$ in space.

An example: The Electromagnetic Fields:—

The most familiar examples of field from classical physics are the electric and magnetic fields, $\overline{E}(\vec{x}, t)$ and $\overline{B}(\vec{x}, t)$. Both of these are spatial 3-vectors. We can derive these two 3-vectors from a single 4 component field.

$$A^\mu(\vec{x}, t) = (\phi, \vec{A}) \quad ; \quad \mu = 0, 1, 2, 3 \longrightarrow (2)$$

This shows that field is a vector in spacetime.

The electric & magnetic fields are given by—

$$\overline{E} = -\nabla\phi - \frac{\partial A}{\partial t} \quad \text{and} \quad \overline{B} = \nabla \times \overline{A} \longrightarrow (3)$$

Lagrangian field theory

Classical Lagrangian Field Theory:-

We consider a system which requires several fields $\phi_r(x), r = 1, 2, \dots, N$ as a characterizing specifying character of system (field), taken as field variable on each point of space Ω at x. The index r may label components of the same field or it may refer to different independent field.

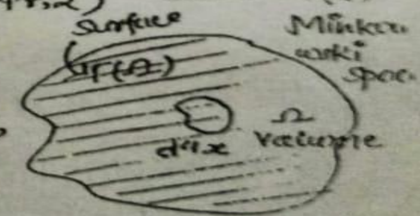
Now we restrict ourselves to theories which can be derived by variational principle from an action integral involve a Lagrangian density –

$$\mathcal{L} = \mathcal{L}(\phi_r, \phi_{r,\alpha}) \quad \text{---} \quad (1)$$

$$\text{where, } \phi_{r,\alpha} = \partial_\alpha \phi_r = \frac{\partial \phi_r}{\partial x^\alpha}$$

We define the action integral $S(\Omega)$ for an arbitrary region Ω of the four dimensional space - time continuum by –

$$S(\Omega) = \int_\Omega d^4x \, \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (2)$$



For an arbitrary region Ω, we consider variation of the fields,

$$\phi_r(x) \longrightarrow \phi_r(x) + \delta\phi_r(x) \longrightarrow (3)$$

which vanish on the surface $\Gamma(\Omega)$ boundary of the region Ω.

Lagrangian field theory

$$\delta\phi_r(x) = 0 \quad \text{on } \Gamma(\Omega) \qquad (4)$$

The fields ϕ_r may be real or complex. In the case of complex field $\phi(x)$, the fields $\phi(x)$ and $\phi^*(x)$ are treated as two independent fields. Alternatively, a complex field $\phi(x)$ can be decomposed into a pair of real fields, which are then treated as independent fields.

For an arbitrary region and the variation, the action has a stationary value, i.e.

$$\delta S(\Omega) = 0 \qquad (5)$$

From equation (2), we get

$$\delta S(\Omega) = \int_\Omega d^4x\, \delta[\mathcal{L}(\phi_r, \phi_{r,\alpha})]$$

$$= \int_\Omega d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_r}\delta\phi_r + \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\delta\phi_{r,\alpha} \right\}$$

$\because$

$$\delta\phi_{r,\alpha} = \frac{\partial}{\partial x^\alpha}\delta\phi_r$$

$\therefore$

$$\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\left(\frac{\partial}{\partial x^\alpha}\delta\phi_r\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\delta\phi_r\right) - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\right)\delta\phi_r$$

(using partial Integration)

Hence,

$$\delta S(\Omega) = \int_\Omega d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\right)\right]\delta\phi_r(x) + \int_\Omega d^4x \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\delta\phi_r(x)\right) \qquad (6)$$

The last term in eqn (6) can be converted into a surface integral over the surface $\Gamma(\Omega)$ using Gauss's divergence theorem in

Lagrangian field theory

four dimensions.

$$\int_{\Omega} d^4x \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r(x)\right) = \int_{\Gamma} dS_\alpha \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r(x)$$

$$= 0 \quad (\text{Since } \delta\phi_r = 0 \text{ on } \Gamma)$$

$\therefore \quad \delta S(\Omega) = 0$ Thus for arbitrary Ω finally we have-

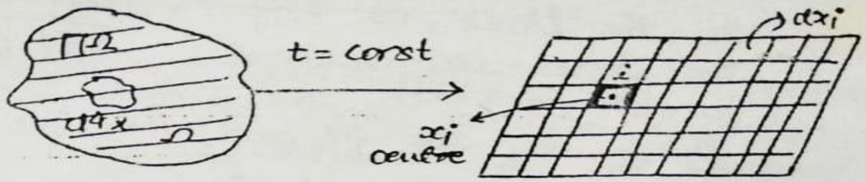
$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\right) = 0}, \quad r = 1, 2, \ldots N \longrightarrow (7)$$

These are the equations of motion of fields (The Euler - Lagrange equations)

We are dealing with a system with a continuous infinite number of degrees of freedom, corresponding to the values of the fields ϕ_r, considered as functions of time, at each point of space x. We shall again approximately the system by one having a countable number of degrees of freedom and ultimately go to the continuum limit.

Consider the system at fixed instant of time t and decomposes the three-dimensional space i.e. the flat-space-like surface t = const., into small cells of equal volume δx_i, labelled by the index $i = 1, 2, \ldots$ We approximate the values of the fields within each cell by their values at the centre of the cell $x = x_i$.

Lagrangian field theory



Flat-space-like Euclidean surface (3 dim)

The system is now described by the discrete set of generalized coordinates –

$$q_{ri}(t) \equiv \phi_r(i,t) \equiv \phi_r(x_i,t) \longrightarrow (8)$$

$$r = 1, 2, \ldots, N \;;\; i = 1, 2, \ldots$$

which are the values of the fields at the discrete lattice sites x_i. If we also replace the spatial derivatives of the fields by their difference coefficients between neighbouring sites, we can write the Lagrangian of the discrete system as –

$$L(t) = \sum_i \delta x_i \, \mathcal{L}_i\left(\phi_r(i,t), \dot{\phi}_r(i,t), \phi_r(i',t)\right) \longrightarrow (9)$$

Hamiltonian Formalism:

We define momenta conjugate to q_{ri} in the usual way as –

$$p_{ri}(t) = \frac{\partial L}{\partial \dot{q}_{ri}} \equiv \frac{\partial L}{\partial \dot{\phi}_r(i,t)} \equiv \pi_r(i,t)\,\delta x_i \longrightarrow (10)$$

$$\text{where } \pi_r(i,t) = \frac{\partial \mathcal{L}_i}{\partial \dot{\phi}_r(i,t)} \longrightarrow (11)$$

The Hamiltonian of the discrete system is given by –

Lagrangian field theory

$$H = \sum_i p_{ri} \dot{q}_{ri} - L$$
$$= \sum_i \delta x_i \{ \pi_r(i,t) \dot{\phi}_r(i,t) - L_i \} \longrightarrow (12)$$

Taking the limit $\delta x_i \to 0$ i.e. letting the cell size and the lattice spacing shrink to zero, we define the fields conjugate to $\phi_r(x)$ as –

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \longrightarrow (13)$$

In the limit $\delta x_i \to 0$, $\pi_r(i,t) \to \pi_r(x_i,t)$ and the discrete Lagrangian and Hamilton functions (9) and (12) becomes –

$$L(t) = \int d^3x\, \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (14)$$

and

$$H = \int d^3x\, \mathcal{H}(x) \longrightarrow (15)$$

where the Hamiltonian density $\mathcal{H}(x)$ is defined by –

$$\mathcal{H}(x) = \pi_r(x) \dot{\phi}_r(x) - \mathcal{L}(\phi_r, \phi_{r,\alpha}) \longrightarrow (16)$$

Integrations in (14) & (15) are all over all space at time t.

$$P^j = \int d^3x\, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j} \longrightarrow (16')$$

Example :– Consider the Lagrangian density –

$$\mathcal{L} = \frac{1}{2}(\phi_{,\alpha}\phi^{,\alpha} - \mu^2\phi^2) \longrightarrow (17)$$

for a single real field $\phi(x)$ with μ a constant, which has the dimension of $(\text{length})^{-1}$.

The equation of motion of this field is the Klein-Gordon equation

$$(\Box + \mu^2)\,\phi(x) = 0 \quad \longrightarrow (18)$$

The conjugate field is –

$$\pi(x) = \frac{1}{c^2}\,\dot{\phi}(x) \quad \longrightarrow (19)$$

and the Hamiltonian dencity is

$$\mathcal{H}(x) = \frac{1}{2}\left[c^2\pi^2(x) + (\nabla\phi)^2 + \mu^2\phi^2\right] \quad \longrightarrow (20).$$

Quantized Lagrangian Field Theory :–

Now it is easy to go from the classical to the quantum field theory by interpreting the conjugate coordinates and momenta of the the discrete lattice approximation, equation (8) and (10), as Heisenberg operators, and subjecting these to the usual canonical commutation relations:

$$\left.\begin{aligned}
&[\phi_r(j,t),\, \pi_s(j',t)] = i\hbar\,\frac{\delta_{rs}\,\delta_{jj'}}{\delta \bar{x}_j} \\
&[\phi_r(j,t),\, \phi_s(j',t)] = [\pi_r(j,t),\, \pi_s(j',t)] = 0
\end{aligned}\right\} \rightarrow (21)$$

Let the lattice spacing go to zero, then (21) becomes –

$$\left.\begin{aligned}
&[\phi_r(\bar{x},t),\, \pi_s(\bar{x}',t)] = i\hbar\,\delta_{rs}(\bar{x}-\bar{x}') \\
&[\phi_r(\bar{x},t),\, \phi_s(\bar{x}',t)] = [\pi_r(\bar{x},t),\, \pi_s(\bar{x}',t)] = 0
\end{aligned}\right\} \rightarrow (22)$$

Lagrangian field theory

In the limit, as $\delta\bar{x}_j \to 0$, $\frac{\delta jj'}{\delta \bar{x}_j}$ becomes the three-dimensional Dirac delta function $\delta(\bar{x}-\bar{x}')$, the points $\bar{x}$ and $\bar{x}'$ lying in the j^{th} and j'^{th} cell respectively.

For the Klein-Gordan fields, equation (22) reduce to the commutation relations—

$$[\phi(\bar{x},t), \dot{\phi}(\bar{x}', t)] = i\hbar c^2 \delta(\bar{x}-\bar{x}')$$

$$[\phi(\bar{x},t), \phi(\bar{x}',t)] = [\dot{\phi}(\bar{x},t), \dot{\phi}(\bar{x}',t)] = 0 \longrightarrow (23)$$

THE KLEIN-GORDAN FIELD.

: The Real Klein-Gordan Field :— (For spin-0)

For particle of rest mass m, energy momentum are related by—

$$E^2 = m^2c^4 + c^2\bar{p}^2 \longrightarrow (1)$$

In quantum mechanics—

$$\bar{p} \longrightarrow -i\hbar\bar{\nabla}, \; \bar{E} \rightarrow i\hbar\, \partial/\partial t$$

Then we have,

$$(\Box + \mu^2)\phi(x) = 0 \longrightarrow (2)$$

where $\mu \equiv mc/\hbar$

From equation (1), energy eigen values are—

$$E^2 = p^2 + m^2 \Rightarrow E = \pm\sqrt{p^2+m^2}$$

(In the unit $c=1$)