**MMS-107** **Topology** **Credit: 4 (3-1-0)**

## UNIT- I

Definitions and examples of topological spaces, Topology induced by a metric, closed sets, Closure, Dense subsets, Neighbourhoods, Interior, Exterior and boundary accumulation points and derived sets, Bases and subbases. **9**

## UNIT- II

Topology generated by the subbases, subspaces and relative topology.Alternative methods of defining a topology in terms of Kuratowski closure operator and neighbourhood systems. Continuous functions and homeomorphism. First and second countable space. Lindelöf spaces. Separable spaces. **9**

## UNIT- III

The separation axioms T0, T1, T2, T3½, T4; their characterizations and basic properties. Urysohn's lemma. Tietz extension theorem. Compactness. Basic properties of compactness. Compactness and finite intersection property. Sequential, countable, and B-W compactness. Local compactness. **9**

## UNIT- IV

Connected spaces and their basic properties. Connectedness of the real line. Components. Locally connected spaces. Tychonoff product topology in terms of standard sub-base and its characterizations. Product topology and separation axioms, connected-ness, and compactness, Tychonoff's theorem, countability and product spaces. **9**

## Books/References

**1.** GF Simmons: Introduction to Topology and Modern Analysis, Mc Graw Hill, 1963
**2.** James R Munkres: Topology, A first course, Prentice Hall, New Delhi, 2000
**3.** JL Kelly: Topology, Von Nostrand Reinhold Co. New York, 1995.
**4.** K.D. Joshi : Introduction to General Topology, Wiley Eastern Ltd.
**5.** J. V. Deshpande: Introduction to Topology, Tata McGraw Hill, 1988.

# SEPARATION AXIOM:

Unit-3 &4

## $T_0$ - SPACE: (DEF).

A topological space $(X,\mathcal{T})$ is said to be $T_0$-space if for each $x,y\in X$ such that $x\neq y$, either there exists an open set $U$ such that $x\in U$ and $y\notin U$ or there exists an open set $V$ such that $y\in V$ and $x\notin V$.

## EXAMPLES:

1) Let $X=\{1,2\}$ and $\mathcal{T}=\{\phi,X,\{1\}\}$, then $\{X,\mathcal{T}\}$ is $T_0$ space.

2). $(X;\mathcal{T}_D)$ is also $T_0$-space.

3). $(X,\mathcal{T}_I)$ is not $T_0$-space.

**THEOREM:** Every subspace of $T_0$ space is $T_0$.

**PROOF:** Let $(X,\mathcal{T})$ be a $T_0$-space and $Y$ be a subspace of $X$. To prove: $Y$ is $T_0$.

Let $x,y\in Y$ such that $x\neq y$.

As $x,y\in Y$ and $Y\subseteq X$ so $x,y\in X$.

Since, $X$ is $T_0$ space, so either there exist an open set $U$ such that $x\in U$ and $y\notin U$ or ~~their~~ there exist an open set $V$ such that $y\in V$ and $x\notin V$.

With out any loss of generality, suppose there exist an open set $U$ such that $x\in U$ and $y\notin U$.

Now as $U$ is open in $X$, so $U\cap Y$ is open in $Y$.

As $x\in U$ and $x\in Y\Rightarrow x\in U\cap Y=U_1$.

As $y\notin U\Rightarrow y\notin U\cap Y=U_1$.

Hence, we have found open set $U_1$ in $Y$

such that $x \in U_1$ and $y \notin U_1$.
$\Rightarrow$ Y is $T_0$ space.

THEOREM: A topological space $(X, \mathcal{T})$ is $T_0$ iff for every $a, b \in X$ such that $a \neq b$ then $\{\bar{a}\} \neq \{\bar{b}\}$.

PROOF: Suppose X is $T_0$-space and $a, b \in X$ such that $a \neq b$. To prove: $\{\bar{a}\} \neq \{\bar{b}\}$.
Since X is $T_0$ and $a, b \in X$ with $a \neq b$, so say, there exists an open set U in X such that $a \in U$ and $b \notin U$.

As, $a \in U$ and $b \notin U \Rightarrow U$ is an open set containing 'a' such that $U \cap \{b\} = \phi$.
$\Rightarrow a \notin \{\bar{b}\}$ ($\bar{A} = \{x :$ for each open set U containing $x$, $U \cap A \neq \phi\}$).
$\Rightarrow \{a\} \nsubseteq \{\bar{b}\}$.
Further as, $\{a\} \subseteq \{\bar{a}\}$
$\Rightarrow \{\bar{a}\} \nsubseteq \{\bar{b}\}$.
$\Rightarrow \{\bar{a}\} \neq \{\bar{b}\}$.

Conversely suppose that for every $a, b \in X$ such that $a \neq b$, then $\{\bar{a}\} \neq \{\bar{b}\}$.
To prove: X is $T_0$-space.
Suppose on the contrary that X is not $T_0$-space. Then, there is a pair $a, b \in X$ such that $a \neq b$ and for every open set U containing 'a' also contain 'b' and for every open set V containing 'b' contains 'a'.

Hence, now for every open set U containing

$a, U \cap \{b\} \neq \phi \Rightarrow a \in \{\bar{b}\} \Rightarrow \{a\} \subseteq \{\bar{b}\}$

$\Rightarrow \{\bar{a}\} \subseteq \{\bar{b}\}$ ($\because$ If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$).

$\Rightarrow \{\bar{a}\} \subseteq \{\bar{b}\}$ ($\because A = \bar{A} \Rightarrow \bar{A} = \bar{A}$)

Similarly, $\{\bar{b}\} \subseteq \{\bar{a}\}$.

$\Rightarrow \{\bar{a}\} = \{\bar{b}\}$.

Which is a contradiction.

$\therefore \{\bar{a}\} \neq \{\bar{b}\}$.

So our supposition is wrong.

Hence, X is $T_0$ space.

## $T_1$ - SPACE: (DEF).

A topological space $(X, \mathcal{T})$ is said to be $T_1$-space if for every $x, y \in X$ such that $x \neq y$, there exists two open sets $U$ and $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

**EXAMPLE:** Let $X = \{1, 2\}$

$\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$

**THEOREM:**

Every subspace of $T_1$-space is $T_1$.

**PROOF:**

Let $(X, \mathcal{T})$ be a $T_1$-space and $Y$ be a subspace of $X$. To prove: $Y$ is $T_1$.

Let $x, y \in Y$ such that $x \neq y$.

Now, $x, y \in Y$ and $Y \subseteq X$, so $x, y \in X$.

As $X$ is $T_1$ space, so there exists two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Now as $U$ and $V$ are open sets in $X$, so $U_1 = U \cap Y$ and $V_1 = V \cap Y$ are open in $Y$.

As $x \in U$ and $x \in Y \Rightarrow x \in U_1$ and as $y \notin U \Rightarrow y \notin U_1$.

As $y \in V$ and $y \in Y \Rightarrow y \in V \cap Y = V_1$ and as $x \notin V \Rightarrow x \notin V \cap Y = V_1$

Hence, we have found two open sets $U_1$ and $V_1$ in $Y$ such that:

$x \in U_1, y \notin U_1$ and $y \in V_1, x \notin V_1$

Hence $Y$ is $T_1$-space.

**THEOREM:** Prove that every $T_1$-space is $T_0$-space.

**PROOF:** Let $X$ be $T_1$-space. To prove: $X$ is $T_0$-space.
Let $x, y \in X$ such that $x \neq y$.
Since, $X$ is $T_1$, so there exists two open sets $U$ and $V$ in $X$ such that $x \in U$; $y \notin U$ and $y \in V, x \notin V$.
Since here we have an open set $U$ in $X$ with $x \in U, y \notin U \Rightarrow X$ is also $T_0$-space.

**REMARK:** Converse of the above theorem is not true in general i.e a $T_0$-space need not necessarily to be $T_1$-space.

**EXAMPLE:** Let $X = \{1, 2\}$, $\mathcal{T} = \{\phi, X, \{1\}\}$.
Hence $X$ is $T_0$-space. But it is not $T_1$.

**THEOREM:** A topological space $(X, \mathcal{T})$ is $T_1$-space iff each singleton subset of $X$ is closed in $X$.

**PROOF:** Let us suppose $X$ is $T_1$-space.
To prove: Each singleton subset of $X$ is closed.
Let $\{x\}$ be the singleton subset of $X$.
To prove: $\{x\}$ is closed.
For this, we prove $\{x\}'$ is open in $X$.
Let $y \in \{x\}' \Rightarrow y \notin \{x\} \Rightarrow x \neq y$.
Hence, we have $x, y \in X$ such that $x \neq y$ and $X$

is $T_1$- Space. So there exists two open sets $U_x$ and $V_y$ in $X$ such that $x \in U_x$, $y \notin U_x$ and $y \in V_y$, $x \notin V_y$.

Now as $V_y \subseteq X$ and $x \notin V_y \Rightarrow V_y \subseteq X - \{x\}$.

As $y \in V_y \subseteq X - \{x\} \Rightarrow \{y\} \subseteq V_y \subseteq \{x\}'$

$$\Rightarrow \bigcup_{y \in \{x\}'} \{y\} \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'.$$

$$\Rightarrow \{x\}' \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'.$$

$$\Rightarrow \{x\}' = \bigcup_{y \in \{x\}'} V_y.$$

Since, $V_y$ is open set and union of any number of open sets is open. So $\bigcup_{y \in \{x\}'} V_y$ is open.

$\Rightarrow \{x\}'$ is open $\Rightarrow \{x\}$ is closed.

Conversely, suppose each singleton subset of $X$ is closed. To prove: $X$ is $T_1$- space.

For this, let $x, y \in X$ such that $x \neq y$.

Then by supposition $\{x\}$ and $\{y\}$ are closed.

$\Rightarrow \{x\}'$ and $\{y\}'$ are open.

Let $U = \{y\}'$ and $V = \{x\}'$

Then $x \in U$, $y \notin U$, $y \in V$, $x \notin V$.

$\Rightarrow X$ is $T_1$- space.

**THEOREM:** Every finite $T_1$- space is discrete.

**PROOF:** Let $X$ be a $T_1$- space. To prove: $X$ is discrete. For this, we will have to show that each subset of $X$ is closed. Let $A \subseteq X$.

If $A=\phi$, then $A$ is closed.
If $A\neq\phi$, then $A$ contains some elements.
Since $X$ is finite. So $A$ is also finite.
Let $A=\{x_1,x_2,\dots,x_n\} \Rightarrow A=\bigcup_{i=1}^{n}\{x_i\}$

Since $X$ is $T_1$. So each singleton subset of $X$ is closed $\Rightarrow$ for each $i$, $1\leq i\leq n$, $\{x_i\}$ is closed.
Since union of finite number of closed sets is closed. So $\bigcup_{i=1}^{n}\{x_i\}$ is closed $\Rightarrow A$ is closed.

$\Rightarrow X$ is discrete.

**THEOREM:** A topological space $(X,\mathcal{T})$ is $T_1$ space iff each subset of $X$ is the intersection of its open supersets.

**PROOF:** Let $X$ be a $T_1$-space and $A\subseteq X$.
To prove: $A$ is the intersection of its open supersets.
Let $y\in X$ such that $y\notin A \Rightarrow y\in A'$
Now, as $X$ is $T_1$-space. So, each singleton subset of $X$ is closed $\Rightarrow \{y\}$ is closed.
$\Rightarrow \{y\}'$ is open. Now as $A\subseteq X$ and $y\notin A$.

$\Rightarrow A\subseteq X-\{y\} \Rightarrow A\subseteq\{y\}'$.
$\Rightarrow \{y\}'$ is an open superset of $A$.
Now, we prove $A=\bigcap_{y\in A'}\{y\}'$

Let $x\in A \Rightarrow x\in\{y'\}$ for all $y\in A'$
$\Rightarrow x\in\bigcap_{y\in A'}y' \Rightarrow A\subseteq\bigcap_{y\in A'}\{y'\} \to ①$

Now, let $x \in \bigcap_{y \in A'} \{y\}' \Rightarrow x \in \{y\}'$ for all $y \in A'$.

$\Rightarrow x \notin \{y\}, \forall y \in A' \Rightarrow x \neq y, \forall y \in A'$.

$\Rightarrow x \notin A' \Rightarrow x \in A \Rightarrow \bigcap_{y \in A'} \{y\}' \subseteq A \rightarrow$ ②

① and ② $\Rightarrow A = \bigcap_{y \in A'} \{y\}'$.

Hence, A is the intersection of its supersets.

Conversely, suppose that in topological space $(X, \mathcal{T})$ each subset of X is the intersection of its open supersets. To prove: X is $T_1$.

Suppose X is not $T_1$.

Then, there exists $x, y \in X$ such that $x \neq y$.

And either each open set containing $x$ also contains $y$ or each open set containing $y$ also contains $x$.

Say, each open set containing $x$ also contains $y$, so by given condition $\{x\}$ is the intersection of its open supersets.

$\Rightarrow \{x\}$ is open $\Rightarrow y \in \{x\} \Rightarrow y = x$.

Which is a contradiction $\because y \neq x$.

So our supposition is wrong.

Hence, X is $T_1$.

**THEOREM:** Let X be $T_1$ space and $A \subseteq X$ and $x \in X$ is the limit point of A. Then every open set containing $x$ contains infinite number of distinct points of A.

**PROOF:**

Suppose given is not true i.e each open set containing $x$ doesnot contain infinite number of distinct points of A. Then, there exists an open set U containing $x$ which contains finite number of distinct points of A. i.e,

$$U \cap A = \{x_1, x_2, x_3, \ldots, x_n\} = B$$

As X is $T_1$ space and B is finite subset of X. So, B is closed.

$\Rightarrow B'$ is open, as $x \notin B$ so $x \in B'$.

$\Rightarrow B'$ is an open set containing $x$.

Also $B' \cap A = \phi \Rightarrow B' \cap A \setminus \{x\} = \phi \Rightarrow x \notin D(A)$

Which is a contradiction. $\because x \in D(A)$

So our supposition is wrong.

And hence, each open set in X containing $x$ contains infinite number of distinct points of A.

**$T_2$ SPACE: (DEF).**

Let $(X, \mathcal{T})$ be a topological space, then it is said to be $T_2$- space or Hausdorff space if for every $x, y \in X$ such that $x \neq y$, then there exists two open sets $U$ and $V$ such that $x \in U, y \in V$ and $U \cap V = \phi$.

**EXAMPLE:** Let $X = \{1, 2\}$, $\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$.

$1 \in \{1\}$, $2 \in \{2\}$, $\{1\} \cap \{2\} = \phi$.

$\Rightarrow (X, \mathcal{T})$ is a $T_2$- space.

**THEOREM:** Every $T_2$ space is $T_1$- space.

**PROOF:**

Let $X$ be $T_2$ space, To prove: $X$ is $T_1$- space.

Let $x, y \in X$ such that $x \neq y$.

As $X$ is $T_2$, so there exists two open sets $U$ and $V$ such that: $x \in U$, $y \in V$ and $U \cap V = \phi$.

Now as $x \in U$ and $U \cap V = \phi \Rightarrow x \notin V$.

As $y \in V$ and $U \cap V = \phi \Rightarrow y \notin U$.

$\Rightarrow X$ is $T_1$-space.

**REMARK:** Converse of the above theorem is not true in general i.e, a $T_1$-space is not necessarily $T_2$-space.

**EXAMPLE:** Let $X = N$ and $\mathcal{F} = \mathcal{F}_c$.

Now, for every $x, y \in X$ such that $x \neq y$. We have:

$x \in X - \{y\}$, $y \notin X - \{y\}$ and $y \in X - \{x\}$, $x \notin X - \{x\}$.

$X - \{x\}$ and $X - \{y\}$ are open in $X$.

Hence, $X$ is $T_1$-space.

But $X$ is not $T_2$-space.

Because, on the contrary if we suppose that $X$ is $T_2$-space, then there exists two open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Now $U \cap V = \phi \Rightarrow (U \cap V)' = \phi'$

$\Rightarrow U' \cup V' = X \Rightarrow U' \cup V' = N$.

Now as $(X, \mathcal{F}_c)$ is cofinite and $U, V$ are open in $X$.

$\Rightarrow U'$ and $V'$ are finite.

$\Rightarrow U' \cup V'$ is finite $\Rightarrow X = N$ is finite

Which is a contradiction.

So our supposition is wrong.

Hence, $X$ is not $T_2$-space.

**THEOREM:** Every subspace of $T_2$-space is $T_2$-space.

**PROOF:** Let $X$ be a $T_2$-space and $Y$ be a subspace of $X$. To prove: $Y$ is $T_2$.

Let $x, y \in Y$ such that $x \neq y$.

As $x, y \in Y$ and $Y \subseteq X \Rightarrow x, y \in X$ such that $x \neq y$.

As $X$ is $T_2$. So, there exist two open sets $U$ and $V$ in $X$ such that: $x \in U$, $y \in V$ and $U \cap V = \phi$.

Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.

then, $U_1$ and $V_1$ are open in $Y$.

As $x \in U$, $x \in Y \Rightarrow x \in U \cap Y \Rightarrow x \in U_1$

As $y \in V$, $y \in Y \Rightarrow y \in V \cap Y \Rightarrow y \in V_1$.

Now, $U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$.

$$= (U \cap V) \cap Y = \phi \cap Y = \phi$$

$$\Rightarrow Y \text{ is } T_2.$$

**THEOREM:** In $T_1$-space, no finite subset has the limit point.

**PROOF:**

Let $X$ be a $T_1$-space and $A$ be a finite subset of $X$. Suppose $x \in D(A)$.

Then, each open set $U$ containing $x$ contains infinite number of distinct points of $A$.

Which is a contradiction.

$\because$ $A$ itself is finite.

So our supposition is wrong.

Hence, in $T_1$-space a finite set has no limit points.

**THEOREM:** Every metric space is $T_2$-space:

**PROOF:** Let $(X,d)$ be the metric space.

To prove: $X$ is $T_2$-space.

Let $x, y \in X$ such that $x \neq y$.

$\Rightarrow d(x, y) > 0$

Let $d(x, y) = r > 0$.

Now consider $U = B(x, r/2)$ and $V = B(y, r/2)$.

$\Rightarrow$ $U$ and $V$ are open sets in $X$ ($\because$ Open balls are open sets).

and $x \in U$, $y \in V$.

Now to show $U \cap V = \phi$.

Suppose, on the contrary that $U \cap V \neq \phi$.

$\Rightarrow z \in U \cap V \Rightarrow z \in U$ and $z \in V$.

$\Rightarrow z \in B(x, r/2)$ and $z \in B(y, r/2)$

$\Rightarrow d(z, x) < r/2$ and $d(z, y) < r/2$.

Now $d(x, y) \leq d(x, z) + d(z, y)$.

$r < r/2 + r/2 = r$

$\Rightarrow r < r$

A contradiction.

So our supposition is wrong.

Hence $U \cap V = \phi$

$\Rightarrow X$ is $T_2$ space.

## PRODUCT TOPOLOGY. (DEF).

Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ be two topological spaces and $X \times Y$ be the cartesian product of $X$ and $Y$. Define a subset $U \times V$ of $X \times Y$ to be open in $X \times Y$. if $U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2$, then the class of all subsets $U \times V$ of $X \times Y$ is the base for the topology $\mathcal{T}$ on $X \times Y$, called product topology on $X \times Y$.

**EXAMPLE:** Let $X = \{1, 2, 3\}$, $\mathcal{T}_1 = \{\phi, X, \{1\}, \{2, 3\}\}$

$Y = \{a, b, c, d\}$, $\mathcal{T}_2 = \{\phi, Y, \{a, b\}\}$.

$\mathcal{B} = \{\phi, X \times Y, \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}, \{(1,a), (1,b), (1,c), (1,d)\}, \{(1,a), (1,b)\}, \{(2,a), (2,b), (2,c), (2,d), (3,a), (3,b), (3,c), (3,d)\}, \{(2,a), (2,b), (3,a), (3,b)\}\}$.

$\mathcal{T} = \{ \cup \gamma : \gamma$ is a subfamily of $\mathcal{B}\}$.

**THEOREM:** The following statements about the topological space are equivalent:
i) $X$ is $T_2$- space. ii) The diagonal $D = \{(x,x) : x \in X\}$ is closed in $X \times X$.

**PROOF:** i) $\Rightarrow$ ii) i.e, we assume that $X$ is $T_2$ space and prove that $D$ is closed in $X \times X$.
For this, we prove that $D'$ is open in $X \times X$.
Let $(x,y) \in D' \Rightarrow x \neq y$.
Hence, we have $x, y \in X$ such that $x \neq y$ and $X$ is $T_2$ space. So, there exists two open sets $U_x$ and $V_y$ such that $x \in U_x$ and $y \in V_y$ and $U_x \cap V_y = \phi$.

Now let $(U,V) \in U_x \times V_y$
$\Rightarrow U \in U_x$ and $V \in V_y$.
As $U_x \cap V_y = \phi \Rightarrow U \neq V \Rightarrow (U,V) \in D'$.
$\Rightarrow U_x \times V_y \subseteq D' \Rightarrow (x,y) \in U_x \times V_y \subseteq D'$.

$$\Rightarrow \{(x,y)\} \subseteq U_x \times V_y \subseteq D'.$$

$$\Rightarrow \bigcup_{(x,y) \in D'} \{(x,y)\} \subseteq \bigcup_{(x,y) \in D'} U_x \times V_y \subseteq D'.$$

$$\Rightarrow D' \subseteq \bigcup_{(x,y) \in D'} U_x \times V_y \subseteq D'$$

$$\Rightarrow D' = \bigcup_{(x,y) \in D'} U_x \times V_y.$$

Now as $U_x$ and $V_y$ are open in $X$. So, $U_x \times V_y$ is open in $X \times X$ and as union of any number of open sets is open. So $D'$ is open.
$\Rightarrow D$ is closed.

ii) $\Rightarrow$ i) i.e, here we assume that $D$ is closed in $X \times X$ and we have to prove that $X$ is $T_2$-space.

Let $x, y \in X$ such that $x \neq y \Rightarrow (x, y) \in D'$.

Now as $D'$ is an open set in $X \times X$.

So, there exists open set $U_x \times V_y$ in $X \times X$ such that $(x, y) \in U_x \times V_y \subseteq D'$.

$\Rightarrow (x, y) \in U_x \times V_y$.

$\Rightarrow x \in U_x$ and $y \in V_y$.

Now to prove $U_x \cap V_y = \phi$.

Suppose on the contrary $U_x \cap V_y \neq \phi$.

Let $z \in U_x \cap V_y \Rightarrow z \in U_x$ and $z \in V_y$.

$\Rightarrow (z, z) \in U_x \times V_y$.

$\Rightarrow (z, z) \in D'$ $(\because U_x \times V_y \subseteq D')$.

$\Rightarrow z \neq z$.

Which is a contradiction.

$\because z = z$

So our supposition is wrong.

Hence, $U_x \cap V_y = \phi$.

$\Rightarrow X$ is $T_2$-space.

## CONVERGENCE: (DEF).

Let $(X, \mathcal{T})$ be a topological space then a sequence $\{x_n\}$ in $X$ is said to converge to a point $x \in X$ if for every open set $U$ containing $x$ there is a natural number $n_0$ such that:

$$x_n \in U, \text{ for all } n \geq n_0.$$

**THEOREM:** Let $X$ be a $T_2$-space. Then, any sequence in $X$ can converge to atmost one point. i.e, in $T_2$ space, limit of the sequence is unique.

**PROOF:** Suppose $\{x_n\}$ is a sequence in $X$ and $x_n \to x$ and $x_n \to y$ and suppose $x \neq y$. As $X$ is $T_2$-space so, then there exists two open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Now as $x_n \to x \in U$, so then, there exists some positive integer $n_0$, such that $x_n \in U$ $\forall n \geq n_0$.

Also as $x_n \to y \in V$, so then there exist some positive integer $n_1$ such that $x_n \in V$, $\forall n \geqslant n_1$.

Let $m = \max(n_0, n_1)$.

Then $\forall n \geqslant m$, $x_n \in U$ and $x_n \in V$.

$\Rightarrow U \cap V \neq \phi$

& Which is a contradiction.

$\because U \cap V = \phi$

So, our supposition is wrong.

Hence $x = y$.

Hence, limit of the sequence is unique.

**THEOREM:** Let $(X, \mathcal{T})$ be a topological space and $Y$ be a $T_2$-space and $f: X \to Y$ is a continuous function, then the graph $G = \{(x,y) : y = f(x)\}$ is closed in $X \times Y$.

**PROOF:** We prove $G'$ is open in $X \times Y$.

Let $(x,y) \in G' \Rightarrow y \neq f(x)$.

As $y, f(x) \in Y$, $y \neq f(x)$ and $Y$ is $T_2$, so then there exists two open sets $V$ and $V_1$ in $Y$ such that: $y \in V$, $f(x) \in V_1$ and $V \cap V_1 = \phi$

Let $U = f^{-1}(V_1)$

As $V$ is open in $Y$ and $f$ is continuous, so inverse image $f^{-1}(V_1) = U$ is open in $X$.

Now, $f(x) \in V_1$

$\Rightarrow x \in f^{-1}(V_1) \Rightarrow x \in U$

$\Rightarrow x \in U$, $y \in V \Rightarrow (x,y) \in U \times V$

Hence, $(x,y) \in U \times V \subseteq G'$

But $U \times V$ is open in $X \times Y$.

$\Rightarrow G'$ is open in $X \times Y$.

$\Rightarrow G$ is closed in $X \times Y$

**THEOREM:** Let $X$ be a topological space and $Y$ be a $T_2$-space and $f, g: X \to Y$ be two continuous functions, then prove that $A = \{x : x \in X \wedge f(x) = g(x)\}$ is closed in $X$.

**PROOF:** We prove, $A'$ is open in $X$.

Let $a \in A' \Rightarrow f(a) \neq g(a)$.

As $a \in X \Rightarrow f(a), g(a) \in Y$ and $f(a) \neq g(a)$ and $Y$ is $T_2$, so then there exists two open sets $V$ and $V_1$ in $Y$ such that:

$f(a) \in V$, $g(a) \in V_1$ and $V \cap V_1 = \phi$.

As $f$ and $g$ are continuous so, $f^{-1}(V)$ and $g^{-1}(V_1)$ are open in $X$

As $f(a) \in V \Rightarrow a \in f^{-1}(V)$

$g(a) \in V_1 \Rightarrow a \in g^{-1}(V_1)$

$\Rightarrow a \in f^{-1}(V) \cap g^{-1}(V_1) \subseteq A'$

As $f^{-1}(V) \cap g^{-1}(V_1)$ is open in $X$.

$\Rightarrow A'$ is open in $X$.

$\Rightarrow A$ is closed in $X$.

**THEOREM:** Let $f: X \to Y$ and $g: X \to Y$ be two continuous functions from a topological space $X$ to a $T_2$ space $Y$ and $f(x) = g(x)$ for all $x \in D$, where $D$ is dense in $X$. Then, $f(x) = g(x)$ $\forall$ $x \in X$.

**PROOF:** If $D = X$, then theorem is trivially proved. If $D \neq X$, then $D \subset X$, then there is $z \in X$ such that $z \notin D$. To prove: $f(z) = g(z)$.

Suppose $f(z) \neq g(z)$. As $f(z), g(z) \in Y$, $f(z) \neq g(z)$ and $Y$ is a $T_2$-space. Then there exists two open sets $U$ and $V$ such that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \phi$.

Put $U_1 = f^{-1}(U)$ and $V_1 = g^{-1}(V)$

Since $U$ and $V$ are open in $Y$ and $f, g$ are continuous functions so then $U_1$ and $V_1$ are open in $X$.

Further $f(x) \in U$ and $g(x) \in V$.

$\Rightarrow x \in f^{-1}(U)$ and $x \in g^{-1}(V)$.

$\Rightarrow x \in U_1$ and $x \in V_1 \Rightarrow x \in U_1 \cap V_1$

Now as $D$ is dense in $X$. So $\overline{D} = X$.

$\Rightarrow x \in \overline{D}$

$\Rightarrow (U_1 \cap V_1) \cap D \neq \phi$

Let $d \in (U_1 \cap V_1) \cap D$

$\Rightarrow d \in U_1$, $d \in V_1$ and $d \in D$.

$\Rightarrow d \in f^{-1}(U)$ and $d \in g^{-1}(V)$ and $d \in D$.

Now $d \in D \Rightarrow f(d) = g(d)$ $(\because \forall x \in D, f(x) = g(x))$.

Also, $d \in f^{-1}(U)$ and $d \in g^{-1}(V)$

$\Rightarrow f(d) \in U$ and $g(d) \in V$.

$\Rightarrow f(d) \neq g(d)$ $(\because U \cap V = \phi)$

Which is a contradiction.

So, our supposition is wrong.

Hence, $f(x) = g(x)$, $\forall x \in X$.

**THEOREM:** A topological space $X$ is $T_2$-space iff for any two distinct points $a, b \in X$, there are closed sets $C_1$ and $C_2$ such that $a \in C_1$, $b \notin C_1$ and $b \in C_2$, $a \notin C_2$ and $C_1 \cup C_2 = X$.

**PROOF:**

Suppose $X$ is $T_2$-space.

As $a, b \in X$ and $a \neq b$. As $X$ is $T_2$-space. So there exist two open sets $U$ and $V$ such that:

$a \in U$, $b \in V$ and $U \cap V = \phi$.

Put $C_1 = V'$ and $C_2 = U'$.

As $U$ and $V$ are open. So $C_1$ and $C_2$ are closed in $X$.

As $a \in U$ and $U \cap V = \phi \Rightarrow a \notin V$.

$b \in V$ and $U \cap V = \phi \Rightarrow b \notin U$.

Now, $a \in U \Rightarrow a \notin U' \Rightarrow a \notin C_2$.

$a \notin V \Rightarrow a \in V' \Rightarrow a \in C_1$.

$b \in V \Rightarrow b \notin V' \Rightarrow b \notin C_1$

$b \notin U \Rightarrow b \in U' \Rightarrow b \in C_2$.

Further, $U \cap V = \phi \Rightarrow (U \cap V)' = \phi'$

$\Rightarrow U' \cup V' = X \Rightarrow C_1 \cup C_2 = X$.

Conversely suppose $C_1 \cup C_2 = X$.

To prove: $X$ is $T_2$

Let $U = C_2'$ and $V = C_1'$.

As $C_1$ and $C_2$ are closed.

So $U$ and $V$ are open in $X$.

As $a \notin C_2 \Rightarrow a \in C_2' \Rightarrow a \in U$.

$b \notin C_1 \Rightarrow b \in C_1' \Rightarrow b \in V$.

Now, $U \cap V = C_2' \cap C_1' = (C_2 \cup C_1)'$

$= X' = \phi$.

$\Rightarrow X$ is $T_2$-space.

**THEOREM:** A topological space $X$ is $T_2$ space iff for every point $a \in X$, $\{a\} = \bigcap_{\alpha \in I} C_\alpha$ where each $C_\alpha$ is a closed set containing an open set $U$ such that $a \in U$.

**PROOF:** Suppose $X$ is $T_2$-space.

To prove: $\{a\} = \bigcap_{\alpha \in I} C_\alpha$

Let $b \in X$ such that $a \neq b$, then there exists two open sets $U$ and $V$ such that $a \in U$, $b \in V$ and $U \cap V = \phi$.

Put $V' = C_\alpha$. As $V$ is open so $V'$ is closed.

Now as $U \cap V = \phi$. So $U \subseteq V'$.

$\Rightarrow a \in U \subseteq C_\alpha$. Now as $b \in V \Rightarrow b \notin V'$

$\Rightarrow b \notin C_\alpha$.

Now as for every point $b \in X$ distinct from $a$, we have a closed set $C_\alpha$ such that $a \in C_\alpha$ and $b \notin C_\alpha$.

$\Rightarrow a \in \bigcap C_\alpha$ and $b \notin \bigcap C_\alpha$

$\Rightarrow \{a\} = \bigcap_{\alpha \in I} C_\alpha$.

Conversely, suppose in a topological space $X$, for every point $a \in X$, $\{a\} = \bigcap_{\alpha \in I} C_\alpha$, where $C_\alpha$ is a closed set containing an open set $U$ such that $a \in U$.

To prove: $X$ is $T_2$-space.

Let $b \in X$ such that $a \neq b$:

$\Rightarrow b \notin \bigcap_{\alpha \in I} C_\alpha$

$\Rightarrow b \notin C_\alpha$ for some $\alpha$.

Put $V = C_\alpha' \Rightarrow b \in V$.

$\Rightarrow a \in U$ and $b \in V$.

Now as $U \subseteq C_\alpha$.

$\Rightarrow U \cap C_\alpha' = \phi \Rightarrow U \cap V = \phi$.

$\Rightarrow X$ is $T_2$-space.

**THEOREM:** A $1^{st}$ countable space $X$ is $T_2$-space if and only if every convergent sequence has a unique limit.

**PROOF:** Suppose $X$ is $1^{st}$ countable space which is $T_2$. To prove: Every convergent sequence has unique limit.

Suppose on the contrary that $x_n \to x$ and $x_n \to y$ and $x \neq y$.

Since, $X$ is $T_2$, so there exists two open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$ $\to$ ①

Now $x_n \to x \in U$, so there exist $n_1 \in N$ such that $x_n \in U$, $\forall\ n \geq n_1$.

As $x_n \to y \in V$, so there exist $n_2 \in N$ such that $x_n \in V$, $\forall\ n \geq n_2$.

Put $n_0 = \max[n_1, n_2]$

$\Rightarrow x_n \in U \quad \forall\ n \geq n_0$

and $x_n \in V \quad \forall\ n \geq n_0$

$\Rightarrow U \cap V \neq \phi \to$ ②.

① and ② gives the contradiction

So our supposition is wrong.

Hence $x = y$.

$\Rightarrow \{x_n\}$ has unique limit.

Conversely suppose in a first countable space $X$, every convergent sequence has a unique limit. To prove: $X$ is $T_2$ space.

Let $a, b \in X$ such that $a \neq b$.

To prove. X is $T_2$ - space.

We suppose X is not $T_2$ - space.

Then, every open set containing 'a' has a non-empty intersection with every open set which contains 'b'.

Let $\{U_n\}$ and $\{V_n\}$ be countable nested bases at 'a' and 'b' respectively.

Then $U_n \cap V_n \neq \phi$.

$\Rightarrow a_n \in U_n \cap V_n, \forall n$.

Then $a_n \rightarrow a$ and $a_n \rightarrow b$.

Which is a contradiction.

$\because$ Every convergent sequence in X has unique limit.

So our supposition is wrong.

Hence, X is $T_2$ space.

**THEOREM:**

Every $T_2$ space is $T_0$ - space.

## REGULAR SPACE: (DEF).

A topological space $(X,\mathfrak{T})$ is said to be regular space if for every $x\in X$ and for any closed subset $A$ of $X$ with $x\notin A$, there exists two open sets $U$ and $V$ such that $x\in U$, $A\subseteq V$ and $U\cap V=\phi$.

**EXAMPLE:** Let $X=\{a,b\}$.

$\mathfrak{T}=\{\phi, X, \{a\}, \{b\}\}$

Then $(X,\mathfrak{T})$ is regular.

**THEOREM:** The following statements about a topological space are equivalent.

1) $X$ is regular.
2) For any open set $U$ in $X$ and $x\in U$, there is an open set $V$ containing $x$ such that $x\in\bar{V}\subseteq U$.
3) Each element of $X$ has a local base containing closed sets.

**PROOF:** 1) $\Rightarrow$ 2) ie, here it is given that $X$ is regular and to prove ②.

Let $U$ be an open set in $X$ with $x\in U$.

To prove: There exist an open set $V$ in $X$ containing $x$ such that $x\in\bar{V}\subseteq U$.

Now as $x\in U$ and $U$ is open set.

$\Rightarrow x\notin U'$ and $U'$ is closed.

Then, by the definition of regular

space, there exist two open sets $V$ and $V_1$ such that $x \in V$, $U' \subseteq V_1$ and $V \cap V_1 = \phi$.

Now, $U' \subseteq V_1 \Rightarrow V_1' \subseteq U$.
Also, $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$.

$$\Rightarrow x \in V \subseteq V_1' \subseteq U.$$

Now as $V_1$ is an open set, so $V_1'$ is closed set. So $V_1'$ is closed superset of $V$. But $\bar{V}$ is the smallest closed superset of $V \Rightarrow x \in V \subseteq \bar{V} \subseteq V_1' \subseteq U$

$$\Rightarrow x \in \bar{V} \subseteq U.$$

**2) ⇒ 3)**: Let $x \in X$. To prove: $X$ has a local base containing closed sets.

Let $U$ be an open set such that $x \in U$. Then, by condition ②, there exist an open set $V$ such that $x \in \bar{V} \subseteq U$.

This shows that local base at $x$ contains sets of the form $\bar{V}$ which is of course closed set.

**3) ⇒ 1)**: Let $x \in X$ and $A$ be closed subset of $X$ such that $x \notin A$.

$\Rightarrow x \in A'$. Further as $A$ is closed. So, $A'$ is open set. Then by ③, there is a closed set $B$ in the local base at $x$ such that $x \in B \subseteq A'$. Now $B \subseteq A' \Rightarrow A \subseteq B'$.

Let $U = B$ and $V = B'$.
Then, $U$ is open as $U$ is in local base.
$V$ is open because $V = B'$ and $B$ is closed.

Further $x \in U$, $A \subseteq V$ and $U \cap V = \phi$ ($\because B \cap B' = \phi$).

Hence, it shows that ①, ② and ③ are equivalent.

## COMPLETELY REGULAR SPACE: (DEF).

A topological space $(X, \mathcal{T})$ is said to be completely regular space if for any closed set $A$ in $X$ and $x \in X$ such that $x \notin A$, there exist a continuous function $f : X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

## EXAMPLE:

Every metric space is completely Regular.

## THEOREM:

Every completely regular space is regular.

**PROOF:**

Let $X$ be a completely regular space.
To prove: $X$ is regular.

Let $x \in X$ and $A$ be closed subset of $X$ such that $x \notin A$. Then, as $X$ is completely regular so there exist a continuous function $f : X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Let $U = [0\ \frac{1}{2}[$ and $V = ]\frac{1}{2}\ \ 1]$
Then $U$ and $V$ are open in $[0\ 1]$.
As $f$ is continuous. So $f^{-1}(U)$ and $f^{-1}(V)$ are open in $X$.

And $x \in f^{-1}(U)$, $A \subseteq f^{-1}(V)$
and $f^{-1}(U) \cap f^{-1}(V) = \phi$.

So, $X$ is regular.

HENCE PROVED.

Theorem: Every subspace of the completely regular space is completely regular.

Proof: Let $X$ be a completely regular space and $Y$ be a subspace of $X$.

To prove: $Y$ is completely regular.

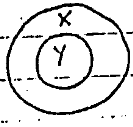
Let $x \in Y$ and $A$ be a closed subset of $Y$ such that $x \notin A$. As $x \in Y$ and $Y \subseteq X$. So $x \in X$. Further as, $A$ is closed in $Y$ and $Y$ is subspace of $X$. So, then there exists a closed subset $B$ in $X$ such that $A = B \cap Y$.

As $x \notin A$ and $x \in Y \Rightarrow x \notin B$.

As $X$ is completely regular, so there exist a continuous function $f: X \to [0\ 1]$ such that:

$f(x) = 0$ and $f(B) = 1$

Now define $g: Y \to [0\ 1]$ by $g(x) = f(x)\ \forall\ x \in Y$.



Then, $x \in Y \Rightarrow g(x) = f(x) = 0$

$g(A) = f(A) \because A \subseteq Y$.

$= f(B \cap Y)$

$= f(B) \cap f(Y) = 1$

$f: A \to B$

$E \subseteq A$

$g: E \to B$

$g(x) = f(x)\ \forall\ x \in E$

$f(E) = 1$

$A = \{x_1, x_3\}$

$E = \{x_1, x_2, x_3\}$

$Y = \{x_1, x_3, x_4, x_5\}$

As $g$ is restriction of $f$ and Then $g$ is restriction of $f$. $f$ is continuous so $g$ is also continuous. Hence $Y$ is completely regular.

Definition: $T_3$ - Space:

A regular $T_1$-space is called $T_3$-space.

Definition: $T_{3\frac{1}{2}}$-Space:

A completely regular $T_1$-space is called $T_{3\frac{1}{2}}$-space or Tychonoff space.

Theorem. A completely regular $T_1$-space is $T_2$-space.

Proof: Let $X$ be a completely regular $T_1$-space.

To prove: $X$ is $T_2$-space.

Let $x, y \in X$ such that $x \neq y$.

As $X$ is $T_1$ space. So each singelton subset in $X$ is closed.

So $A = \{y\}$ is closed set in $X$ and $x \notin A$.

As $X$ is completely regular so then there exists a continuous function $f: X \Rightarrow [0\ 1]$ such that:

$$f(x) = 0 \;,\; f(A) = 1 \Rightarrow f(y) = 1$$

Now let $U = [0\ \tfrac{1}{2}[$ and $V = ]\tfrac{1}{2}\ 1]$ be two open sets in $[0\ 1]$, Then as $f$ is continuous so $f^{-1}(U)$ and $f^{-1}(V)$ are open in $X$. Further as:

$$f(x) = 0 \in U \Rightarrow x \in f^{-1}(U)$$

$$f(y) = 1 \in V \Rightarrow y \in f^{-1}(V).$$

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$$

$$= f^{-1}(\phi) = \phi$$

$\Rightarrow X$ is $T_2$ space.

Theorem: A subspace of $T_{3\frac{1}{2}}$-Space is $T_{3\frac{1}{2}}$-Space.

Proof: Let $X$ be a $T_{3\frac{1}{2}}$ space and $Y$ be a subspace of $X$.

To prove: $Y$ is $T_{3\frac{1}{2}}$.

As $X$ is $T_{3\frac{1}{2}}$ space so $X$ is completely regular and $X$ is $T_1$ space.

A subspace of completely regular space is completely regular. As subspace of $T_1$-Space is $T_1$. So $Y$ is completely regular and $T_1$-space.

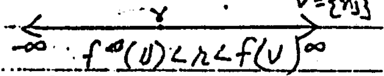
$\Rightarrow Y$ is $T_{3\frac{1}{2}}$.

Hence Proved.

Set of all continuous functions from $X \to R$.

Theorem: For any topological space $X$ if $C(X,R)$ separates the points of $X$, then $X$ is $T_2$-space.

Proof: Let $x, y \in X$ such that $x \neq y$.
And let $f \in C(X,R)$
i.e $f$ is a continuous function from $X \to R$.
Now by the given condition: $f(x) \neq f(y)$.
Say $f(x) < f(y)$
As $f(x), f(y) \in R$ and $f(x) < f(y)$.
So then there exist $r \in R$ s.t.
(there is a real no. b/w two real nos.) $f(x) < r < f(y)$.
Put $U = \{u : u \in X$ and $f(u) < r\} \Rightarrow U < f^{-1}(r)$
$V = \{v : v \in X$ and $f(v) > r\} \Rightarrow v > f^{-1}(r)$
then $x \in U$ and $y \in V$. $X = \{x_1, x_2, x_3\}$ $U = \{x_1, x_2\}$ $V = \{x_3\}$
Further, $U = f^{-1}(]-\infty, r[)$.
and $V = f^{-1}(]r, \infty[)$.



Now as $]-\infty, r[$ and $]r, \infty[$ are open sets in $R$ and $f$ is continuous. So inverse images of open sets is also open. So $U$ and $V$ are open in $X$. Also by definition of $U$ and $V$:
$U \cap V = \phi$
Hence, we have found two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
$\Rightarrow X$ is $T_2$.

Definition: NORMAL SPACE:
A topological space $(X, \mathcal{T})$ is said to be normal space if for any two closed disjoints subsets $A$ and $B$ of $X$ there are open sets $U$ and $V$ in $X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Theorem. Every discrete space with atleast two points is normal.

Proof: Let $X$ be a discrete space with atleast two points. To prove: $X$ is normal.

As $X$ is discrete, so, each subset of $X$ is open as well as closed.

Let $A$ and $B$ be two disjoint closed sets in $X$. Let $U = A$ and $V = B$.

Then $U$ and $V$ are open and $U \cap V = \phi$.

$$A \subseteq U, \; B \subseteq V$$

$$\Rightarrow X \text{ is normal.}$$

Theorem: Every subspace of a regular space is regular.

Proof: Let $X$ be a regular space and $Y$ be a subspace of $X$.

To prove: $Y$ is regular.

Let $x \in Y$ and $A$ be a closed set in $Y$ such that $x \notin A$.

Now as $A$ is closed in $Y$ and $Y$ is subspace of $X$, so then there exists a closed set $B$ in $X$, such that $A = B \cap Y$.

Further $x \notin A \Rightarrow x \notin B \cap Y$

$$\Rightarrow x \notin B \quad (\because x \in Y).$$

Further $x \in Y \subseteq X \Rightarrow x \in X$.

So $x \in X$ and $B$ is closed set in $X$ such that $x \notin B$ and $X$ is regular so then there exists two open sets $U$ and $V$ in $X$ such that:

$$x \in U, \; B \subseteq V \text{ and } U \cap V = \phi.$$

Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.

As $U$ and $V$ are open in $X$ so, $U_1$ and $V_1$ are open in $Y$.

Now as $x \in U$, $x \in Y \Rightarrow x \in U \cap Y \Rightarrow x \in U_1$.

Also $B \subseteq V \Rightarrow B \cap Y \subseteq V \cap Y \Rightarrow A \subseteq V_1$.

And $U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$

$= (U \cap V) \cap Y$

$= \phi \cap Y = \phi$

$\Rightarrow Y$ is regular.

Theorem: Every metric space is normal space.

Proof: Let $(X,d)$ be a metric space.

To prove: $X$ is normal space.

Let $A$ and $B$ be the two disjoint non-empty closed sets in $X$. Let $a \in A$, Then

$$d(a,B) = \inf_{b \in B} d(a,b)$$

definition of a distance of a point:

As, $A \cap B = \phi$ and $a \in A$ so $a \notin B$.

Further as $B$ is closed. So, $d(a,B) \geq 0$

Let $d(a,B) = r_a$.

Similarly let $b \in B$ and $d(b,A) = r_b$.

Now consider the open balls,

$B(a, r_{a/3})$ and $B(b, r_{b/3})$.

Put $U = \bigcup_{a \in A} B(a, r_{a/3})$, $V = \bigcup_{b \in B} B(b, r_{b/3})$.

Then $U$ and $V$ are open, ($\because$ Union of open balls is open).

with $A \subseteq U$ and $B \subseteq V$.

Now we prove that $U \cap V = \phi$.

Suppose on the contrary that $U \cap V \neq \phi$.

$\Rightarrow x \in U \cap V \Rightarrow x \in U$ and $x \in V$.

$\Rightarrow x \in \bigcup_{a \in A} B(a, r_a/3)$ and $x \in \bigcup_{b \in B} B(b, r_b/3)$

$\Rightarrow x \in B(a_1, r_{a_1}/3)$ and $x \in B(b_1, r_{b_1}/3)$.

$\Rightarrow d(x, a_1) < r_{a_1/3}$ and $d(x, b_1) \leq r_{b_1/3}$.

As $a_1 \in A$, $b_1 \in B$ and $A \cap B = \phi \Rightarrow a_1 \neq b_1$

$\Rightarrow d(a_1, b_1) > 0$

اگر closed set ہو تو اسکے باہر کے ہر element کا greater dist ہو نا چاہیے۔
~~[illegible]~~ یا گا greater ہو " " " " " " " " ہو تو "
119 ۔ Zero equal to ہو گا۔

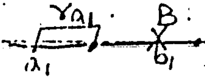
Let $d(a_1, b_1) = r$ — $a_1$ and

Clearly, $r_{a_1} \le \underset{\text{arbitrary}}{r}$ and $r_{b_1} \le r$ — $r_{..}$ are min dis from each other.

Now $r = d(a_1, b_1) \le d(a_1, x) + d(x, b_1)$

$< r_{a_1}/3 + r_{b_1}/3$

$\le r/3 + r/3 = 2r/3$ — $r_{a_1} = d(a, B)$



$\Rightarrow r < 2r/3$

$\Rightarrow 3r < 2r$.

Which is a contradiction.
So our supposition is wrong
Hence $U \cap V = \phi$
$\Rightarrow X$ is normal.

DEFINITION: $T_4$ SPACE.
A normal $T_1$-space is called $T_4$-space.

Theorem: $T_4$ space is regular space.
Proof: Let X be a $T_4$ space.
To prove: X is regular.
As X is $T_4$, so X is normal and X is $T_1$ as well.
Let $x \in X$ and A be a closed set in X s.t. $x \notin A$.
As X is $T_1$ so $\{x\}$ is closed.
Hence $\{x\}$ and A are two disjoint closed sets in X.
As X is normal so then there exists two open sets U and V in X such that:

$\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$.

$\Rightarrow x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

$\Rightarrow X$ is regular.

V-Imp
Theorem: A topological space $(X, \mathcal{T})$ is normal iff for any closed set A and open set U containing A, there

is atleast one open set V containing 'A' such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proof: Let X be a normal space. And let 'A' be a closed set in X, U be an open set in X with $A \subseteq U$. To prove: There is atleast one open set 'V' in X with $A \subseteq V \subseteq \overline{V} \subseteq U$.

Now, $A \subseteq U \Rightarrow A \cap U' = \phi$.

As U is open so U' is closed in X. So, we have A and U' two closed disjoint sets in X. As, X is normal so then There exists Two open sets V and $V_1$ in X such That $A \subseteq V$, $U' \subseteq V_1$ and $V \cap V_1 = \phi$.

Now, $U' \subseteq V_1 \Rightarrow V_1' \subseteq U$. Also as $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$.

$\Rightarrow A \subseteq V \subseteq V_1' \subseteq U$.

Now, as $V_1$ is open, so $V_1'$ is closed. So it means $V_1'$ is The closed superset of V.

But as $\overline{V}$ is the smallest closed superset of V so $V \subseteq \overline{V} \subseteq V_1'$

$\Rightarrow A \subseteq V \subseteq \overline{V} \subseteq V_1' \subseteq U$.

$\Rightarrow A \subseteq V \subseteq \overline{V}_1 \subseteq U$.

Conversely, let it is given That in a topological space $(X, \mathcal{T})$ for any closed set A in X, U is an open set in X with $A \subseteq U$, There is atleast one open set V in X such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

To prove: X is normal. Let A and B be The Two closed disjoint sets in X. As $A \cap B = \phi \Rightarrow A \subseteq B'$

As B is closed so B' is open.

Now, by given condition, There is an open set V in X such That $A \subseteq V \subseteq \overline{V} \subseteq B'$

Now, $A \subseteq V$ and $\overline{V} \subseteq B' \Rightarrow A \subseteq V$ and $B \subseteq \overline{V}'$.

Now as $\overline{V}$ is closed so $\overline{V}'$ is open.

Further as $V \subseteq \overline{V} \Rightarrow V \cap (\overline{V})' = \phi$

Hence, we have found two open sets V and $\overline{V}'$

in $X$ such that $A \subseteq V$, $B \subseteq \overline{V}'$ and $V \cap \overline{V}' = \phi$.

$\Rightarrow$ $X$ is normal.

Theorem: Every closed subspace of a normal space is normal.

Proof: Let $X$ be a normal space and $Y$ be a closed subspace of $X$.

To prove: $Y$ is normal.

Let $A_1$ and $A_2$ be the Two disjoint closed sets in $Y$. As $Y$ is subspace. So then there are Two disjoint closed sets $B_1$ and $B_2$ in $X$ such That:

$A_1 = B_1 \cap Y$, $A_2 = B_2 \cap Y$.

Now as $B_1$ and $B_2$ are two disjoint closed sets in $X$ and $X$ is normal. So Then, There are two open sets $U_1$ and $U_2$ such that $B_1 \subseteq U_1$, $B_2 \subseteq U_2$ and $U_1 \cap U_2 = \phi$.

Put $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$.

$\Rightarrow$ $V_1$ and $V_2$ are open in $Y$.

As $B_1 \subseteq U_1 \Rightarrow B_1 \cap Y \subseteq U_1 \cap Y \Rightarrow A_1 \subseteq V_1$.

Also $B_2 \subseteq U_2 \Rightarrow B_2 \cap Y \subseteq U_2 \cap Y \Rightarrow A_2 \subseteq V_2$.

$$\begin{aligned} V_1 \cap V_2 &= (U_1 \cap Y) \cap (U_2 \cap Y) \\ &= (U_1 \cap U_2) \cap Y \\ &= \phi \cap Y = \phi \end{aligned}$$

$\Rightarrow$ $Y$ is normal.

Theorem: Every metric space is completely regular.

Proof: Let $(X,d)$ be a metric space.

To prove: $X$ is completely regular.

Let $A$ be a closed subset of $X$ and $x \in X$ such that $x \notin A$. And we have to find out a continuous function $f: X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Define $g: X \to [0\ 1]$ by $g(y) = d(y, B)$ where $B$ is any other closed set in $X$ with $A \cap B = \phi$ and $x \in B$.

Then, i) $g(x) = d(x, B) = 0$.

ii) $g(A) = d(A, B) > 0$ ($\because$ $A$ and $B$ are closed).

Let $d(A, B) = K$.

iii) Now for any $\varepsilon > 0$, we chose $\delta = \varepsilon$ such that whenever $d(y, y') < \delta$.

Then $|g(y) - g(y')| = |d(y, B) - d(y', B)|$.

$$\le d(y, y')$$

$$< \delta = \varepsilon.$$

$\Rightarrow |g(y) - g(y')| < \varepsilon$. $(|d(x, y) - d(x, z)| \le d(y, z)|$

$\Rightarrow g$ is continuous on $X$.

Now $f: X \to [0\ 1]$ by $f(y) = \frac{1}{K} g(y)$.

As $g$ is continuous, so $f$ is continuous with

$$f(x) = \frac{1}{K} g(x) = \frac{1}{K}(0) = 0$$

$$f(A) = \frac{1}{K} g(A) = \frac{1}{K} d(A, B) = \frac{1}{K} \cdot K = 1.$$

$\Rightarrow X$ is completely regular.

Open Function:

A function '$f$' is said to be open function if image of each open set is open.

Example:

Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_X = \{\phi, X, \{1, 2\}, \{3, 4\}\}$

$Y = \{a, b, c, d\}$, $\mathcal{T}_Y = \{\phi, Y, \{b, c\}, \{a, d\}\}$

Define $f: X \to Y$ as $f(1) = b$, $f(2) = c$, $f(3) = a$, $f(4) = d$. Then $f$ is open

Closed function:

A function 'f' is said to be closed if image of each closed set is closed.

Theorem: A closed and continuous image of a normal space is normal.

Proof: Let $X$ be a normal space and $Y = f(X)$ is its closed continuous image.

To prove: $Y = f(X)$ is normal.

Let $A_1$ and $B_1$ be the two disjoint closed in $Y = f(X)$.

As $f$ is continuous so inverse image of each closed set is closed.

So $f^{-1}(A_1)$ and $f^{-1}(B_1)$ are closed in $X$.

Let $A = f^{-1}(A_1)$ and $B = f^{-1}(B_1)$.

Then, $A$ and $B$ are closed in $X$ and

$$
\begin{aligned}
A \cap B &= f^{-1}(A_1) \cap f^{-1}(B_1) \\
&= f^{-1}(A_1 \cap B_1) \\
&= f^{-1}(\phi) = \phi
\end{aligned}
$$

$\Rightarrow$ $A$ and $B$ are two disjoint closed sets in normal space $X$. So, then there exists two open sets $U$ and $V$ in $X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

As $U$ and $V$ are open in $X$. So, $U'$ and $V'$ are closed in $X$.

Further as $f$ is closed.

So $f(U')$ and $f(V')$ are closed

As $Y = f(X)$

Put $U_1 = Y \setminus f(U')$

$V_1 = Y \setminus f(V')$

$\Rightarrow$ $U_1$ and $V_1$ are open in $Y = f(X$,

Now we show $A_1 \subseteq U_1$.

Let $x \in A_1$

$\Rightarrow x \in f(A)$

$\Rightarrow f^{-1}(x) \in A \subseteq U$

$\Rightarrow f^{-1}(x) \in U$

$\Rightarrow f^{-1}(x) \notin U'$

$\Rightarrow x \notin f(U')$

$\Rightarrow x \in Y \setminus f(U')$

$\Rightarrow x \in U_1$

$\Rightarrow A_1 \subseteq U_1$

$\because A = f^{-1}(A_1)$

$\Rightarrow f(A) = A_1$

Similarly $B_1 \subseteq V_1$

Now, $U_1 \cap V_1 = (f(U'))' \cap (f(V'))'$

$= [f(U') \cup f(V')]'$

$= [f(U' \cup V')]'$

$= [f(X)]'$

$= Y' = \phi$

$\Rightarrow f(X)$ is normal.

# "COMPACTNESS IN TOPOLOGICAL SPACES":

DEFINITION: Compactness:

A topological space $(X,\mathcal{T})$ is said to be compact if every open cover for X has a finite subcover.

Examples:

1) If X is any set with indiscrete topology Then X is compact.
2) If X is finite set Then for any topology $\mathcal{T}$ on X, $(X,\mathcal{T})$ is compact.
3) If X is any set with $A \subseteq X$, then $\mathcal{T}=\{\phi, X, A, A'\}$ then, $(X,\mathcal{T})$ is compact.

REMARK:

If $(X,\mathcal{T})$ is compact space then it is Lindelof. But converse is not true because e.g. If $X=\mathbb{N}$ and $\mathcal{T}=\mathcal{T}_D$

Then $(X,\mathcal{T})$ is Lindelof space but $(X,\mathcal{T})$ is not compact because $\{\{x\}: x\in\mathbb{N}\}$ is an open cover for X, which has no finite subcover. $\{\overset{\text{open cover}}{\bigcup\{x\}=X}$ all are open$\}$

THEOREM: Let X be an infinite set with cofinite topology then X is compact.

PROOF: Let $\gamma=\{U_\alpha : \alpha\in I\}$ be an open cover for X. We have to find a finite subcover of $\gamma$ for X.

Since $\gamma$ is an opencover for X.

$$\text{So } X=\bigcup\gamma=\bigcup_{\alpha\in I}U_\alpha.$$

Now, for any $U_\alpha\in\gamma$, $\Rightarrow U_\alpha$ is an open set.

$\Rightarrow U'_\alpha$ is finite.

$\Rightarrow U'_\alpha=\{x_1, x_2, x_3, \dots, x_n\}$

Now as $\gamma$ is an open cover for X ie,

$$X=\bigcup_{\alpha\in I}U_\alpha.$$

So, for any $x_i \in U'_\alpha$ , $1 \le i \le n$.

$$\Rightarrow x_i \in \bigcup_{\alpha\in I} U_\alpha$$

$$\Rightarrow x_i \in U_{\alpha_i} \text{ for some } \alpha_i \in I$$

$$\Rightarrow \{x_i\} \subseteq U_{\alpha_i}$$

$$\Rightarrow \bigcup_{i=1}^{n} \{x_i\} \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow U'_\alpha \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$$

$$\Rightarrow U_\alpha \cup U'_\alpha \subseteq U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

$$\Rightarrow X \subseteq U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \subseteq X$$

$$\Rightarrow X = U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$

$\Rightarrow \{U_\alpha, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite open $\overset{\text{sub}}{\text{cover}}$ for X.

Hence, X is compact space.

THEOREM: The real line $\mathbb{R}$ is not compact with usual topology:

topology کو سے respect metric space کی Real line

کہتے Usual topology on ہیں define

PROOF: Let $\gamma = \{U_n = ]-n\ n[ : n \in \underline{N}\}$ be an open cover for X.

To prove: $\mathbb{R}$ is not compact.

Suppose on the contrary that $\mathbb{R}$ is compact, then by the definition of compact space, $\gamma$ has a finite subcover for $\mathbb{R}$.

Let $\{U_{n_1}, U_{n_2}, U_{n_3}, \ldots U_{n_r}\}$ be the finite subcover for $\mathbb{R} \Rightarrow \mathbb{R} = \bigcup_{i=1}^{r} U_{n_i}$

$U_1 = ]-1\ 1[$ , $U_2 = ]-2\ 2[$

$U_1 \cup U_2 = U_2 = ]-2\ 2[$

Let $m = \max(n_1, n_2, n_3, \ldots, n_r)$

$U_1 \cup U_2 \cup U_3 = U_3$

then $m \in \mathbb{N}$ and $\bigcup_{i=1} U_{n_i} = ]-m\ m[$

$\bigcup_{n=1}^{\infty} U_n = ]-\infty\ \infty[ = \mathbb{R}$.

$\Rightarrow \mathbb{R} = ]-m \; m[$ which is a contradiction.
So, our supposition is wrong.
Hence, $\mathbb{R}$ is not compact.

FINITE INTERSECTION PROPERTY:

Let $(X, \mathcal{T})$ be a topological space and $\gamma = \{U_\alpha : \alpha \in I\}$ be a collection of some subsets of $X$, then $\gamma$ is said to have finite intersection property if each finite subcollection of $\gamma$ has non-empty intersection e.g. Let $X = \mathbb{N}$, $\mathcal{T} = \mathcal{T}_c$ and $\gamma = \{\{1\}, \{2\}, \{1,2,3\}, \{1,2,3,4\}, \dots\}$ then $\gamma$ satisfies finite intersection property.

THEOREM: A topological space $(X, \mathcal{T})$ is compact if and only if every collection of closed sets in $X$ which satisfy finite intersection property has non-empty intersection.

PROOF: Let $X$ be compact and $\{U_\alpha : \alpha \in I\}$ be the collection of closed sets which satisfy finite intersection property.

To prove: $\bigcap_{\alpha \in I} U_\alpha \neq \phi$.

Suppose on the contrary that $\bigcap_{\alpha \in I} U_\alpha = \phi$.

$$\Rightarrow \left(\bigcap_{\alpha \in I} U_\alpha\right)' = \phi' \Rightarrow \bigcup_{\alpha \in I} U_\alpha' = X.$$

Now as $\{U_\alpha : \alpha \in I\}$ is the collection of closed sets so $\{U_\alpha' : \alpha \in I\}$ is the collection of open sets with $\bigcup_{\alpha \in I} U_\alpha' = X$.

$\Rightarrow \{U_\alpha' : \alpha \in I\}$ is an open cover of $X$, where $X$

is compact space.

$\Rightarrow \{U'_{\alpha_1}, U'_{\alpha_2}, U'_{\alpha_3}, \ldots, U'_{\alpha n}\}$ is a finite open subcover for X.

$\Rightarrow \bigcup_{i=1}^{n} U'_{\alpha i} = X$.

$\Rightarrow \left(\bigcup_{i=1}^{n} U'_{\alpha i}\right)' = X' \Rightarrow \bigcap_{i=1}^{n} U_{\alpha i}{}'' = \phi$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha n}\}$ is a finite subcollection of $\{U_\alpha : \alpha \in I\}$ with empty intersection.

$\Rightarrow \{U_\alpha : \alpha \in I\}$ does not satisfy finite intersection property. Which is a contradiction.

$\because \{U_\alpha : \alpha \in I\}$ satisfies finite intersection property.

Sow our supposition is wrong.

And hence, $\bigcap_{\alpha \in I} U_\alpha \neq \phi$

Conversely, let in a topological space $(X, \mathcal{T})$ each collection of closed sets in X which satisfies finite intersection property has non-empty intersection.

To prove: X is compact.

Let $\{O_\alpha : \alpha \in I\}$ be an open cover for X.

i.e, $\bigcup_{\alpha \in I} O_\alpha = X \Rightarrow \left(\bigcup_{\alpha \in I} O_\alpha\right)' = X' \Rightarrow \bigcap_{\alpha \in I} O'_\alpha = \phi$.

$\Rightarrow \{O'_\alpha : \alpha \in I\}$ is a collection of closed sets with empty intersection.

Then, by given hypothesis $\{O'_\alpha : \alpha \in I\}$ does not satisfy finite intersection property. Then, there exists a finite subcollection $\{O'_{\alpha_1}, O'_{\alpha_2}, \ldots, O'_{\alpha n}\}$ with empty intersection.

$\Rightarrow \bigcap_{i=1}^{m} O'_{\alpha i} = \phi \Rightarrow \left(\bigcap_{i=1}^{m} O'_{\alpha i}\right)' = \phi' \Rightarrow \bigcup_{i=1}^{m} O_{\alpha i} = X$

$\Rightarrow \{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha m}\}$ is a finite open subcover for X $\Rightarrow$ X is compact.

THEOREM: Every closed subspace of a compact space is compact.

PROOF: Let $X$ be compact and $Y$ be a closed subspace of $X$.

To prove: $Y$ is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for $Y$.

As $U_\alpha$, $\alpha \in I$, is an open set in $Y$ and $Y$ is subspace of $X$, so then There is an open set $V_\alpha$ in $X$ such that:

$$U_\alpha = V_\alpha \cap Y$$

$$\Rightarrow U_\alpha \subseteq V_\alpha$$

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow Y \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\text{Now, } X = Y \cup Y' \subseteq \left(\bigcup_{\alpha \in I} V_\alpha\right) \cup Y' \subseteq X$$

$$\Rightarrow X = \left(\bigcup_{\alpha \in I} V_\alpha\right) \cup Y'.$$

As $Y$ is closed so $Y'$ is open in $X$.

$\Rightarrow \{Y', V_\alpha : \alpha \in I\}$ is an open cover for $X$.

As $X$ is compact so this open cover has a finite subcover $\{Y', V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$

i.e $X = \left(\bigcup_{i=1}^{n} V_{\alpha i}\right) \cup Y'$

$A \subseteq B \cup C \not\Rightarrow A \subseteq B$ or $A \subseteq C$

$A \subseteq B \cup A' \Rightarrow A \subseteq B$.

$A \subseteq B \Rightarrow A = B \cap A$.

$$\text{Now, } Y \subseteq X = \left(\bigcup_{i=1}^{n} V_{\alpha i}\right) \cup Y'$$

$$\Rightarrow Y \subseteq \bigcup_{i=1}^{n} V_{\alpha i} \quad \because Y \cap Y' = \phi.$$

$$\Rightarrow Y = \left(\bigcup_{i=1}^{n} V_{\alpha i}\right) \cap Y$$

$$\Rightarrow Y = \bigcup_{i=1}^{n} (V_{\alpha i} \cap Y) \Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha i}$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcover for Y.

$\Rightarrow$ Y is compact.

V-Imp.

THEOREM: Continuous image of a compact space is compact.

PROOF: Let $f: X \to Y$ be a continuous function from a compact space X to a topological space Y.

To prove: f(X) is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for f(X), where f(X) is the subspace of Y.

As, $\{U_\alpha : \alpha \in I\}$ is an open set in f(X) and f(X) is subspace of Y.

So, then, there exists an open set $V_\alpha$ in Y such that

$$U_\alpha = V_\alpha \cap f(X).$$

$$\Rightarrow U_\alpha \subseteq V_\alpha \Rightarrow \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right).$$

$$\Rightarrow X \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha) \subseteq X.$$

$V_\alpha \subseteq Y$

$f^{-1}(V_\alpha) \subseteq X.$

$$\Rightarrow X = \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$$

As $V_\alpha, \alpha \in I$ is open in Y and $f: X \to Y$ is continuous.

$\Rightarrow f^{-1}(V_\alpha), \alpha \in I$, is open in X ($\because$ Inverse image of open set is open).

$\Rightarrow \{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover for X.

Since, X is compact.

So, this open cover has a finite subcover, $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots f^{-1}(V_{\alpha_n})\}$ for X.

$$\Rightarrow \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) = X.$$

$$\Rightarrow f^{-1}\left(\bigcup_{i=1}^{n} V_{\alpha i}\right)=X$$

$$\Rightarrow X=f^{-1}\left(\bigcup_{i=1}^{n} V_{\alpha i}\right)$$

$$\Rightarrow f(X)\subseteq \bigcup_{i=1}^{n} V_{\alpha i}$$

$$\Rightarrow f(X)=\left(\bigcup_{i=1}^{n} V_{\alpha i}\right)\cap f(X).$$

$$\Rightarrow f(X)=\bigcup_{i=1}^{n}\left(V_{\alpha i}\cap f(X)\right)$$ (Distributive property).

$$=\bigcup_{i=1}^{n} U_{\alpha i}$$

$\Rightarrow \{U_{\alpha 1}, U_{\alpha 2}, \dots, U_{\alpha n}\}$ is a finite subcover for $f(X)$.

Hence, $f(X)$ is compact.

THEOREM: Prove that in a $T_2$ space any point and disjoint compact subspace of $X$ can be separated by open sets, in the since they have disjoint neighbourhoods.

PROOF: Let $x\in X$ and '$C$' be a compact subspace of $X$ such that $x\notin C$.

To prove: $x$ and '$C$' can be separated by open sets.

Let $y\in C \Rightarrow x\neq y$, then as $x, y\in X$, $x\neq y$, $X$ is $T_2$ space, so then there exists two open sets $U_y$ and $V_y$ such that $x\in U_y$, $y\in V_y$ and $U_y\cap V_y=\phi$.

Now as, $y\in V_y \Rightarrow \{y\}\subseteq V_y$

$$\Rightarrow \bigcup_{y\in C}\{y\}\subseteq \bigcup_{y\in C} V_y$$

$$\Rightarrow C\subseteq \bigcup_{y\in C} V_y \Rightarrow C=\left(\bigcup_{y\in C} V_y\right)\cap C$$

$\Rightarrow C = \bigcup_{y \in C} (V_y \cap C)$

As for every $y \in C$, $V_y$ is open in $X$, so $V_y \cap C$ is open in $C$.

$\Rightarrow \{V_y \cap C : y \in C\}$ is an open cover for $C$.

As $C$ is compact. So, this open cover has a finite subcover $\{V_1 \cap C, V_2 \cap C, V_3 \cap C, \ldots, V_n \cap C\}$

$\Rightarrow C = \bigcup_{i=1}^{n} (V_i \cap C) = (\bigcup_{i=1}^{n} V_i) \cap C$

$\Rightarrow C \subseteq \bigcup_{i=1}^{n} V_i$

Put $U = \bigcap_{i=1}^{n} U_i$ and $V = \bigcup_{i=1}^{n} V_i$.

$\Rightarrow x \in U,\ C \subseteq V$

Now to prove $U \cap V = \phi$.

Suppose on the contrary, $U \cap V \neq \phi$.

$\Rightarrow z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcap_{i=1}^{n} U_i$ and $z \in \bigcup_{i=1}^{n} V_i$

Then, there is an $i$, $1 \leq i \leq n$ such that $z \in U_i$ and $z \in V_i$.

$\Rightarrow U_i \cap V_i \neq \phi$.

Which is a contradiction.

So our supposition is wrong.

Hence $U \cap V = \phi$.

THEOREM: Compact subspace of a $T_2$-space is closed.

PROOF: Let $X$ be a $T_2$ space and $C$ be a compact subspace ~~set~~ of $X$.

To prove: $C$ is closed.

We prove $C'$ is open.

If $C' = \phi$, then $C'$ is open.

If $C' \neq \phi$, then $x \in C' \Rightarrow x \notin C$.

Then, by a well known theorem, In a $T_2$-space any point and a disjoint subspace of $X$ can be separated by open sets in the sense they have disjoint neighbourhoods, there exists two open sets $U_x$ and $V_x$ such that $x \in U_x$, $C \subseteq V_x$ and $U_x \cap V_x = \phi$.

Now, $x \in U_x$

$\Rightarrow \{x\} \subseteq U_x \subseteq V_x'$ $(\because U_x \cap V_x = \phi)$

Also, $C \subseteq V_x \Rightarrow V_x' \subseteq C'$.

$\Rightarrow \{x\} \subseteq U_x \subseteq V_x' \subseteq C'$.

$\Rightarrow \{x\} \subseteq U_x \subseteq C'$

$$\Rightarrow \bigcup_{x \in C'} \{x\} \subseteq \bigcup_{x \in C'} U_x \subseteq C'.$$

$$\Rightarrow C' \subseteq \bigcup_{x \in C'} U_x \subseteq C'$$

$$\Rightarrow C' = \bigcup_{x \in C'} U_x.$$

As, $U_x$ is open so $\bigcup_{x \in C'} U_x$ is open.

$\Rightarrow C'$ is open.

$\Rightarrow C$ is closed.

DEFINITION: HOMEOMORPHISM.

A function $f: X \rightarrow Y$ is said to be homeomorphism if:

i) $f$ is continuous ii) $f$ is open iii) $f$ is bijective.

THEOREM: A 1-1 continuous mapping from a compact space $X$ onto a $T_2$-space $Y$ is Homeomorphism.

PROOF: As given 'f' is continuous and bijective so, we have just to prove that 'f' is open.

Let $G$ be an open set in $X$.

$\Rightarrow G'$ is closed set in $X$.

As closed subspace of a compact space is compact. So, $G'$ is compact.

Further as, continuous image of a compact space is compact. So $f(G')$ is compact.

$\Rightarrow f(G')$ is a compact subspace of $Y$.

As compact subspace of $T_2$ space is closed.

And $f(G')$ is a compact subspace of $T_2$-space.

$\Rightarrow f(G')$ is closed in $Y$.

$$\begin{aligned} \text{Now, } f(G') &= f(X-G). \\ &= f(X) - f(G). \\ &= Y - f(G) \quad (\because f \text{ is onto}) \\ &= [f(G)]' \end{aligned}$$

$\Rightarrow [f(G)]'$ is closed in $Y$.

$\Rightarrow f(G)$ is open in $Y$.

$\Rightarrow f$ is open function.

Hence, $f$ is homeomorphism.

THEOREM: A topological space $X$ is compact if and only if every class of closed sets with empty intersection has a finite sub class with empty intersection.

PROOF: Given $X$ is compact and $\{C_\alpha : \alpha \in I\}$ be a class of closed sets in $X$ with $\bigcap_{\alpha\in I} C_\alpha = \phi$. To prove: There is a finite subclass of $\{C_\alpha : \alpha \in I\}$ with empty intersection.

Now as $\bigcap_{\alpha\in I} C_\alpha = \phi$.

$$\Rightarrow \left(\bigcap_{\alpha\in I} C_\alpha\right)' = \phi' \Rightarrow \bigcup_{\alpha\in I} C'_\alpha = X$$

As $C_\alpha$ is closed so, $C'_\alpha$ is open.

$\Rightarrow \{C'_\alpha : \alpha \in I\}$ is an open cover for $X$.

As $X$ is compact so this open cover has a finite subcover $\{C'_{\alpha_1}, C'_{\alpha_2}, \ldots, C'_{\alpha_n}\}$.

$$\Rightarrow \bigcup_{i=1}^{n} C'_{\alpha_i} = X$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} C'_{\alpha_i}\right)' = X'$$

$$\Rightarrow \bigcap_{i=1}^{n} C_{\alpha_i} = \phi .$$

$\Rightarrow \{C_{\alpha_1}, C_{\alpha_2}, \ldots, C_{\alpha_n}\}$ is a finite subclass $\{C_\alpha : \alpha \in I\}$ with empty intersection.

Conversely, suppose in a Topological space $X$ each class $\{C_\alpha : \alpha \in I\}$ of closed sets with empty intersection has a finite subclass with empty intersection.

To prove: $X$ is compact.

Now, let $\{U_\alpha : \alpha \in I\}$ be an open cover for $X$.

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = X \Rightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right)' = X' \Rightarrow \bigcap_{\alpha \in I} U'_\alpha = \phi .$$

$\Rightarrow \{U'_\alpha : \alpha \in I\}$ is a class of closed sets with empty intersection. Then, by given condition there is a finite subclass $\{U'_{\alpha_1}, U'_{\alpha_2}, \ldots, U'_{\alpha_n}\}$ with,

$$\bigcap_{i=1}^{n} U'_{\alpha_i} = \phi$$

$$\Rightarrow \left(\bigcap_{i=1}^{n} U'_{\alpha_i}\right)' = \phi' \Rightarrow \bigcup_{i=1}^{n} U_{\alpha_i} = X$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_n}\}$ is a finite open subcover for $X$.

$\Rightarrow X$ is compact.

Hence Proved.

THEOREM: Every compact $T_2$-space is normal.

PROOF: Let X be a compact $T_2$-space.
To prove: X is normal.
Let A and B be the two closed disjoint subsets of X. We have to prove there exists two open set U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.
Let $x \in A$. As A and B are disjoint i.e $A \cap B = \phi$.
So $x \notin B$. Then, as X is compact and B is closed in X so, B is also compact.
Further as X is also $T_2$ and in a $T_2$ space a point and a disjoint compact subspace can be separated by open sets, so then there exists two open sets $U_x$ and $V_x$ such that:
$x \in U_x$, $B \subseteq V_x$ and $U_x \cap V_x = \phi$.
Now, $x \in U_x \Rightarrow \{x\} \subseteq U_x \Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x$

$$\Rightarrow A \subseteq \bigcup_{x \in A} U_x$$

$$\Rightarrow A = \left(\bigcup_{x \in A} U_x\right) \cap A \Rightarrow A = \bigcup_{x \in A} (U_x \cap A)$$

$\Rightarrow \{U_x \cap A : x \in A\}$ is an open cover for A.
Again as A is a closed subspace of compact space X, so A is compact.
So, the open cover $\{U_x \cap A : x \in A\}$ for A has a finite subcover $\{U_{x_1} \cap A, U_{x_2} \cap A, \ldots, U_{x_n} \cap A\}$ for A.

$$\Rightarrow A = \bigcup_{i=1}^{n} (U_{x_i} \cap A)$$
$$= \left(\bigcup_{i=1}^{n} U_{x_i}\right) \cap A.$$
$$\Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{x_i}.$$

Put $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcap_{i=1}^{n} V_{x_i}$.

Then $U$ and $V$ are open.

$A \subseteq U$, $B \subseteq V$.

Now to prove only $U \cap V = \phi$.

Suppose on the contrary $U \cap V \neq \phi$

$\Rightarrow z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcup_{i=1}^{n} U_{x_i}$ and $z \in \bigcap_{i=1}^{n} V_{x_i}$

$\Rightarrow z \in U_{x_i}$ for some $i$.

$z \in V_{x_i}$ for all $i$.

$\Rightarrow U_{x_i} \cap V_{x_i} \neq \phi$. A contradiction. So our supposition is wrong and hence $U \cap V = \phi$.

$\Rightarrow X$ is normal.

V-Imp

HEINE BOREL THEOREM:

Statement: Every closed and bounded subspace of a real line $\mathbb{R}$ is compact.

PROOF: Closed and bounded subspace of $\mathbb{R}$ is some closed interval $[a\ b]$.

Case I: If $a = b$, then $[a, b] = \{a\}$, which is compact.

Case II: If $a < b$, then the class of all intervals $[a\ c[$; $]c\ b]$ is an open subbase for $[a\ b]$. Similarly, the class of all intervals $[a\ c]$, $[d\ b]$ is a closed subbase for $[a\ b]$.

Let $S = \{[a\ c_i], [d_j\ b]\}$ be the class of those subbasic closed sets which satisfy finite intersection property.

Now here arises the following cases.

i) If $S$ contains only the interval of the form $[a\ c_i]$ then always $a \in \cap S$.

$\Rightarrow \cap S \neq \phi$. So closed interval $[a\ b]$ is compact.

ii) If $S$ contains only the intervals of the form $[d_j, b]$, then always $b \in \cap S \Rightarrow \cap S \neq \phi$. So $[a\ b]$ is compact.

iii) If $S$ contains the intervals of both forms. Then, put $d = \sup\{d_j\}$

To prove. $d \leq c_i, \forall i$.

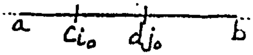
Suppose on the contrary that it is not true.

Then for some $i_0$, $d > c_{i_0}$

$\Rightarrow c_{i_0} < d = \sup\{d_j\}$.

Then, there exists some $d_{j_0}$ such that $c_{i_0} < d_{j_0}$.

Then, $[a\ c_{i_0}] \cap [d_{j_0}\ b] = \phi$.

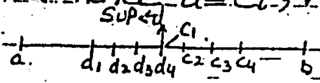


$\Rightarrow S$ doesnot satisfy finite intersection property. Which is a contradiction.

So, our supposition is wrong and hence $d \leq c_i, \forall i$.

$\Rightarrow \cap S \neq \phi$

$\Rightarrow [a\ b]$ is compact.



Real Analysis.

**THEOREM:** Every compact subspace of real line $\mathbb{R}$ is closed and bounded.

**PROOF:** Let '$C$' be a compact subspace of $\mathbb{R}$.

To prove: $C$ is closed and bounded.

As $\mathbb{R}$ is $T_2$-space and compact subspace of a $T_2$-space is closed. So, $C$ is closed.

Now it remains only to prove that $C$ is bounded.

Let $U_K = B(0,K), K \in \mathbb{N}$. Then, $\{U_K : K \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$. $B(0,K) = ]-K\ K[$. $]-1\ 1[ \cup ]-2\ 2[ \cup \dots ]-\infty\ \infty[ \Rightarrow \mathbb{R}$.

Now as $U_K$ is open in $\mathbb{R}$ and $C$ is subspace of $\mathbb{R}$. So, $U_K \cap C$ is a open set in $C$.

Let $\{U_K \cap C : K \in \mathbb{N}\}$ be an open cover for $C$. As $C$ is compact. So this open cover has a finite subcover, $\{U_{k_1} \cap C, U_{k_2} \cap C, \ldots, U_{k_n} \cap C\}$.

$$\Rightarrow C = \bigcup_{i=1}^{n} (U_{k_i} \cap C)$$

$$= (\bigcup_{i=1}^{n} U_{k_i}) \cap C$$

$$\Rightarrow C \subseteq \bigcup_{i=1}^{n} U_{k_i}.$$

Put $m = \max(k_1, k_2, \ldots, k_n)$.

Then $\bigcup_{i=1}^{n} U_{k_i} = U_m = ]-m \ m[$.

$$\Rightarrow C \subseteq ]-m \ m[.$$

$\Rightarrow C$ is bounded.

Real Analysis

THEOREM: Prove that compact subspace of $R^n$ is closed and bounded.

PROOF: Let $C$ be a compact subspace of $R^n$.

To prove: $C$ is closed and bounded.

As $\mathbb{R}^n$ is $T_2$ and $C$ is a compact subspace of $T_2$ space $\mathbb{R}^n$. So $C$ is closed.

Now, let $U_k = B(0, k)$ (where $0 = (0, 0, \ldots, 0)$ & $k \in N$)

then $\{U_k : k \in N\}$ is an open cover for $R^n$.

As $U_k$ is an open set in $\mathbb{R}^n$.

So, $U_k \cap C$ is open set in $C$.

Let $\{U_k \cap C : k \in N\}$ be an open cover for $C$.

As '$C$' is compact. So this open cover has a finite subcover $\{U_{k_1} \cap C, U_{k_2} \cap C, \ldots, U_{k_n} \cap C\}$.

$$\Rightarrow C = \bigcup_{i=1}^{n} (U_{k_i} \cap C).$$

$$\Rightarrow C = (\bigcup_{i=1}^{n} U_{k_i}) \cap C.$$

$$\Rightarrow C \subseteq \bigcup_{i=1}^{n} U_{k_i} \subseteq U_{n^*} \quad \text{where } n^* = \max(k_1, k_2, \ldots, k_n).$$

$\Rightarrow C \subseteq U_n^*$.

As $U_n^*$ is bounded so $C$ is bounded.

Real Analysis

THEOREM: A continuous real valued function defined on a compact space is bounded and attains its bounds.

PROOF: Let $f: X \to \mathbb{R}$ be a continuous function from a compact space $X$.

To prove: $f$ is bounded.

For this, we prove $f(X)$ is bounded.

As $X$ is compact and $f$ is continuous and continuous image of a compact space is compact.

So; $f(X)$ is compact. Hence $f(X)$ is a compact subspace of $\mathbb{R}$.

As a compact subspace of $\mathbb{R}$ is closed and bounded, so $f(X)$ is closed and bounded.

Let $M = \operatorname{Sup} f(X)$ and $m = \operatorname{Inf} f(X)$

As $f(X)$ is bounded so $M$ and $m$ exists.

As sup and Inf of a set are its limit points.

So $M$ and $m$ are the limit points of $f(X)$.

As $f(X)$ is closed.

So $M, m \in f(X)$.

DEFINITION: COUNTABLY COMPACT SPACE:

A topological space $X$ is said to be countably compact space if every countable open cover for $X$ has a finite subcover for $X$.

REMARK:

Every compact space is also countably compact space.

THEOREM: A topological space X is countably compact if and only if every countable collection of closed sets in X which satisfy finite intersection property has non-empty intersection.

PROOF: Suppose that X is countably compact space and $\{U_n : n \in N\}$ be the countable collection of closed sets which satisfy finite intersection property.

To prove: $\bigcap_{n\in N} U_n$ is non empty.

Suppose on the contrary, $\bigcap_{n\in N} U_n = \phi$

$$\Rightarrow \left(\bigcap_{n\in N} U_n\right)' = \phi' \Rightarrow \bigcup_{n\in N} U_n' = X.$$

As $U_n$ is closed so $U_n'$ is open for all $n$.

$\Rightarrow \{U_n' : n\in N\}$ is a countable open cover for X.

As X is countably compact so this countable open cover has a finite subcover $\{U'_1, U'_2, --- U'_n\}$.

$$\Rightarrow \bigcup_{i=1}^{n} U_i' = X$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} U_i'\right)' = X' \Rightarrow \bigcap_{i=1}^{n} U_i = \phi$$

$\Rightarrow$ The class $\{U_n : n\in N\}$ does not satisfy finite intersection property.

Which is a contradiction.

$\because \{U_n : n\in N\}$ satisfy finite intersection property.

So our supposition is wrong.

Hence, $\bigcap_{n\in N} U_n \neq \phi$.

Conversely, suppose that for every countable collection of closed sets $\{U_n : n\in N\}$ which satisfy finite intersection property has non empty intersection.

To prove: X is countably compact.

Suppose, X is not countably compact.

Let $\{U_n : n \in N\}$ be a countably open cover for X.

As X is not countably compact space so for every finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\{U_n : n \in N\}$.

$$\bigcup_{i=1}^{n} U_i \neq X.$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} U_i\right)' \neq X'.$$

$$\Rightarrow \bigcap_{i=1}^{n} U_i' \neq \phi.$$

As $\{U_n : n \in N\}$ be collection of open sets with

$$\bigcup_{n \in N} U_n = X$$

$\Rightarrow \{U_n' : n \in N\}$ is the collection of closed sets with $\bigcap_{n \in N} U_n' = \phi$.

$\Rightarrow \{U_n' : n \in N\}$ is the class of closed sets

THEOREM: Let X be a topological space, then any infinite subset of X has a limit point if and only if every countably infinite subset of X has a limit point.

PROOF: Let us suppose in topological space X every infinite subset of X has limit point Then trivially every countably infinite subset of X also has a limit point.

Conversely, suppose every countably infinite subset of X has the limit point.

To prove: Every infinite subset of X has a limit point.

[illegible] ---- are irrational nos.
[illegible] ---- ⊆ irrational nos. and 1-1 correspondence with prime no. and prime no. ⊆ $\mathbb{N}$ and $\mathbb{N}$ is countably.

144

Let 'A' be an infinite subset of X. Then, by a well known result of a set theory, A has a countable infinite subset B.

Then, by the hypothesis B has the limit point say $\alpha$. Then, for every open set U in X containing $\alpha$,

$$U \cap B \setminus \{\alpha\} \neq \phi$$

$$\Rightarrow U \cap A \setminus \{\alpha\} \neq \phi \quad \because B \subseteq A$$

$\Rightarrow$ $\alpha$ is also the limit point of A.

THEOREM: Let X be a countably compact space then every infinite subset of X has a limit point.

PROOF: Let A be an infinite subset of a countably compact space X. To prove A has a limit point.

Suppose, A has no limit point.

Let $B = \{\alpha_1, \alpha_2, \ldots\}$ be a countably infinite subset of A. Then B has no limit point. Now, consider the subset $C_n = \{\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots\}$, $n \in \mathbb{N}$.

Then, as for every $C_n$, $D(C_n) = \phi \subseteq C_n$. (If $D(A) \subseteq A$ then A is closed.)

Hence, $C_n$ is closed for all $n$.

$\Rightarrow$ $\{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property.

Because, for every finite subcollection $\{C_{n_1}, C_{n_2}, \ldots, C_{n_r}\}$, $\bigcap_{i=1}^{r} C_{n_i} = C_{n'} \neq \phi$ where, $n' = \max(n_1, n_2, \ldots, n_r)$.

Hence, $\{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property and $\bigcap_{i=1}^{\infty} C_n = \phi \Rightarrow$ X is not countably compact.

Which is a contradiction.

So, our supposition is wrong.

Hence, A has a limit point.

BOLZANO WEIERSTRASS PROPERTY:

A space X is said to satisfy, B.W property if and only if every infinite subset of X has a limit point in X.

COROLLARY: Every countably compact space satisfies B.W Property.

PROOF: Let X be a countably compact space and A be an infinite subset of X. Then, by a well known theorem (Previous), A has limit point. So, X satisfies B.W property.

SEQUENTIALLY COMPACT SPACE:

A space 'X' is said to be sequentially compact space if and only if every sequence in X has a convergent subsequence.

THEOREM: A metric space is sequentially compact if and only if it satisfies B.W. Property.

PROOF: A metric space is sequentially compact.

To prove: X satisfies B.W property.

Let A be an infinite subset of X.

To prove: A has limit point in X.

Let $\{x_n\}$ be a sequence in A. As $A \subseteq X$, so $\{x_n\}$ is also a sequence in X.

As, X is sequentially compact, so this sequence $\{x_n\}$ has a convergent subsequence.

Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$

such that $x_{n_k} \to x \in X$.

Let B be the set of the point of $\{x_{n_k}\}$.

Then, $x$ is the limit point of B.

As $B \subseteq A$. So, $x$ is the limit point of A.

$\Rightarrow$ X satisfy B.W Property.

Conversely, suppose X satisfies B.W property.

To prove: X is sequentially compact.

Let $\{x_n\}$ be a sequence in X.

If a point $x$ in $\{x_n\}$ repeated infinitely many times then $(x, x, x, \cdots) \to x$ is convergent subsequence of $\{x_n\}$.

If no point repeated infinitely many times then set A be the set of the points of sequence $\{x_n\}$. As X satisfies B.W property. So, A has limit point '$x$'. Then we can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$.

$\Rightarrow$ X is sequentially compact.

# "CONNECTED SPACES"

DEFINITION: DISCONNECTED:

A topological space $(X, \mathcal{T})$ is said to be disconnected if there exists two non-empty open (or closed) sets $A$ and $B$ in $X$ such that $A \cup B = X$ and $A \cap B = \phi$.

e.g. If $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\phi, X, \{1, 3\}, \{2, 4\}\}$

Then, $(X, \mathcal{T})$ is disconnected because we have $A = \{1, 3\}$, $B = \{2, 4\}$. $A$ and $B$ are open,

$A \cup B = X$, $A \cap B = \phi$.

DEFINITION: CONNECTED SPACES:

A topological space $(X, \mathcal{T})$ is said to be connected if it is not disconnected.

e.g. If $X = \{1, 2, 3\}$, $\mathcal{T} = \{\phi, X, \{2\}\}$. Then $(X, \mathcal{T})$ is connected.

REMARK: For any set $X$ if $\mathcal{T} = \mathcal{T}_I$, then $(X, \mathcal{T})$ is connected.

THEOREM: For any $X$ with more than two points $(X, \mathcal{T}_D)$ is disconnected.

PROOF: Since $\mathcal{T} = \mathcal{T}_D$, so every subset of $X$ is open (as well as closed).

Proper subset Let $A \subsetneq X$. Then $A' = X \setminus A \neq \phi$.

Also $A'$ is open ($\because$ $A$ is also closed).

So we have two open sets $A$ and $B = A'$ such that $A \cup B = X$ and $A \cap B = \phi$.

So $(X, \mathcal{T}_D)$ is disconnected.

Hence proved.

**THEOREM:** If $X$ is infinite. Then $(X, \mathcal{F}_c)$ is connected.

**PROOF:** On the contrary suppose $(X, \mathcal{F}_c)$ is disconnected. Then, there exists two open sets (or closed sets) $A$ and $B$ such that $A \cup B = X$ and $A \cap B = \phi$

Now as $A$ and $B$ are open and $\mathcal{F} = \mathcal{F}_c$. So $A'$ and $B'$ are finite. Now $A \cap B = \phi$

$\Rightarrow (A \cap B)' = \phi'$

$\Rightarrow A' \cup B' = X$.

$\Rightarrow X$ is finite ($\because$ Union of two finite sets is finite).

Which is a contradiction.

$\because X$ is infinite.

So, our supposition is wrong.

Hence $(X, \mathcal{F}_c)$ is connected.

**THEOREM:** Continuous image of a connected space is connected.

**PROOF:** Let $X$ be a connected space and $f : X \to Y$ be a continuous function.

To prove: $f(X)$ is connected.

Suppose on the contrary that $f(X)$ is disconnected. Then, there exists two non-empty open sets $A$ and $B$ in $f(X)$ such that $A \cup B = f(X)$ and $A \cap B = \phi$.

Now $A \cup B = f(X)$

Now as $A$ and $B$ are open in $f(X)$ and $f$ is continuous function.

So, $f^{-1}(A)$ and $f^{-1}(B)$ are open in $X$.
Further $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(X)) = X$
And, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$

$\Rightarrow X$ is disconnected.
Which is a contradiction.
$\therefore X$ is connected.
So, our supposition is wrong.
Hence, $f(X)$ is connected.

THEOREM: The space $\mathbb{Q}$ as subspace of $\mathbb{R}$ is disconnected.

PROOF: Let $r$ be any irrational number.
Then, $]-\infty\ r[$, $]r\ \infty[$ are open in $\mathbb{R}$.

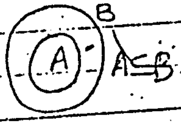
$\Rightarrow\ ]-\infty\ r[ \cap \mathbb{Q}$, $]r\ \infty[ \cap \mathbb{Q}$ are open sets in $\mathbb{Q}$ with $(]-\infty\ r[ \cap \mathbb{Q}) \cup (]r\ \infty[ \cap \mathbb{Q})$

$= (]-\infty\ r[ \cup ]r\ \infty[) \cap \mathbb{Q}$ (By distributive law)

$= (\mathbb{R} \setminus \{r\}) \cap \mathbb{Q}$

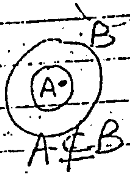
$A \subseteq B$. $A \cap B = A$

$= \mathbb{Q}$



And $(]-\infty\ r[ \cap \mathbb{Q}) \cap (]r\ \infty[ \cap \mathbb{Q})$
$= (]-\infty\ r[ \cap ]r\ \infty[) \cap \mathbb{Q}$

$= \phi \cap \mathbb{Q} = \phi$



$\Rightarrow \mathbb{Q}$ is disconnected.

THEOREM: A topological space X is disconnected if and only if X contains a non-empty subset A which is both open and closed.

PROOF: Suppose X is disconnected and $A \neq \phi$ be a subset of X.

To prove: A is both open and closed.

As X is disconnected then There exists Two open sets A and B such that $A \cup B = X$ and $A \cap B = \phi$.

Now A is open.

Now as $A \cup B = X$ and $A \cap B = \phi$. So, by law of complements $B = A'$.

Also B is open $\Rightarrow$ $A'$ is open $\Rightarrow$ A is closed.

$\Rightarrow$ A is both open and closed.

Conversely, suppose that for topological space X, there is non-empty subset A of X which is both open and closed.

To prove: X is disconnected.

Let $B = A'$. So A is closed.

$\Rightarrow$ $A'$ is open $\Rightarrow$ B is open.

$\Rightarrow$ A and B are open in X with $A \cup B = A \cup A' = X$

And $A \cap B = A \cap A' = \phi$

$\Rightarrow$ X is disconnected.

THEOREM: A space X is connected if and only if there doesnot exists a continuous surjective function from X to discrete Two point space.

PROOF. Let X be connected.
To prove: There doesnot exists a continuous surjective function from X to discrete Two point space $Y = \{a, b\}$.

Suppose on the contrary that there exists a function $f : X \to Y = \{a, b\}$; Y is discrete, which is continuous and onto. As Y is discrete so $\phi, \{a\}, \{b\}, \{a, b\}$ are open sets.

As f is continuous so $f^{-1}(\phi), f^{-1}(\{a\}), f^{-1}(\{b\})$ and $f^{-1}(\{a, b\})$ all are open in X.

Now as f is onto. So $f(X) = Y \Rightarrow X = f^{-1}(Y)$

$$\begin{aligned} \Rightarrow X &= f^{-1}(\{a, b\}) \\ &= f^{-1}(\{a\} \cup \{b\}) \\ &= f^{-1}(\{a\}) \cup f^{-1}(\{b\}). \end{aligned}$$

Further $f^{-1}(\{a\}) \cap f^{-1}(\{b\})$

$= f^{-1}(\{a\} \cap \{b\}) = f^{-1}(\phi) = \phi$

$\Rightarrow$ X is disconnected.

Which is a contradiction.

$\because$ X is connected.

So, our supposition is wrong.

Hence, there doesnot exist a continuous function from X onto discrete Two point space Y.

Conversely suppose there doesnot exist a continuous function from $X$ onto discrete two point space $Y$.

To prove: $X$ is connected.

Suppose on the contrary that $X$ is not connected. Then $X$ is disconnected. So, there exists two non-empty open (or closed) sets $A$ and $B$ in $X$ such that $A \cup B = X$ and $A \cap B = \phi$.

Now define a function $f: X \to Y = \{a, b\}$ by $f(A) = a$ and $f(B) = b$

$$\Rightarrow A = f^{-1}(\{a\}) \;,\; B = f^{-1}(\{b\})$$

Now as $Y = \{a, b\}$ is discrete so, $\phi, \{a\}, \{b\}, \{a, b\}$ are open in $Y$.

$$\begin{aligned} \text{Now } f^{-1}(\phi) &= f^{-1}(\{a\} \cap \{b\}) \\ &= f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = A \cap B = \phi \end{aligned}$$

$$\begin{aligned} f^{-1}(Y) &= f^{-1}(\{a\} \cup \{b\}) \\ &= f^{-1}(\{a\}) \cup f^{-1}(\{b\}) \\ &= A \cup B = X \end{aligned}$$

So $f$ is continuous. ($\because$ Inverse image of each open set is open and here all four open sets of $Y$ have open inverse images). Which is a contradiction.

So our supposition is wrong.
Hence $X$ is connected.
Hence Proved

**THEOREM:** A topological space X is disconnected iff there exist a continuous function from X onto discrete two points space

**THEOREM:** A topological space X is said to be connected iff every continuous function from X to discrete space Y reduces to a constant function.

**PROOF:** Suppose X is connected.
Then, X has no its proper subset which is both open and closed.
Let $a \in Y$. As Y is discrete. So $\{a\}$ is both open and closed.

$\Rightarrow f^{-1}(\{a\})$ is open and closed in X.
$\Rightarrow f^{-1}(\{a\})$ is not a proper subset of X.
$\Rightarrow f^{-1}(\{a\}) = \phi$ or $f^{-1}(\{a\}) = X$.

But $f^{-1}(\{a\}) \neq \phi$.
So, $f^{-1}(\{a\}) = X \Rightarrow f(X) = \{a\}$.
$\Rightarrow f$ is constant function.

Conversely suppose every continuous function f from X to discrete space Y reduces to a constant function.

To prove: X is connected.
Suppose, X is disconnected. Then, There exists a continuous function $f : X \to Y = \{a, b\}$. Which is continuous and is onto and Y is discrete $\Rightarrow f(X) = Y \Rightarrow f$ is not constant.

Which is a contradiction.
So, our supposition is wrong.
And hence, X is connected!

**THEOREM:**

Let X be disconnected space with disconnection $\{A, B\}$ and C is a connected subspace of X. Then, either $C \subseteq A$ or $C \subseteq B$.

**PROOF:**

Suppose on the contrary that $C \notin A$ and $C \notin B$. Then, $C \cap A$ and $C \cap B$ are non empty open sets in C, ($\because$ C is subspace so $C \cap A$ is open in C) with $(C \cap A) \cup (C \cap B) = C \cap (A \cup B)$
$= C \cap X = C$.
and $(C \cap A) \cap (C \cap B) = C \cap (A \cap B)$.
$= C \cap \phi = \phi$

$\Rightarrow$ C is disconnected.
A contradiction.
$\because$ C is connected.
So our supposition is wrong.
Hence, $C \subseteq A$ or $C \subseteq B$.

**THEOREM:**

Let $X = \cup X_\alpha$ where each $X_\alpha$ is connected and $\bigcap_{\alpha \in I} X_\alpha \neq \phi$. Then, X is connected.

**PROOF:** Suppose X is disconnected then there exists two non-empty open sets A and B in X such that $A \cup B = X$ and $A \cap B = \phi$.
Now as $X = \bigcup_{\alpha \in I} X_\alpha \Rightarrow$ for each $\alpha \in I$, $X_\alpha \subseteq X$.

As for each $\alpha \in I$, $X_\alpha$ is connected. So, by well known theorem, for each $\alpha \in I$,

either $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$

But as $\bigcap_{\alpha \in I} X_\alpha \neq \phi$,

So $\bigcup_{\alpha \in I} X_\alpha \subseteq A$ or $\bigcup_{\alpha \in I} X_\alpha \subseteq B$.

$\Rightarrow X \subseteq A$ or $X \subseteq B$

If $X \subseteq A \Rightarrow A = X \Rightarrow B = \phi$.

If $X \subseteq B \Rightarrow B = X \Rightarrow A = \phi$.

Which is a contradiction

$\because$ Both A and B are non empty.

So, our supposition is wrong.

And, hence X is connected.

**THEOREM:** A topological space X is connected iff for every pair of points in X there is some connected subspace of X which contains both.

**PROOF:** Suppose X is connected and $x, y \in X$ such that $x \neq y$. To prove: there is some connected subspace of X which contains both x and y. Then X itself is the connected subspace of X which contains both x and y.

Conversely, suppose in a topological space X, for every pair of points $x, y \in X$ such that $x \neq y$, there is some connected subspace of X which contains both $x$ and $y$.

To prove: X is connected:

Now let, $a \in X$ be some fixed point such that for $x \in X$, $a \neq x$. Then, by the hypothesis, there is a connected subspace $C_{a,x}$ of X such that $a, x \in C_{a,x}$.

Then we have a collection $\{C_{a,x} : x \in X\}$ of connected subspace of X such that,

$$\bigcap_{x \in X} C_{a,x} \neq \phi \text{ and } \bigcup_{x \in X} C_{a,x} = X$$

Then, by a well known Theorem, X is connected.

**THEOREM**: Let C be a connected subspace of X and for some subset A of X, $C \subseteq A \subseteq \overline{C}$. Then, A is connected. In particular $\overline{C}$ is connected.

**Proof**: Suppose on the contrary that A is disconnected. Then, there exists two non-empty open sets $U_1$ and $V_1$ of A such that:

$U_1 \cup V_1 = A$ and $U_1 \cap V_1 = \phi$.

As $U_1$ and $V_1$ are open in A and A is subspace of X. So, then there exists two disjoint open sets U and V in X such that:

$U_1 = U \cap A$ and $V_1 = V \cap A$.

Now, $C \subseteq A = U_1 \cup V_1 \subseteq U \cup V$

$\Rightarrow C \subseteq U \cup V$ and C is connected and $U \cap V = \phi$. Then, by a well known Theorem: either $C \subseteq U$ or $C \subseteq V$.

With out any loss of generality, suppose

$C \subseteq U$.

As $U \cap V = \phi \Rightarrow U \subseteq V'$.

$\Rightarrow C \subseteq U \subseteq V' \Rightarrow C \subseteq V'$.

As $V$ is open $\Rightarrow$ $V'$ is closed.

So $V'$ is the closed superset of $C$.

But $\overline{C}$ is the smallest closed superset of $C$. So $\overline{C} \subseteq V'$.

$\Rightarrow C \subseteq A \subseteq \overline{C}$ (Given)

$\Rightarrow C \subseteq A \subseteq \overline{C} \subseteq V'$.

$\Rightarrow A \subseteq V' \Rightarrow A \cap V = \phi \Rightarrow V_1 = \phi$.

Which is a contradiction.

$\therefore V_1 \neq \phi$.

So, our supposition is wrong.

Hence, $A$ is connected.

Now to prove $\overline{C}$ is connected.

As $C \subseteq \overline{C} \subseteq \overline{C}$ So, by the above argument $\overline{C}$ is connected.

**THEOREM:** A subspace $X$ of a real line $R$ is connected if and only if $X$ is an interval.

**PROOF:** Suppose, $X$ is connected. To prove: $X$ is an interval. Suppose $X$ is not an interval, then there exists $x, y, z$ such that $x < y < z$ and $x, z \in X$ but $y \notin X$.

Now $]-\infty, y[$ and $]y\ \infty[$ are open in $R$

$\Rightarrow ]-\infty, y[ \cap X$ and $]y\ \infty[ \cap X$ are open in $X$ with

$$(]-\infty\ y[ \cap X) \cup (]y\ \infty[ \cap X) = (]-\infty\ y[ \cup ]y\ \infty[) \cap X$$
$$= (R|\{y\}) \cap X = X.$$

and $(]-\infty\ y[\cap X) \cap (]y\ \infty[\cap X)$

$= (]-\infty\ y[\cap ]y\ \infty[) \cap X$

$= \phi \cap X = \phi$

$\Rightarrow$ X is disconnected.

Which is a condradiction.

$\because$ X is connected.

So, our supposition is wrong.

Hence, X is an interval.

Conversely suppose X is an interval.

To prove: X is connected. On the contrary suppose X is disconnected. Then there exists two non empty open disjoint subsets A and B of X such that $A \cup B = X$ and $A \cap B = \phi$.

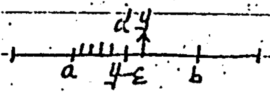
Let $a \in A$ and $b \in B$.

As $A \cap B = \phi \Rightarrow a \neq b$.

Let $a < b$. Put $y = \sup([a,b] \cap A)$.

Then, by the definition of supremum for every $\varepsilon > 0$, there is some point $a'$ in A such that $y - \varepsilon < a'$

$\Rightarrow y - a' < \varepsilon$.



$\Rightarrow d(y, a') < \varepsilon$.

$\Rightarrow a' \in B(y, \varepsilon)$

So, every open ball with centre at y contains a point of A different from y.

$\Rightarrow$ y is the limit point of A.

As A is also closed.

So, $y \in A$. Similarly $y \in B \Rightarrow A \cap B \neq \phi$.

Which is a contradiction.
$\because A \cap B = \phi$
So, our supposition is wrong.
Hence, X is connected.

## COMPONENT: (DEF).

The maximal connected subspace of topological space X is called component of X. i.e, a connected subspace of topological space X is called component of X if it is not contained in any other connected subspace of X.

**THEOREM:** Let X be a topological space, then:

i) Each $x \in X$ is contained in exactly one component of X.

ii) Each connected subspace of X is contained in exactly one component of X.

iii) Each connected subspace of X which is both open and closed is component of X.

iv) Every component of X is closed in X.

**PROOF:** i) Let $\gamma = \{C_\alpha : \alpha \in I$ and $x \in C_\alpha\}$ be a collection of all connected subspace of X which contains $x$.

Then, $\bigcap_{\alpha \in I} C_\alpha \neq \phi$.

Then, by a well known theorem,
$C = U\gamma = \bigcup_{\alpha \in I} C_\alpha$ is connected

subspace of $X$ and $x \in C$ and for every $\alpha \in I$, $C_\alpha \subseteq C$. This shows that $C$ is component of $X$. Now we show that $C$ is the only component of $X$ containing $x$. On the contrary let $C^*$ be another component of $X$ containing $x$.

Now as, $C^*$ is the component of $X$ containing $x$ and $C$ is connected subspace of $X$. So, $C \subseteq C^*$. Also as $C^*$ is connected subspace of $X$ containing $x$. So $C^* \in \gamma$

$\Rightarrow C^* \subseteq \cup\gamma = C$.

$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$.

This shows that $C$ is the only component of $X$ containing $x$.

ii) Let $A$ be a connected subspace of $X$ and to prove: $A$ is contained only in one component of $X$. Let $\{C_\alpha : \alpha \in I\}$ is a collection of all connected subspaces of $X$ containing $A$. Then, $\bigcap_{\alpha \in I} C_\alpha \neq \phi$ and $\bigcup_{\alpha \in I} C_\alpha = C$, which

is connected subspace of $X$. Also, $A \subseteq C$

$\Rightarrow$ $C$ is connected subspace of $X$ containing $A$. Also $C = \bigcup_{\alpha \in I} C_\alpha$ so $C$ is such maximal

connected subspace of $X$.

$\Rightarrow$ $C$ is component of $X$ containing $A$. Now we show $C$ is the only component of $X$ containing $A$.

For this, let $C^*$ be another component of $X$ containing $A$. Now, as $C^*$ is maximal connected subspace of X containing A and C is connected subspace of X containing A so $C \subseteq C^*$. Further also as $C^*$ is connected subspace of X containing A. So, $C^* \in \{C_\alpha : \alpha \in I\}$

$$\Rightarrow C^* \subseteq \bigcup_{\alpha \in I} C_\alpha = C$$

$$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$$

Hence, C is only component of X containing A.

iii) Let A be a connected subspace of X which is both open and closed.

To prove: A is component of X.

Suppose, A is not component of X. Then, A is contained in exactly one component of X, say C. As C is component of X, $A \subseteq C$ and A is not component of X.

$\Rightarrow$ A is proper subset of C.

Then $A \cap C$ and $A' \cap C$ are both non-empty.

Now, as A is open in X, so $A \cap C$ is open in C. Also, as A is closed in X $\Rightarrow$ A' is open in X.

$\Rightarrow A' \cap C$ is open in $C \Rightarrow A \cap C$ and $A' \cap C$ are ~~with~~ open in C with:

$$(A \cap C) \cap (A' \cap C) = (A \cap A') \cap C$$
$$= \phi \cap C = \phi$$

And $$(A \cap C) \cup (A' \cap C) = (A \cup A') \cap C$$
$$= X \cap C = C$$

$\Rightarrow$ C is disconnected.

Which is a contradiction.

$\therefore$ C is component of X.

So our supposition is wrong.

Hence, A is component of X.

iv). Let $C$ be a component of $X$.
To prove: $C$ is closed.
For this, we prove $C = \bar{C}$.
Suppose $C \neq \bar{C}$.

Now as $C \subseteq \bar{C}$ and $C \neq \bar{C} \Rightarrow C \subset \bar{C}$.
Now as $C$ is connected, then by a well known theorem, $\bar{C}$ is connected $\Rightarrow \bar{C}$ is connected subspace of $X$ containing $C \Rightarrow C$ is not component of $X$.
A contradiction.
$\because C$ is component of $X$.
So our supposition is wrong and hence $C = \bar{C}$.
$\Rightarrow C$ is closed.

## TOTALLY DISCONNECTED: (DEF).

A topological space $X$ is called totally disconnected if for each pair of points $x, y \in X$ we can form a disconnection $\{A, B\}$ of $X$ such that $x \in A$ and $y \in B$.

**THEOREM:** Every discrete space is totally disconnected.

**PROOF:** Let $X$ be a discrete space.
To prove: $X$ is totally disconnected.
Let $x, y \in X$ such that $x \neq y$.
Let $U = \{x\}$ and $V = X - \{x\}$.
As $X$ is discrete so $U$ and $V$ are open in $X$. Also clearly,
$x \in X$, $y \in V$, $U \cap V = \phi$, $U \cup V = X$.
$\Rightarrow X$ is totally disconnected.

**THEOREM:** Every totally disconnected is $T_2$ space.

**PROOF:** Let $X$ be a totally disconnected.
To prove: $X$ is $T_2$ space.
Let $x, y \in X$ such that $x \neq y$.
As $X$ is totally disconnected so then there exist two open sets $U$ and $V$ in $X$ such That. $x \in U$, $y \in V$, $U \cap V = \phi$ and $U \cup V = X$.
ie, we have two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
$\Rightarrow X$ is $T_2$ space.

**THEOREM:** A subspace $Q$ of rationals in real line $R$ is totally disconnected.

**PROOF:** To prove: $Q$ is totally disconnected in $R$.
Let $r_1, r_2 \in Q$ such that $r_1 \neq r_2$. Without any loss of generality, suppose $r_1 < r_2$.

Now as by a well known Theorem of calculus here is an irrational number between every two rational numbers. There is an irrational number '$t$' such that $r_1 < t < r_2$.
Now $]-\infty\ t[$ and $]t\ \infty[$ are two open sets in $R$. Now as $Q$ is subspace of $R$. So,
$U = Q \cap ]-\infty\ t[$ and $V = Q \cap ]t\ \infty[$ are open in $Q$.
Also $r_1 \in U$, $r_2 \in V$, $U \cap V = \phi$, $U \cup V = Q$.

$\Rightarrow Q$ is totally disconnected.

Hence Proved.

**THEOREM:** The components of totally disconnected space are its singleton subsets.

**Proof:** Let $X$ be a totally disconnected space.
To prove: Components of $X$ are its singleton subset.
For this, we show that no two points subspace of $X$ is connected.

Let $x, y \in X$ such that $x \neq y$ and $C = \{x, y\}$ be a subspace of $X$. As $X$ is totally disconnected and $x, y \in X$ such that $x \neq y$, so then there exist two open set $U$ and $V$ in $X$ such that,

$$x \in U,\ y \in V,\ U \cup V = X,\ U \cap V = \phi.$$

Now as $U$ and $V$ are open in $X$ and $C$ is subspace of $X$ so,

$U \cap C$ and $V \cap C$ are open in $C$.

Also, $x \in U$ and $x \in C \Rightarrow x \in U \cap C$.

$y \in V$ and $y \in C \Rightarrow y \in V \cap C$.

$$(U \cap C) \cup (V \cap C) = (U \cup V) \cap C$$
$$= X \cap C = C.$$
$$(U \cap C) \cap (V \cap C) = (U \cap V) \cap C$$
$$= \phi \cap C = \phi.$$

$\Rightarrow C$ is disconnected.

HENCE PROVED.

**THEOREM:** If a $T_2$-space has an open base whose sets are also closed. Then, $X$ is totally disconnected.

**PROOF:**

Let $x, y \in X$ such that $x \neq y$. As $X$ is $T_2$ space. So then, there exists two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Let $B$ be an open base for $X$ whose elements are also closed. As $x \in U$, $U$ is an open set, $\mathcal{B}$ is base, so then there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$

Now as $B \subseteq U$ and $U \cap V = \phi$

So $B \cap V = \phi \Rightarrow V \subseteq B' = W$

As $B \in \mathcal{B}$, so $B$ is closed.

$\Rightarrow B'$ is open. Now $B$ and $W$ are two open sets in $X$ with $x \in B$. As $y \notin B$

$\Rightarrow y \in B' = W \Rightarrow y \in W$.

Also $B \cap W = B \cap B' = \phi$

$B \cup W = B \cup B' = X$

$\Rightarrow X$ is totally disconnected.

**THEOREM:** Let $X$ be a compact housdroff space then $X$ is totally disconnected iff it has an open base whose sets are also closed.

**Proof:** Let the compact $T_2$ space has an open base whose sets are closed. To prove: $X$ is totally disconnected.

Let $x, y \in X$ such that $x \neq y$. As $X$ is $T_2$ so then there is an open set $U$ such that $x \in U$ and $y \notin U$. Now as $x \in U$ and $U$ is open in $X$ with $X$ has base $B$. Then, there is $G \in B$ such that $x \in G \subseteq U$.

As $y \notin U$ and $G \subseteq U \Rightarrow y \notin G \Rightarrow y \in G'$

As $G \in B$. So $G$ is also closed.

$\Rightarrow G'$ is open. Put $G' = H$

Hence, we have two open sets $G$ and $H$ in $X$ such that: $x \in G$, $y \in H$

$G \cup H = G \cup G' = X$ and $G \cap H = G \cap G' = \phi$.

$\Rightarrow$ X is totally disconnected.

Conversely, suppose X is totally disconnected. (Where X is also compact and $T_2$).
To prove: X has an open base whose sets are also closed.
Let $\mathcal{B}$ be an open base for X.
To prove: elements of $\mathcal{B}$ are also closed.
Let $x \in X$ and G be an open set in X such that $x \in G$.

Case I: If $G = X$, then $B_x = X \in \mathcal{B}$ such that $x \in B_x \subseteq G$. Clearly $B_x$ is both open and closed.

Case II: If $G \neq X \Rightarrow G \subset X$. Now as G is an open set so $G'$ is closed. As $x \in G$ so $x \notin G'$.
Now as, $G'$ is closed subspace of X and X is compact. So $G'$ is also compact ($\because$ Closed subspace of a compact space is compact).
As X is totally disconnected so $\forall\ y \in G'$. such that $x \neq y$, there is subset $H_y$ of X which is both open and closed, such that:
$y \in H_y$ and $x \notin H_y$. Then, the set
$\{H_y : y \in X\}$ is an open cover for $G'$.

As $G'$ is compact, so this open cover has a finite subcover $\{H_1, H_2, \dots, H_n\}$
$$\Rightarrow G' = \bigcup_{i=1}^{n} H_i = H \Rightarrow G' \subseteq H$$

Clearly H is both open and closed.
Further as $x \notin H_y \Rightarrow x \notin \bigcup_{i=1}^{n} H_i = H \Rightarrow x \notin H.$
$\Rightarrow x \in H' = B_x.$

Here $B_x$ is both open and closed.

Now let $z \in B_x$.

$\Rightarrow z \in H' \Rightarrow z \notin H \Rightarrow z \notin G' \Rightarrow z \in G$

$\Rightarrow B_x \subseteq G \Rightarrow x \in B_x \subseteq G$.

$\therefore$ The collection of all such $B_x$ form an open base whose elements are also closed.

HENCE PROVED.

## DEFINITION:

Let X be a topological space and A and B are subsets of X. Then, A and B are said to be separated if and only if $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$.

**THEOREM:** Let X be a topological space and A, B are the subsets of X if A and B are separated in X then $A \cup B$ is disconnected.

**PROOF:** Let $Y = A \cup B$.

Now as, $A$ and $B$ are separated in $X$,
So, $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$.
Now let $G = \bar{B}'$ and $H = \bar{A}'$.

Then as $\bar{A}$ and $\bar{B}$ are closed.
$\Rightarrow \bar{A}'$ and $\bar{B}'$ are open.
$\Rightarrow H$ and $G$ are open in $X$
$\Rightarrow Y \cap G$ and $Y \cap H$ are open in $Y$ ($\because Y$ is subspace).

Now $A \cap \bar{B} = \phi \Rightarrow A \subseteq \bar{B}' \Rightarrow A \subseteq G$.
Further as, $B \subseteq \bar{B} \Rightarrow B \cap \bar{B}' = \phi$.

Now, $Y \cap G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G)$.
$\Rightarrow Y \cap G = A \cup (B \cap \bar{B}')$
$= A \cup \phi = A$.

Similarly, $Y \cap H = B$.

Now, $Y \cap G$ and $Y \cap H$ are open in $Y$ with
$(Y \cap G) \cup (Y \cap H) = A \cup B = Y$.

and $(Y \cap G) \cap (Y \cap H) = A \cap B = \phi$.

$\Rightarrow Y$ is disconnected.

**THEOREM:** Let $G$ and $H$ be the disconnection of a subset $A$ of a topological space $X$ then, show that $A \cap G$ and $A \cap H$ are separated.

PROOF: To prove: $A\cap G$ and $A\cap H$ are separated.
i.e, $(A\cap G)\cap(\overline{A\cap H})=\phi$ and $(\overline{A\cap G})\cap(A\cap H)=\phi$.

First we prove, if $x\in D(A\cap G)$ then;
$x\notin A\cap H$.
Suppose on the contrary; $x\in D(A\cap G)$
$\Rightarrow x\in A\cap H$
$\Rightarrow x\in A$ and $x\in H$
Now, $A\cap G\subseteq G$ and $A\cap G\subseteq A$
$\Rightarrow D(A\cap G)\subseteq D(G)$ and $D(A\cap G)\subseteq D(A)$

So $x\in D(A\cap G)\Rightarrow x\in D(G)\Rightarrow x\in G$
($\because G$ is closed)
Now $x\in H$ and $x\in G\Rightarrow x\in G\cap H$
$\Rightarrow G\cap H\neq\phi$.
Which is a contradiction.
So, our supposition is wrong.

Hence for $x\in D(A\cap G)\Rightarrow x\notin A\cap H$
$\Rightarrow D(A\cap G)\cap(A\cap H)=\phi$
Also, $(A\cap G)\cap(A\cap H)=A\cap(G\cap H)$
$=A\cap\phi=\phi$
$\Rightarrow [D(A\cap G)\cap(A\cap H)]\cup[(A\cap G)\cap(A\cap H)]=\phi$

$\Rightarrow [D(A\cap G)\cup(A\cap G)]\cap(A\cap H)=\phi$ $\big((X\cap Z)\cup(Y\cap Z)=(X\cup Y)\cap Z\big)$

$\Rightarrow (\overline{A\cap G})\cap(A\cap H)=\phi$.

Similarly, $(A\cap G)\cap(\overline{A\cap H})=\phi$

$\Rightarrow (A\cap G)$ and $(A\cap H)$ are separated.

**THEOREM**: Show that a topological space X is connected if and only if every non empty proper subspace has a non-empty boundary.

**PROOF**: We know that.
i) A topological space X is disconnected iff it has a subset A which is both open and closed.
ii) If $(X,\mathcal{T})$ is topological space and $A \subseteq X$ then boundary of A is empty iff A is both open and closed.

Now given X is connected and A is non empty proper subspace of X.
To prove: boundary of A is non-empty.
Suppose, boundary of A is empty.
i.e, $b(A) = \phi$ then, by (ii), A is both open and closed, but by (i) X is disconnected.
Which is a contradiction.
So our supposition is wrong.
And hence, $b(A) \neq \phi$.

Conversely suppose in a topological space X every non empty proper subset of X has non-empty boundary.

To prove: X is connected.
Suppose X is disconnected then by (i), there is subset A of X, which is both open and closed.

then by (ii), $b(A) = \phi$.

A contradiction.

$\therefore b(A) \neq \phi$.

So, our supposition is wrong.

And Hence, $X$ is connected.

**THEOREM:** If $X$ and $Y$ are connected topological spaces then, $X \times Y$ is also connected.

**PROOF:** Let $x \in X$ and $y \in Y$.

Then, $\{x\} \times Y$ and $X \times \{y\}$ are two topological spaces with $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X$.

$\Rightarrow$ As $X$ and $Y$ are connected.

So $\{x\} \times Y$ and $X \times \{y\}$ are connected for all $x \in X$ and $y \in Y$.

Also $(x, y) \in (\{x\} \times Y) \cap (X \times \{y\})$.

$\Rightarrow (\{x\} \times Y) \cap (X \times \{y\}) \neq \phi$.

$\Rightarrow (\{x\} \times Y) \cup (X \times \{y\})$ is connected.

($\because$ The union of T.S is connected provided their intersection $\neq \phi$)

Furthermore,

$$\bigcap_{x \in X} T_x \neq \phi \text{ and } \bigcup_{x \in X} T_x = X \times Y.$$

where $T_x = [(\{x\} \times Y) \cup (X \times \{y\})]$

$\Rightarrow X \times Y$ is connected.