GAME THEORY

Basic terms:

→ **Player:** An opponent is refered as player.

#. **Strategies:-** Each player has a number of choices, these are called strategies.

#. **Outcome (Payoff):** Outcome of a game when different alternatives are adopted by the competing players, the gains or losses are called pay-off or outcome.

#. **Two person zero sum game:-** When only two players are involved in the game and the gains made by one player are equal to loss of the other, it is called two person zero sum game.

This may be the case when just two players in the game are assuming that there are only two types of beverages, tea and coffee. Any market share gained by the tea is equal to loss of the market share of coffee. Since the sum of gains and losses is zero. This situation is called two person zero sum game.

n-person game :- A game in which n-person are involved is called n-person game.

→ **Pay-offs matrix** :- When gain and losses which result from different action of the competitors are represented in form of matrix table, is known as Pay-offs matrix.

Decision of a game :- In Game theory, the decision criterion of optimallity is adopted i.e. a player who wants to maximized is outcome, maxmin is used and who wants to minimize his outcome, minmax is used.

#. **Pure Strategy** :- If the player select the same strategy each time, then it is referred as pure strategy.

#). **Mixed strategy:-** When the players use a combination of strategies and each player always kept guessing as to which course of action is to be selected by the other player at a particular occasion then this is known as mixed strategy.

#). **Maximin Criterion:** → The player who is maximizing his outcome or payoff finds out his minimum gains from each strategy and selects the maximum value of these minimum gains.

#). **Minimax Criterion:-** The player who is minimizing his outcome or payoff finds out his maximum losses from each strategy and select the minimum value of these losses.

Example → Consider the following pay off matrix to illustrate the concept of saddle points.

$$\begin{array}{ccc}
 & \text{Player B} & \\
 & \begin{array}{cc} B_1 & B_2 \end{array} & \text{Row minima.} \\
\begin{array}{c} \text{Player} \\ A \end{array} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{cc} 100 & 80 \\ 150 & 180 \\ 180 & 200 \end{array}\right] & \begin{array}{c} 80 \\ 150 \\ \underset{\uparrow \atop \text{maxmin}}{\boxed{180}} \end{array} \\
\text{Column maxima} & \begin{array}{cc} \underset{\uparrow \atop \text{minimax.}}{\boxed{180}} & 200 \end{array} &
\end{array}$$

$\because$ maximin = minimax = 180

Hence saddle point exist as (3, 1).

Hence the optimal strategy for A is A_3 and for B is B_1.

The value of the game is 180 for A and −180 for B.

Player B by selecting second strategy B_2 is minimizing his maximum loss, Player B's selection is called **Minimax strategy.**

→ **Value of Game** → It is expected payoff of the game when all the players of the game follow their optimum (best) strategies. It is denoted by v.

#. **Fair game** :→ The game whoes value is zero is called fair game.

Unfair game :— The game whoes value is non-zero called unfair games

Saddle points :— It is the position in the pay-off matrix of a game where the maximum of Row minima is equal to the minimum of column maxima.

Example :- Let us consider a two person zero sum game involving maximum player A and assuming player B. The strategies available to player A are A_1, A_2, A_3 and to the player B are B_1 and B_2. The pay-off matrix is given below

Player B

$$\begin{array}{llccc}
 & & B_1 & B_2 & \text{Row minima} \\
\text{player A} & A_1 & 12 & 4 & 4 \\
 & A_2 & 10 & 6 & 6 \\
 & A_3 & 8 & 10 & \textcircled{8} \\
 & & & & \uparrow \\
\text{Column maxima} & & 12 & \textcircled{10} & \text{maximin} \\
 & & & \uparrow & \\
 & & & \text{minimax.} &
\end{array}$$

Here, Player A by selecting strategy A_3 is maximizing his minimum gains. Player A's selection is called maximin strategy.

Example Solve the following two person zero sum game

$$\begin{array}{ccc} & & \text{Player } B \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{ccc} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{array}\right] \end{array}$$

Solution : Since the given pay-off matrix is

$$\begin{array}{cccc} & & B & \\ & & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{Row minima} \\ A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{ccc} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{array}\right] & \begin{array}{l} 2 \\ \boxed{5} - \text{maximin} \\ -7 \end{array} \\ \begin{array}{c}\text{column} \\ \text{maxima}\end{array} & & \begin{array}{ccc} 15 & \boxed{5} & 7 \\ & \uparrow & \\ & \text{minimax.} & \end{array} & \end{array}$$

$\therefore$ minimax = maximin = 5.

Hence saddle point exist as (2, 2).

Hence optimal strategy for player A is A_2 and for player B is B_2. Value of the game is 5 for player A and -5 for player B.

Example 3 : Determine the range of value of p and q that will make the payoff element a_{22}. a saddle point for game whoes payoff matrix is given below -

$$\begin{array}{cc|ccc|} & & & \text{Player B.} & \\ & & B_1 & B_2 & B_3 \\ & A_1 & 2 & 4 & 5 \\ \text{Player A} & A_2 & 10 & 7 & q \\ & A_3 & 4 & p & 8 \end{array}$$

Solution : First of all ignore the values of p and q and determine the maximin and minimax values of payoff matrix.

Here the pay-off matrix is

$$\begin{array}{cccccc} & & B & & & \\ & B_1 & B_2 & B_3 & & \underline{\text{Row minima.}} \\ A_1 & 2 & 4 & 5 & & 2 \\ A_2 & 10 & 7 & q & & \textcircled{7} \leftarrow \text{maximin} \\ A_3 & 4 & p & 8 & & 4 \\ \text{Column maxima} & 10 & \textcircled{7} & 8 & & \\ & & \uparrow & & & \\ & & \text{minimax.} & & & \end{array}$$

$\because$ maximin value = minimax value = 7

Hence there exist a saddle point at position (2, 2).

This impose the condition on q as $q \geq 7$ and on p as $p \leq 7$.

Hence required range of values of p and q is

$$\boxed{p \leq 7, \quad q \geq 7}$$

Notes :

1. Saddle point may or may not exist in a game.

2. If there are more than one saddle point, then the problem has more than one solution.

3. The value of the game may be (+) ve or (-) ve.

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4. A game is said to be strictly determinable if maximin value = minimax value, otherwise game is said to be non-strictly determinable.

Exercise

Q. Determine which of the following two person zero sum games are strictly determinable and also find the value of game

i).

$$\begin{array}{ccc} & & \text{Player B.} \\ & B_1 & B_2 \\ \text{Player A} \quad A_1 & -5 & 2 \\ A_2 & -7 & -4 \end{array}$$

ii).

$$\begin{array}{cccc} & & Ⓑ & \\ & B_1 & B_2 & B_3 \\ Ⓐ \quad A_1 & 10 & 6 & 8 \\ A_2 & 8 & 2 & 4 \end{array}$$

iii). For what value of λ, the game with following payoff matrix is strictly determinable?

$$\begin{array}{cc} & Ⓑ \\ Ⓐ & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix} \end{array}$$

Q. Solve the following two person zero sum games :

a). Player B.

$$\text{Player A} \begin{bmatrix} -2 & 12 & -4 \\ 1 & 4 & 8 \\ -5 & 2 & 3 \end{bmatrix}$$

b). Player B

$$\text{player A} \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

c). Player B

$$\text{player A} \begin{bmatrix} 7 & 8 & 4 & 6 & 8 \\ -8 & 6 & 1 & 9 & 6 \end{bmatrix}$$

#). Games without saddle points (Mixed strategies):

Consider a 2x2 two person zero sum game without any saddle point having the pay off matrix

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$$

and $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$

where,

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \quad p_1 + p_2 = 1$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}, \quad q_1 + q_2 = 1$$

and the value of game is

$$v = \frac{a_{11} a_{22} - a_{12} a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}.$$

Example: Solve the following pay off matrix, determine the optimal strategies and the value of game

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution: consider the pay-off matrix be

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

then we get

$a_{11} = 5$, $a_{12} = 1$, $a_{21} = 3$

$a_{22} = 4$

the optimal mixed strategies

$$S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix} \quad \text{and} \quad S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

where,

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{4 - 3}{(4+5) - (1+3)} = \frac{1}{5}$$

$\because \quad P_1 + P_2 = 1$

$\Rightarrow \quad P_2 = 1 - P_1 = 1 - \frac{1}{5} = \frac{4}{5}.$

Again, $\quad q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{4-1}{(5+4)-(1+3)}$

$= \frac{3}{5}.$

$\Rightarrow \boxed{q_1 = \frac{3}{5}.}$

$\because \quad q_1 + q_2 = 1$

$\Rightarrow q_2 = 1 - q_1 = 1 - \frac{3}{5} = \frac{2}{5}.$

Hence optimum mixed strategies for player A is $S_A = \left(\frac{1}{5}, \frac{4}{5}\right)$ and for player B is $S_B = \left(\frac{3}{5}, \frac{2}{5}\right)$.

$\because$ value of game $= \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$

$= \frac{(5\times4) - (1\times3)}{(5+4) - (1+3)} = \frac{17}{5}$

$\Rightarrow \boxed{v = 17/5}$

Exercise

Solve the following games and determine the value of game

a).

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

b).

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$$

c).

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & 1 \\ 2 & 3. \end{bmatrix} \end{array}$$

#. Dominance property :→.

The general rules for dominance are –

i). If all the elements of a row say k^{th} row are less than or equal to the corresponding elements of any other row, say r^{th} row then k^{th} row is dominated by the r^{th} row.

ii). If all the elements of a column say k^{th} column are greater than or equal to the corresponding elements of any other columns say r^{th} column, then k^{th} column is dominated by the r^{th} column.

iii). Dominated rows and columns may be deleted to reduce size of pay off matrix as the optimal strategies will remain unaffacted.

iv). If some linear combinations of some rows dominate i^{th} row, then i^{th} row will be deleted. Similar arguments follow for column.

Example. Solve the following games by dominance property whoes pay-off matrix is given below –

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & 2 & 4 & 0 \\ 2 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix} \end{array}$$

Solution: Here the given payoff matrix is

$$\begin{array}{cccccc} & & A & & & \\ & & & & & \text{Row minima} \\ & \lceil\ 3 & 2 & 4 & 0\ \rceil & 0 \\ & \ 2 & 4 & 2 & 4\ & 2 \\ A & \ 4 & 2 & 4 & 0\ & 0 \\ & \lfloor\ 0 & 4 & 0 & 8\ \rfloor & 0 \\ \text{Column maxima} & 4 & 4 & 4 & 8 & \end{array}$$

$\because$ maximin value $= 0$

minimax value $= 4$

$\Rightarrow$ maximin value $\neq$ minimax value.

$\Rightarrow$ given game can't be solved by Saddle point method..

Step 2 : Here $R_1 \le R_3$, So neglecting R_1 the pay-off matrix reduces to the form.

$$\begin{bmatrix} 2 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$$

Step 3: Here $C_1 \geq C_3$. So neglecting C_1 the pay-off matrix reduces to the form

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

Step 4: Here $C_2 + C_3 \geq C_1$, so neglecting C_1 we get

$$\begin{bmatrix} 2 & 4 \\ 4 & 0 \\ 0 & 8 \end{bmatrix}$$

Step 5: Again, $R_2 + R_3 \geq R_1$, So neglecting R_1, we get

$$\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$$

Now we can solve it by mixed strategy method.

Let the optimum strategy for A be S_A and for B be S_B, where

$$S_A = \begin{pmatrix} A_3 & A_4 \\ p_1 & p_2 \end{pmatrix}, \qquad p_1 + p_2 = 0$$

$$S_B = \begin{pmatrix} B_3 & B_4 \\ q_1 & q_2 \end{pmatrix} \qquad q_1 + q_2 = 0$$

Now, $$p_1 = \frac{8-0}{(4+8)-(0+0)} = \frac{8}{12} = \frac{2}{3}.$$

$$q_1 = \frac{8-0}{(4+8)-(0+0)} = \frac{8}{12} = \frac{2}{3}$$

$$\because \quad p_2 = 1 - p_1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{3} = \frac{1}{3}.$$

Hence optimal strategy for player A is

$$S_A = \begin{pmatrix} A_3 & A_4 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \text{ or } \left(0, 0, \frac{2}{3}, \frac{1}{3}\right).$$

Similary optimal strategy for player B is

$$S_B = \begin{pmatrix} B_3 & B_4 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \text{ or } \left(0,\ 0,\ \frac{2}{3},\ \frac{1}{3}\right).$$

Now, value of the game $= \dfrac{4\times 8 - 0\times 0}{(4+8)-(0+0)}$

$$= \frac{32}{12} = \frac{8}{3}.$$

$$\boxed{v = \frac{8}{3}}$$

Exercise :

Use the dominance property to solve the rectangular game with following pay-off :

a).

$$\begin{array}{cc|cccc} & & \multicolumn{4}{c}{\text{player B}} \\ & & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline & 1 & 18 & 4 & 6 & 4 \\ \text{player} & 2 & 6 & 2 & 13 & 7 \\ \text{A} & 3 & 11 & 5 & 17 & 3 \\ & 4 & 7 & 6 & 12 & 2 \end{array}$$

ⓑ.

Player B

$$\text{player A}\quad\begin{array}{c|ccccc} & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \hline 1 & 2 & 4 & 3 & 8 & 4 \\ 2 & 5 & 6 & 3 & 7 & 4 \\ 3 & 6 & 7 & 9 & 8 & 7 \\ 4 & 4 & 2 & 8 & 4 & 3 \end{array}$$

ⓒ.

player B

$$\text{player A}\quad\begin{bmatrix} 10 & 5 & -2 \\ 13 & 12 & 15 \\ 16 & 14 & 10 \end{bmatrix}$$

#). Graphical method for 2xn OR mx2 Games :-

Consider the following 2xn games

$$
\begin{array}{c|cccc}
 & B_1 & B_2 & \cdots\cdots & B_n \\
\hline
A_1 & a_{11} & a_{12} & \cdots\cdots & a_{1n} \\
A_2 & a_{21} & a_{22} & \cdots\cdots & a_{2n}
\end{array}
$$

Ⓑ above the columns, Ⓐ beside the rows.

Let the mixed strategy for player A is given by $S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix}$ where $P_1 + P_2 = 1$.

Now for each of pure strategies available to B, expected pay-off for player A would be as follows

B's pure strategy	A's expected pay off $E(P)$
B_1	$E_1(P) = a_{11}P_1 + a_{21}P_2$
B_2	$E_2(P_2) = a_{12}P_1 + a_{22}P_2$
$\vdots$	$\vdots$
B_n	$E_n(P) = a_{1n}P_1 + a_{2n}P_2$

The player B would like to choose that pure move Bj against S_A for which $E_j(P)$ is a minimum for $j = 1, 2, ---- n$.

Let us denote this minimum expected pay off for A by

$$\gamma = \text{Min}\,(E_j(P)) \quad , \quad j = 1, 2, --- n.$$

The objective of the player A is two select P_1 and P_2 in such a way that j is as Large as possible. This may be done by plotting straight lines.

$$E_j(P) = a_{1j} P_1 - a_{2j} P_2, \quad j = 1, 2, --- n.$$

The height point on the lower boundary of these lines will give the maximum value among the minimum expected pay-off on the lower boundary and the optimum value of P_1 and P_2.

Now ' two strategies of the player B corresponding to those lines which passes through the maximum point can be determined.

Example : Solve the following games by graphical method.

Player B

$$\text{Player A} \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix}$$

Solution :

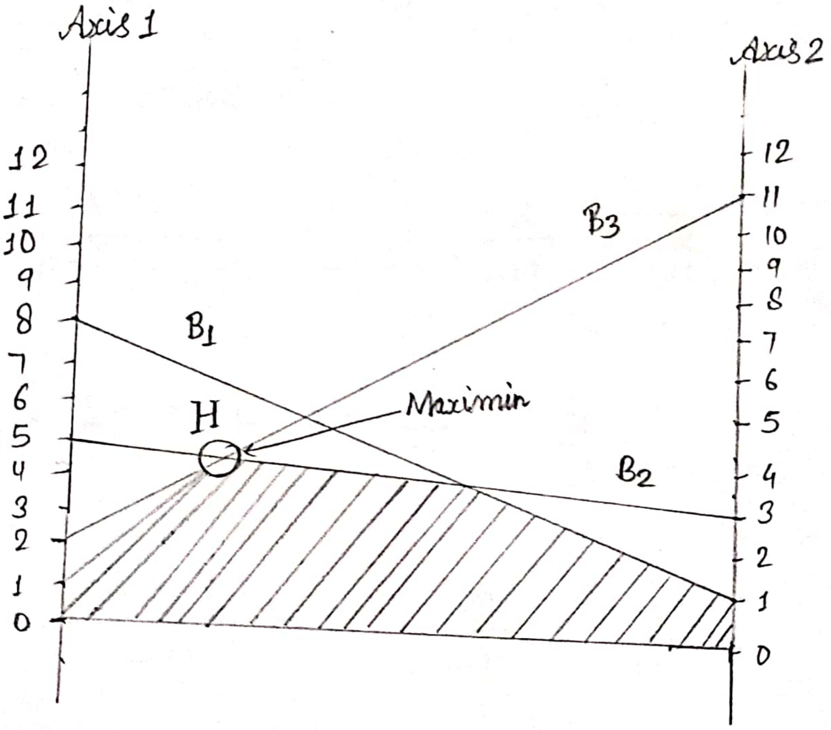
Since the problem does not posses any saddle point let the player A play the mixed strategy.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}, \qquad p_1 + p_2 = 1$$

The A's expected pay-off against B's pure strategy is given by-

B's pure strategy	A's expected payoff
B_1	$E_1(P_1) = P_1 + 8(1-P_1) = 7P_1+8$
B_2	$E_2(P_1) = 3P_1 + 5(1-P_1) = -2P_1+5$
B_3	$E_3(P_1) = 11P_1 + 2\cdot(1-P_1) = 9P_1+2$

Now these expected payoff equations are the plotted as function of P_1 which shows payoff of each column respresented as points on two vertical lines (axis) 1 and 2 of unit distance apart.



From above, the solution to the original 2×3 game reduce to.

$$\begin{array}{ccc} & B_2 & B_3 \\ A_1 & 3 & 11 \\ A_2 & 5 & 2 \end{array}$$

The optimum strategy for A and B are given by -

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}, \quad P_1 + P_2 = 1$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{bmatrix}, \quad q_1 + q_2 = 1$$

$$\because \quad P_1 = \frac{2-5}{(3+2)-(11+5)} = \frac{-3}{-11} = \frac{3}{11}.$$

$$\Rightarrow \quad P_2 = 1 - P_1 = 1 - \frac{3}{11} = \frac{8}{11}.$$

$$\because \quad q_1 = \frac{2-11}{-11} = \frac{-9}{-11} = \frac{9}{11}$$

then $q_2 = 1 - q_1 = 1 - \frac{9}{11} = \frac{2}{11}.$

Hence optimal strategy for player A is $\begin{pmatrix} A_1 & A_2 \\ 3/11 & 8/11 \end{pmatrix}$ or $\left(\frac{3}{11}, \frac{8}{11}\right)$. and optimal strategy for player B is $\begin{pmatrix} B_1 & B_2 & B_3 \\ 0 & \frac{9}{11} & \frac{2}{11} \end{pmatrix}$ or $\left(0, \frac{9}{11}, \frac{2}{11}\right)$.

value of the game is

$$v = \frac{6-55}{5-16} = \frac{49}{11} .$$

Example :→ Solve the game whoes pay-off matrix is given below.

$$\begin{array}{c|cccc} & & B & & \\ & B_1 & B_2 & B_3 & B_4 \\ A_1 & 4 & -2 & 3 & -1 \\ A_2 & -1 & 2 & 0 & 1 \\ A_3 & -2 & 1 & -2 & 0 \end{array}$$

Solution : Here, $R_3 \le R_2$. Hence R_3 is dominated by R_2, we get reduced pay-off matrix as-

$$\begin{array}{cc|cccc} & & & B & & \\ & & b_1 & B_2 & B_3 & B_4 \\ A & A_1 & 4 & -2 & 3 & -1 \\ & A_2 & -1 & 2 & 0 & 3 \end{array}$$

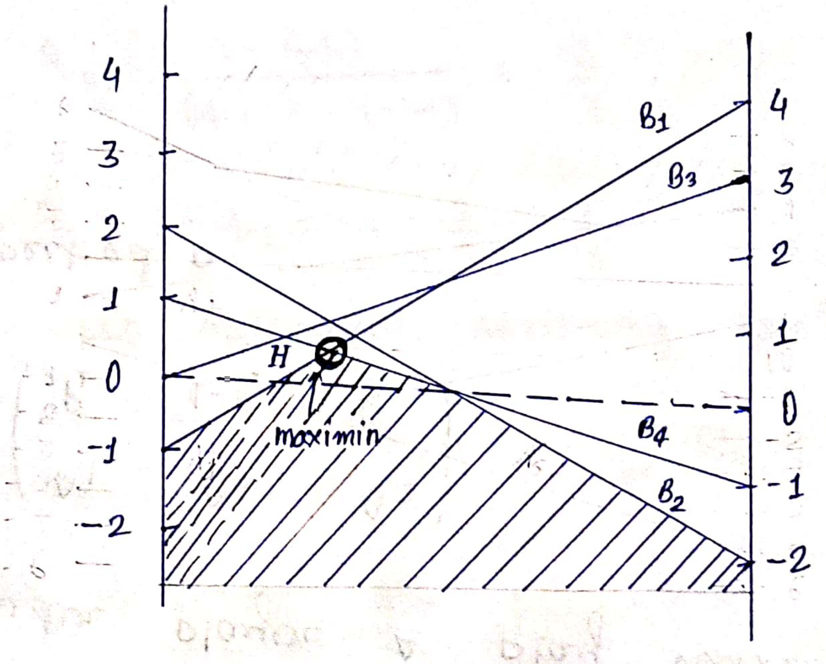
Consider player A plays mixed strategy

$$S_A = \begin{pmatrix} A_1 & A_2 & A_3 \\ P_1 & P_2 & 0 \end{pmatrix}, \quad \& \; P_1 + P_2 = 1.$$

Now A's expected payoff against B's pure moves (strategy)

(B's pure strategy)	A's expected pay-off
B_1	$E_1(P_1) = 4P_1 + (1-P_1) = 3P_1 + 1$
B_2	$E_2(P_2) = -2P_1 + 2.(1-P_1) = -4P_1 + 2$
B_3	$E_3(P_1) = 3 \cdot P_1 + 0.(1-P_1) = 3P_1$
B_4	$E_4(P_1) = -P_1 + (1-P_1) = -2P_1 + 1.$

Now these are represented by graphically as



Therefore we get the reduced pay-off matrix as -

$$\begin{array}{cc|cc} & & \text{Ⓑ} & \\ & & B_1 & B_4 \\ \text{Ⓐ} & A_1 & 4 & -1 \\ & A_2 & -1 & 1 \end{array}$$

Now the optimum strategy for A and B is given by -

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \qquad p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_4 \\ q_1 & q_2 \end{pmatrix}, \qquad q_1 + q_2 = 1.$$

$$\therefore \quad p_1 = \frac{1-(-1)}{(4+1)-(-1-1)} = \frac{2}{7}$$

then $p_2 = 1 - p_1 = 1 - \frac{2}{7} = \frac{5}{7}$.

Again, $q_1 = \frac{1-(-1)}{(4+1)-(-1-1)} = \frac{2}{7}$.

$$q_2 = 1 - q_1 = 1 - \frac{2}{7} = \frac{5}{7}.$$

Hence we get optimum strategy for player A is $\begin{pmatrix} A_1 & A_2 & A_3 \\ 2/7 & 5/7 & 0 \end{pmatrix}$ or $(2/7, 5/7, 0)$.

✓ optimum strategy for player B is

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 2/7 & 0 & 0 & 5/7 \end{bmatrix}. \quad \text{or}$$

$$\left(\frac{2}{7}, 0, 0, \frac{5}{7} \right).$$

Value of game –

$$v = \frac{4 \times 1 - (-1).(-1)}{(4+1) - (-1-1)} = \frac{3}{7}.$$

$$\boxed{v = 3/7}$$

Exercise :

- Solve the following problem by graphical method -

a). Player B

Player A
$$\begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix}$$

b). B

A
$$\begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}$$

c). Player B.

Player A
$$\begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix}$$

Imp. #>. Solution of a game using LPP :-

Consider the following game

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 2 & -3 & 4 \\ A_2 & -3 & 4 & -5 \\ A_3 & 4 & -5 & 6 \end{array}$$

Solution:

Notes: if some of the element in payoff matrix is negative then choose the smallest negative element. Adding the same amount of positive quantity in all elements of pay-off matrix;

Here, we get -5 is the smallest negative element in the payoff matrix. Hence by adding 5 to each element of the payoff matrix, we get

$$\begin{bmatrix} 7 & 2 & 9 \\ 2 & 9 & 0 \\ 9 & 0 & 11 \end{bmatrix}.$$

Now, we have

$$\begin{bmatrix} 7 & 2 & 9 \\ 2 & 9 & 0 \\ 9 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

LPP is–

minimize $Z = x_1 + x_2 + x_3$

s.t.

$7x_1 + 2x_2 + 9x_3 \geq 1 \quad \dashrightarrow y_1$

$2x_1 + 9x_2 \geq 1 \quad \dashrightarrow y_2$

$9x_1 + 11x_3 \geq 1 \quad \dashrightarrow y_3$

and $x_1, x_2, x_3 \geq 0$.

Maximize $Z_w = y_1 + y_2 + y_3$

s.t.

$7y_1 + 2y_2 + 9y_1 \leq 1$

$2y_1 + 9y_2 \leq 1$

$9y_1 + 11y_3 \leq 1$

where $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

As we can solve any one program and obtain the solution of both.

Here we solve second problem.

Standard form -

$$\text{Max. } Z_w = y_1 + y_2 + y_3 + 0 \cdot y_4 + 0 \cdot y_5 + 0 \cdot y_6$$

s.t.

$$7y_1 + 2y_2 + 9y_3 + y_4 = 1$$

$$2y_1 + 9y_2 + y_5 = 1$$

$$9y_1 + 11y_3 + y_6 = 1$$

7	2	9	x_1
2	9	0	x_2
9	0	11	x_3

Now LPP is:

(*i*) Minimize $z = x_1 + x_2 + x_3$

subject to

$$7x_1 + 2x_2 + 9x_3 \geq 1$$
$$2x_1 + 9x_2 \geq 1$$
$$9x_1 + 11x_3 \geq 1$$

and $\quad x_1, x_2, x_3 \geq 0$

and

$$\text{Maximize } \omega = y_1 + y_2 + y_3$$

Subject to

$$7y_1 + 2y_2 + 9y_3 \leq 1$$
$$2y_1 + 9y_2 \leq 1$$
$$9y_1 + 11y_3 \leq 1$$

and $\quad y_1, y_2, y_3 \geq 0$

As we can solve any one program and obtain the solution to both, we solve the 2nd problem. The standard form is

$$\text{Maximize } w = y_1 + y_2 + y_3 + 0y_4 + 0y_5 + 0y_6$$

Subject to

$$7y_1 + 2y_2 + 9y_3 + y_4 = 1$$
$$2y_1 + 9y_2 + \qquad + y_5 = 1$$
$$9y_1 + 11y_3 + y_6 = 1$$

And $y_i's \geq 0, i = 1, 2, 3, 4, 5, 6.$

where y_4, y_5 and y_6 are slack variables.

Basic Variables		$c_j \rightarrow$	1	1	1	0	0	0	Min Ratio $(X_B/Y_k) Y_k > 0$
	C_B	X_B	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	
y_4	0	1	7	2	9	1	0	0	1/7
y_5	0	1	2	9	0	0	1	0	½
y_6	0	1	$\boxed{9}$	0	11	0	0	1	1/9←
$y_1 = y_2 = y_3 = 0$	$z = 0$		–1↑	–1	–1	0	0	0↓	$\leftarrow \Delta_j = z_j - c_j$
y_4	0	2/9	0	2	4/9	1	0	–7/9	1/9
y_5	0	7/9	0	$\boxed{9}$	–22/9	0	1	–2/9	$\frac{7/9}{9} = \frac{7}{81} \leftarrow$
y_1	1	1/9	1	0	11/9	0	0	1/9	–
$y_3 = y_2 = y_6 = 0$	$z = 1/9$		0	–1↑	2/9	0	0↓	1/9	$\leftarrow \Delta_j = z_j - c_j$

y_4	0	4/81	0	0	$\boxed{80/81}$	1	$-2/9$	$-59/81$		4/81←
y_2	1	7/81	0	1	$-22/81$	0	$1/9$	$-2/81$		—
y_1	1	1/9	1	0	$11/9$	0	0	$1/9$		1/11
$y_3 = y_4 = y_6 = 0$	$z = 16/81$ 0		0	–	0↓ 4/81↑	1/9	7/81←	$\Delta j = z_j - c_j$		
y_3	1	1/20	0	0	1	$81/80$	$-9/40$	$-59/80$		
y_2	1	1/10	0	1	0	$11/40$	$1/20$	$-9/40$		
y_1	1	1/20	1	0	0	$-99/80$	$11/40$	$81/80$		
$y_4 = y_5 = y_6 = 0$	$z^* = 1/5$		0	0	0	1/20	1/10	1/20	← $\Delta j = z_j - c_j$	

The value of the game is $v = \dfrac{1}{z^*} - 5 = \dfrac{1}{1/5} - 5 = 0$

The optimal strategies for the player B are

$$q_1 = \frac{1/20}{z^*} = \frac{1/20}{1/5} = \frac{1}{4}; \qquad q_2 = \frac{1/10}{z^*} = \frac{1/10}{1/5} = \frac{1}{2};$$

$$q_3 = \frac{1/20}{z^*} = \frac{1/20}{1/5} = \frac{1}{4}$$

Since the linear program for the player A's expected gain is the dual of the linear program given above, the optimal solution to the player A can be obtained from the last row of the final optimal table as given above. Thus

$$p_1 = \frac{1/20}{1/5} = \frac{1}{4}, \; p_2 = \frac{1/10}{1/5} = \frac{1}{2}$$

and $$p_3 = \frac{1/20}{1/5} = \frac{1}{4}$$