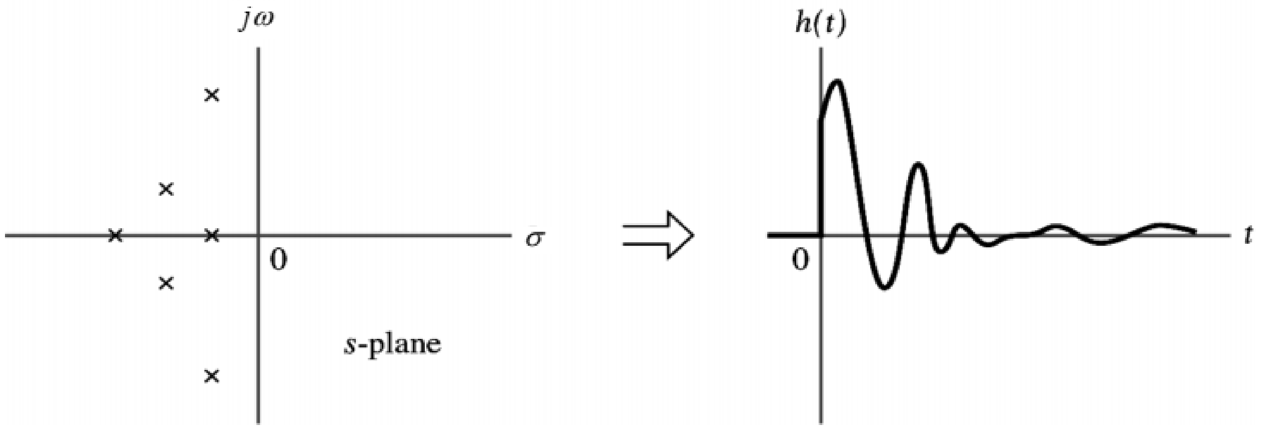
Causal and Stable LTI System

- To obtain a unique inverse transform of $H(s)$, we must know the ROC or have other knowledge(s) of the impulse response
- The relationships between the poles, zeros, and system characteristics can provide some additional knowledges
- **Systems that are stable and causal must have all their poles in the left half of the s-plane**:



A system has the transfer function $H(s) = \frac{2}{s+3} + \frac{1}{s-2}$

Find the impulse response, (a) assuming that the system is stable; (b) assuming that the system is causal; (c) can this system be both stable and causal?

<Sol.>

(a) This system has poles at $s = -3$ and $s = 2$.

Stable ➔ the ROC contains $j\omega$-axis.
the pole at $s=-3$ contributes a right-sided term;
the pole at $s=2$ contributes a left-sided term.

➔ $h(t) = 2e^{-3t}u(t) - e^{2t}u(-t)$

(b) This system has poles at $s = -3$ and $s = 2$.

Causal ➔ right-sided

➔ $h(t) = 2e^{-3t}u(t) + e^{2t}u(t)$

(c) This system has poles at $s = -3$ and $s = 2$ ➔ this system cannot be both stable and causal

Properties of B.L.T.:

- **Linearity Property**

  If $x(t) \longleftrightarrow X(s)$ & $y(t) \longleftrightarrow Y(s)$

  then linearity property states that

  $ax(t)+by(t) \longleftrightarrow aX(s)+bY(s)$

- **Time Shifting Property**

  If $x(t) \longleftrightarrow X(s)$

  then time shifting property states that

  $x(t-t_0) \longleftrightarrow e^{-st_0} X(s)$

$$\boldsymbol{e^{-2t}u(t) \leftrightarrow X(s) = \frac{1}{s+2}} \quad \textbf{ROC: Re}\{\boldsymbol{X(s)}\} > -2$$

$$\boldsymbol{e^{-2(t-3)}u(t-3) \leftrightarrow X(s) = \frac{e^{-3s}}{s+2}} \quad \textbf{ROC: Re}\{\boldsymbol{X(s)}\} > -2$$

Note: ROC doesn't change because it is defined by the pole location.

- **Frequency Shifting Property**

  If $x(t) \longleftrightarrow X(s)$

  then frequency shifting property states

  $e^{-s_0 t} x(t) \longleftrightarrow X(s-s_0)$

- **Time Reversal Property**

  If $x(t) \longleftrightarrow X(s)$

  then time reversal property states

  $x(-t) \longleftrightarrow X(-s)$

continue….:

- **Time Scaling Property**

If x(t) ⟷ X(s)

then x(at)⟷$\frac{1}{|a|}$1X$\left(\frac{s}{a}\right)$

- **Differentiation property**

If x(t) ⟷ X(s)

then $\frac{\boldsymbol{dx(t)}}{\boldsymbol{dt}}$⟷sX(s) and

$\frac{\boldsymbol{d^n x(t)}}{\boldsymbol{dt^n}}$⟷$s^n$X(s)

- **Integration property**

If x(t) ⟷ X(s)

then ∫x(t)dt⟷$\frac{1}{s}$X(s)

∭…∫x(t)dt⟷ $\frac{1}{s^n}$X(s)

If *h*(*t*) is a right-sided sequence, then the ROC extends outward from the outermost pole in *H*(*s*)

continue….:

- **Multiplication by 't' Properties/Frequency differentiation**

If $x(t) \longleftrightarrow X(s)$

then $tx(t) \longleftrightarrow -\frac{dF(s)}{dS}$

- **Multiplication and Convolution Properties**

If $x(t) \longleftrightarrow X(s)$ and $y(t) \longleftrightarrow Y(s)$

then multiplication property states that

$$x(t).y(t) \longleftrightarrow \frac{1}{2\pi j} X(s) * Y(s)$$

and convolution property states that

$$\mathbf{x(t) * y(t) \longleftrightarrow X(s).Y(s)}$$

Determine the Laplace transform of $x(t) = e^{at}u(t)$, and depict the ROC and locations of poles and zeros in the s-plane. Assume that a is real.

<Sol.>

1. First find the LT of $x(t)$: $X(s) = \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s-a)t}dt = \frac{-1}{s-a}e^{-(s-a)t}\Big|_{0}^{\infty}$

2. To evaluate the limit value, we use $s = \sigma + j\omega$ to re-write $X(s)$:

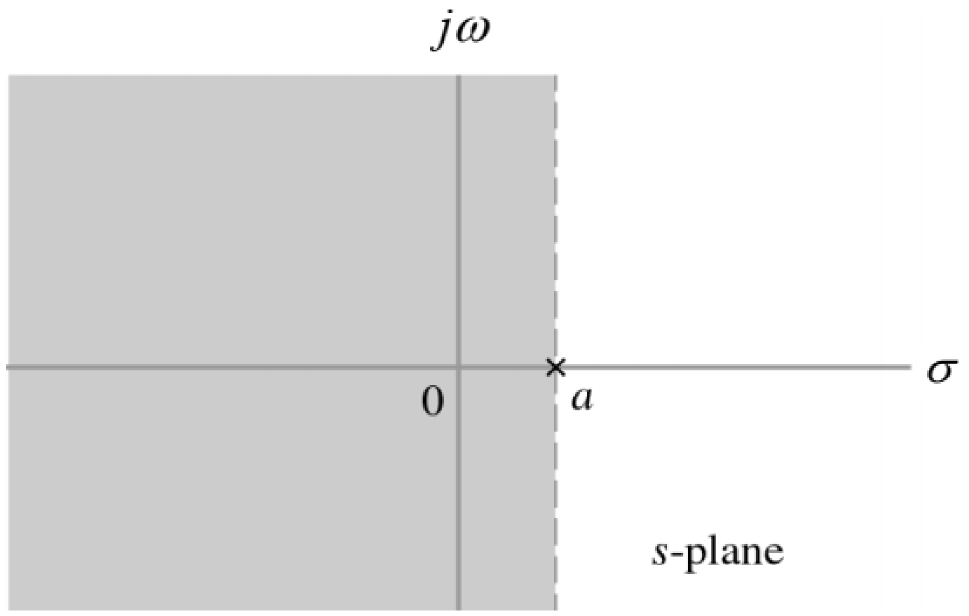
$$X(s) = \frac{-1}{\sigma + j\omega - a}e^{-(\sigma-a)t}e^{j\omega t}\Big|_{0}^{\infty}$$

if $\sigma > a$, then $e^{-(\sigma-a)t}$ goes to zero at $t \to \infty$

3. ROC: $\sigma > a$ or $\mathrm{Re}(s) > a$, and

$$X(s) = \frac{-1}{\sigma + j\omega - a}(0-1)$$

$$= \frac{1}{\sigma + j\omega - a} = \frac{1}{s-a}, \quad \mathrm{Re}(s) > a$$



Determine the Laplace transform of $y(t) = -e^{at}u(-t)$, and depict the ROC and locations of poles and zeros in the *s*-plane. Assume that a is real.

<Sol.>

1. First find the LT of *y*(*t*): $Y(s) = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0} -e^{-(s-a)t}dt = \frac{1}{s-a}e^{-(s-a)}\Big|_{-\infty}^{0}$

2. To evaluate the limit value, we use $s = \sigma + j\omega$ to re-write *Y*(*s*):

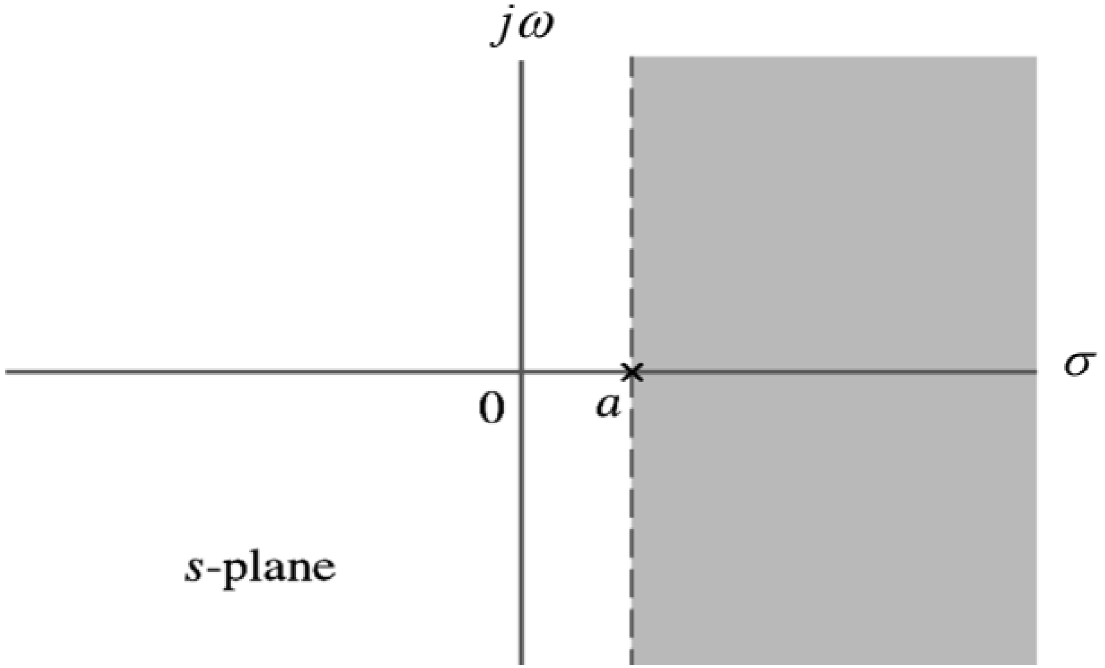
$$Y(s) = \frac{1}{\sigma + j\omega - a}e^{-(\sigma-a)t}e^{j\omega t}\Big|_{-\infty}^{0}$$

⇒ if $\sigma<a$, then $e^{-(\sigma-a)t}$ goes to zero at $t \to -\infty$

3. ROC: $\sigma<a$ or $\text{Re}(s)<a$, and

$$Y(s) = \frac{1}{\sigma + j\omega - a}(1-0)$$

$$= \frac{1}{\sigma + j\omega - a} = \frac{1}{s-a}, \quad \text{Re}(s) < a$$



Find the Laplace transform of $x(t)=\frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right)$

<Sol.>

1. We know from Ex. 6.1 that $e^{-3t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+3}$ with ROC $\mathrm{Re}(s)>-3$

2. The time-shift property implies that

$$e^{-3(t-2)}u(t-2) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+3}e^{-2s} \quad \text{with} \quad \text{ROC} \quad \mathrm{Re}(s)>-3$$

3. Apply the time-differentiation property twice, we obtain

$$x(t)=\frac{d^2}{dt^2}\left(e^{-3(t-2)}u(t-2)\right) \stackrel{\mathscr{L}}{\longleftrightarrow} X(s)=\frac{s^2}{s+3}e^{-2s} \quad \text{with} \quad \text{ROC} \quad \mathrm{Re}(s)>-3$$

Properties of U.L.T.:

The properties of U.L.T. and B.L.T are same except:

- **Time Shifting Property**

If x(t)⟷X(s)

$x(t-t_0) \leftrightarrow e^{-st_0}X(s)$ B.L.T

$x(t-t_0)u(t-t_0) \leftrightarrow e^{-st_0}X(s)$ U.L.T

- **Differentiation property**

If x(t) ⟷ X(s)

then $\frac{dx(t)}{dt} \leftrightarrow sX(s)$ B.L.T

$\frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0^-)$ U.L.T

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^nx(t)}{dt^n} \leftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}x'(0^-) \ldots\ldots\ldots\ldots\ldots\ldots - x^{n-1}(0^-)$$

- Note that ULT and LT are equivalent for signals that are causal.

$e^{at}u(t) \xleftrightarrow{ULT} \frac{1}{s-a}$ equivalent to $e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a}$ with ROC $\text{Re}\{s\} > a$

Since one-sided, do not specify ROC

Find the unilateral Laplace transform of $x(t) = \left(-e^{3t}u(t)\right) * \left(tu(t)\right)$

<Sol.>

Since $-e^{3t}u(t) \xleftrightarrow{\mathscr{L}_u} \dfrac{-1}{s-3}$ and $u(t) \xleftrightarrow{\mathscr{L}_u} \dfrac{1}{s}$

Apply the *s*-domain differentiation property, we have $tu(t) \xleftrightarrow{\mathscr{L}_u} 1/s^2$

Now, from the convolution property, we obtain

$$x(t) = \left(e^{3t}u(t)\right) * \left(tu(t)\right) \xleftrightarrow{\mathscr{L}_u} X(s) = \frac{-1}{(s-3)s^2}$$