# Numerical Methods for Scientific Computations MMS - 203

***By - Suchi Bhatt***

**Mathematics and Scientific Computing**

***Madan Mohan Malaviya University of Technology Gorakhpur*** ***(UP State Govt. University)***

**Email: 1995suchibhatt@gmail.com**

## UNIT- I

# Unit –1

# Solution of algebraic and transcendental equations

The equations of the form $f(x) = 0$ where $f(x)$ is purely a polynomial in $x$ e.g. $x^6 - x^4 - x^3 - 1 = 0$ is called an **algebraic equation.** But, if $f(x)$ involves trigonometrical, arithmetic or exponential terms in it, then it is called **transcendental equation.**
E.g. $xe^x - 2 = 0$ and $x \log_{10}x - 1.2 = 0$.

**Basic Properties and Observations of an Algebraic Equation and its Roots:**

(*i*) If $f(x)$ is exactly divisible by $(x - \alpha)$, then $\alpha$ is a root of $f(x)$.

(*ii*) Every algebraic equation of $n$th degree has $n$ and only $n$ real or imaginary roots. Conversely, if $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ be the $n$ roots of the $n$th degree equation $f(x) = 0$, then

$$f(x) = A(x - \alpha_1)(x - a_2)\ldots\ldots\ldots(x - \alpha_n).$$

(*iii*) If $f(x)$ is continuous in the interval $[a, b]$ and $f(a)$, $f(b)$ have different signs, then the equation have at least one root between $x = a$ and $x = b$ (oftenly known as *Intemediate Value Theorem.*)

(*iv*) In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e. if $\alpha + i\beta$ is root of $f(x) = 0$, then $\alpha - i\beta$ is also its root. Similarly, if $\alpha + \sqrt{\beta}$ is an irrational root of $f(x) = 0$, then $\alpha - \sqrt{\beta}$ is also its roots.

# Fixed point iteration method

Consider the equation $f(x) = 0$ ... (1)

We rewrite the equation in the form

$$x = \phi(x) \quad \text{.... (2)}$$

Let us draw two curves

$$y = x \text{ and } y = \phi(x)$$

The point of intersection of two curves is the root of (1).

Let $x = x_0$ be an initial approximate root, then first approximation $x_1$ is found by

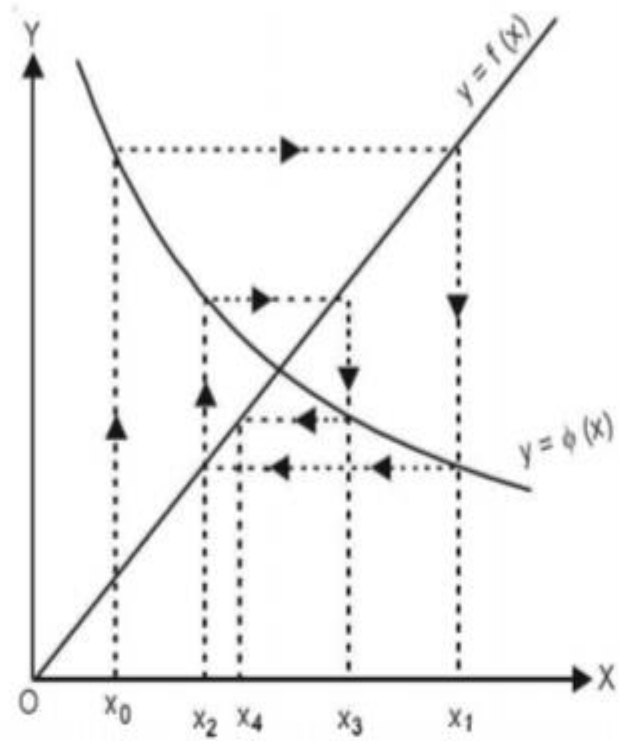
$$x_1 = \phi(x_0)$$

Now taking $x_1$ as initial value, $x_2$ second approximation is given by

$$x_2 = \phi(x_1) \quad \text{and so on.}$$

$$x_{n+1} = \phi(x_n)$$

This is also known as successive approximation method.



# Test for Convergence

For convergence it is convenient to identify an interval that contains the root and for which $\phi'(x)$ has small magnitude.

$$x = \phi(x)$$

$$\alpha = \phi(\alpha) \quad \ldots (1)$$

$$x_n = f(x_{n-1}) \quad \ldots (2)$$

Subtracting (1) from (2), we have

$$x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \quad \ldots (3)$$

By Mean value theorem

$$\frac{\phi(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi), \text{ where } x_{n-1} < \xi < \alpha \quad \ldots (4)$$

Substituting the value of $\phi(x_{n-1}) - \phi(\alpha)$ from (4) in (3), we get

$$x_n - \alpha = (x_{n-i} - \alpha)\,\phi'(\xi)$$

$$|x_n - \alpha| \le k|\,x_{n-1} - \alpha| \qquad [\text{If } \phi'(x_i) \le k < 1 \text{ for all } i] \quad \ldots (5)$$

Similarly, $|\,x_{n-1} - \alpha| \le k\,|x_{n-2} - \alpha\,|$ ... (6)

Putting the value of $|x_{n-1} - \alpha|$ from (6) in (5), we have

$$|x_n - \alpha| \le k^2\,|x_{n-2} - \alpha\,|$$

..............................................

$$|x_n - \alpha| \le k^n |x_0 - \alpha|$$

$$|x_n - \alpha| = 0 \qquad \left[ \lim_{n \to \infty} k^n = 0 \right]$$

So, the approximation converges by this method.

**Note. 1.** The rate of convergence is more if the value of $\phi'(x)$ is smaller.

**2.** For real roots, the method is very useful.

**Remember.** The equation $f(x) = 0$ is written as $x = \phi(x)$.

This form $x = \phi(x)$ can be choosen in many ways. We have to choose $\phi(x)$ in such a way that initial approximation $x_0$ should satisfy the condition $|\phi'(x_0)| < 1$.

Then $x_0, x_1, x_2 \ldots\ldots\ldots\, x_n$ converge to the root $\alpha$ of the equation $f(x) = 0$.

# Problem -1

- **Apply fixed point iteration method to find the real root of $xe^x = 1$ correct to three decimals, assume initial approximation as $x_0 = 0.5$**

**Solution.**

The condition for the convergence of the iterative scheme is

$$|\phi'(x_k)| < 1.$$

Here $\quad xe^x = 1 \quad \Rightarrow x = e^{-x} \quad \dots (1)$

$\Rightarrow \quad \phi(x) = e^{-x}$

Putting $\quad x = 0.5$ in (1), we get

$$x_1 = e^{-0.5} = 0.6065$$

Again putting $\quad x = 0.6065$ in (1), we have

$$x_2 = e^{-0.6065} = 0.5453$$

Similarly putting the successive values of $x$ in (1), we get

$$x_3 = e^{-.5453} = 0.5797$$
$$x_4 = e^{-.5797} = 0.5601$$
$$x_5 = e^{-.5601} = 0.5712$$
$$x_6 = e^{-.5712} = 0.5648$$
$$x_7 = e^{-.5648} = 0.5685$$
$$x_8 = e^{-.5685} = 0.5664$$

$$\begin{cases} \phi'(x) = -e^{-x} \\ \phi'(0.5) = -e^{-0.5} \\ \qquad = -0.6065 \\ |\phi'(0.5)| < 1 \end{cases}$$

$$x_9 = e^{-.5664} = 0.5676$$

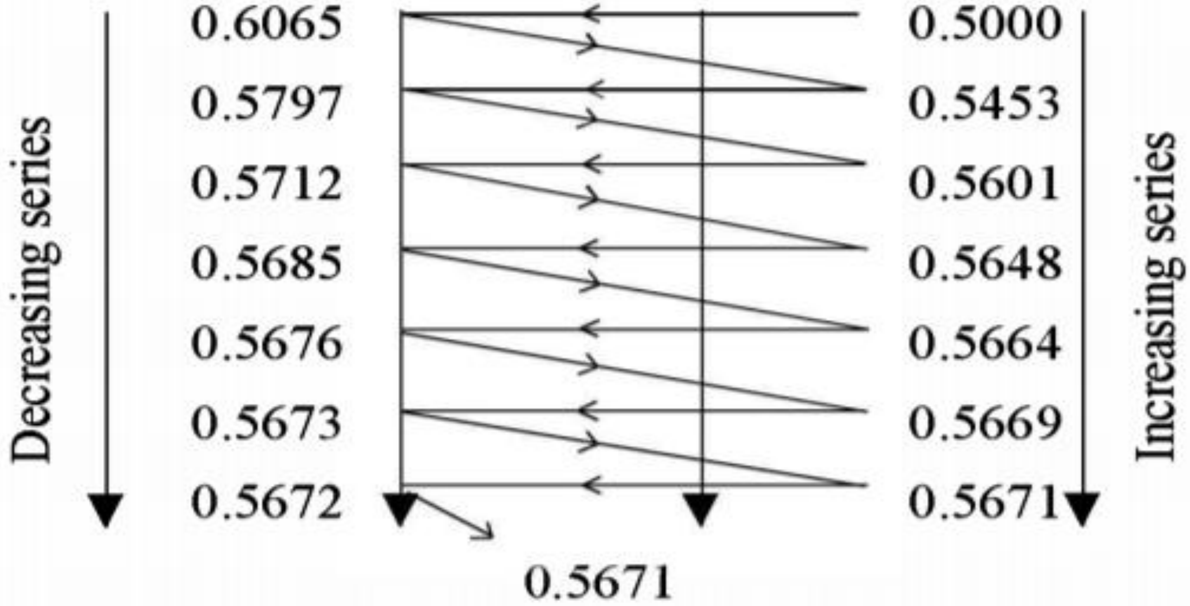
$$x_{10} = e^{-.5676} = 0.5669$$

$$x_{11} = e^{-.5669} = 0.5673$$

$$x_{12} = e^{-0.5673} = 0.5671$$

$$x_{13} = e^{-0.5671} = 0.5672$$

The above iterates are written to show the convergence of the iterates.



# Newton-Raphson method

Let $x_0$ be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_0 + h) = 0$

**To find *h***, we expand $f(x_0 + h)$ by Taylor's Series

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots \qquad [f(x_0 + h) = 0]$$

$$0 = f(x_0) + h f'(x_0) \qquad \text{[Neglecting the second and higher order derivative]}$$

$$h = -\frac{f(x_0)}{f'(x_0)}$$

But $x_1 = x_0 + h$

Putting the value of $h$, we get $\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1$ is better approximation than $x_0$. $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2$ is better approximation than $x_1$.

Successive approximations are $x_3, x_4, \ldots\ldots\ldots\, x_{n+1}$.

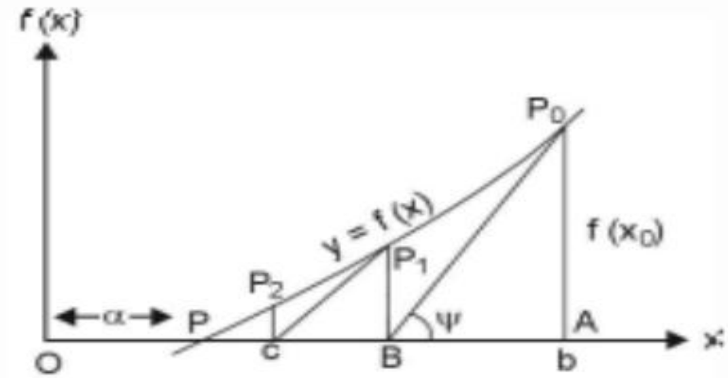
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

# Geometrical Interpretation

Let $P_0 P$ be a curve $y = f(x)$.

Slope of the tangent $P_0 B$ to the curve at the point $P_0\,(x_0, y_0) = f'(x_0)$.

Tangent $P_0 B$ cuts the $x$-axis at $B$ *i.e.* $(x_1, 0)$.




$$\begin{aligned} x_1 &= OB \\ &= OA - AB \\ &= x_0 - P_0 A \cot \Psi \\ &= x_0 - \frac{P_0 A}{\tan \psi} \quad \left[ AN = f(x_0),\ \frac{BA}{AN} = \cot \psi \right] \\ &= x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{(First approximation)} \end{aligned}$$

The tangent to the curve at $P_1$ (corresponding to $x_1$) cuts the axis at $C\,(x_2, 0)$.

Using $x_1$ as the starting point, then

Similarly $\quad x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Now $x_2$ is nearer to $\alpha$ than $x_1$ (second approximation).

The process can be repeated and the root $\alpha$ is approached very fast.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

# Convergence of Newton-Raphson Formula

By Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $$\phi(x) = x - \frac{f(x)}{f'(x)} = \frac{x f'(x) - f(x)}{f'(x)}$$

On differentiating both sides w.r.t '$x$', we get

$$\phi'(x) = \frac{f'(x)\left[f'(x) + x f''(x) - f'(x)\right] - \left[x f'(x) - f(x)\right] f''(x)}{\left[f'(x)\right]^2}$$

$$\Rightarrow \quad \phi'(x) = \frac{\left[f'(x)\right]^2 + x f'(x) f''(x) - \left[f'(x)\right]^2 - x f'(x) f''(x) + f(x) f''(x)}{\left[f'(x)\right]^2}$$

$$\Rightarrow \qquad \phi'(x) = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

For convergence, $|\phi'(x)|<1$

$$\frac{f(x).f''(x)}{[f'(x)]^2} < 1$$

$$f(x)\,f''(x) < [f'(x)]^2$$

# Rate of Convergence of Newton – Raphson Mathod

Let $x_n$ (approximate root) differs from the actual root $\alpha$ by a small quantity $h_n$.

So $$x_n = \alpha + h_n \quad \text{... (1)}$$

$$x_{n+1} = \alpha + h_{n+1} \quad \text{... (2)}$$

By Newton-Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{... (3)}$$

Putting the values of $x_{n+1}$ and $x_n$ from (1) and (2) in (3), we get

$$\alpha + h_{n+1} = \alpha + h_n - \frac{f(\alpha + h_n)}{f'(\alpha + h_n)} \Rightarrow h_{n+1} = h_n - \frac{f(\alpha + h_n)}{f'(\alpha + h_n)}$$

On expanding $f(\alpha + h_n)$ and $f'(\alpha + h_n)$ by Taylor's Series, we get

$$h_{n+1} = h_n - \frac{f(\alpha) + h_n f'(\alpha) + \frac{1}{2!} h_n^2 f''(\alpha) + .....}{f'(\alpha) + h_n f''(\alpha) + .....}$$

We know that $f(\alpha) = 0$, so

$$h_{n+1} = h_n - \frac{h_n f'(\alpha) + \frac{1}{2} h_n^2 f''(\alpha) + \ldots}{f'(\alpha) + h_n f''(\alpha) + \ldots\ldots}$$

$$= \frac{h_n f'(\alpha) + h_n^2 f''(\alpha) - h_n f'(\alpha) - \frac{1}{2} h_n^2 f''(\alpha) + \ldots}{f'(\alpha) + h_n f''(\alpha) + \ldots\ldots} = \frac{\frac{1}{2} h_n^2 f''(\alpha)}{f'(\alpha) + h_n f''(\alpha)}$$

$$h_{n+1} = h_n^2 \left( \frac{f''(\alpha)}{2 f'(\alpha)} \right) \text{ approximately} \qquad [f''(\alpha) \text{ neglected}]$$

$$h_{n+1} \propto h_n^2 \qquad \left( \frac{f''(\alpha)}{2 f'(\alpha)} \text{ constant} \right)$$

**1.** It means that subsequent error $h_{n+1}$ at each step is proportional to the square of the previous error $h_n$. So, the number of correct decimal is approximately doubled at each iteration if $\frac{f''(\alpha)}{2 f'(\alpha)}$ is not too large.

**2.** Convergence is of quadratic order *i.e.* P = 2.

# Problem-2

- **Using Newton-Raphson method, find the real root of $xlog_{10}x = 1.2$ correct to five decimal places.**

**Solution.** Let

$$f(x) = x\log_{10} x - 1.2$$

$$f(1) = -1.2 = -\text{ve},$$

$$f(2) = 2\log_{10} 2 - 1.2 = -0.59794 = -\text{ve}$$

and

$$f(3) = 3\log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ve}$$

$$f(2)\,.\,f(3) < 0$$

So, a root of $f(x) = 0$ lies between 2 and 3.

Let us take $x_0 = 2$.

Also,

$$f'(x) = \log_{10} x + x.\frac{1}{x}\log_{10} e = \log_{10} x + 0.43429$$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} = \frac{x_n \log_{10} x_n + 0.43429 x_n - x_n \log_{10} x_n + 1.2}{\log_{10} x_n + 0.43429}$$

$$= \frac{0.43429x_n + 1.2}{\log_{10} x_n + 0.43429} \quad \ldots (1)$$

Putting $x_0 = 2$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429}$$

$$= \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (1), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74065$$

$$x_4 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly, $x_3 = x_4$

Hence, the required root is 2.74065 correct to five decimal places. **Ans.**

# Solution of linear system of equations

## INTRODUCTION

We have already solved simultaneous equations of two or three unknowns. When the number of unknowns in simultaneous equations is large, then it becomes tedious to solve them by the known methods. Simultaneous equations of large number of unknowns are very important in the field of science and engineering. Now, we will use the following methods to solve such simultaneous equations.

**1. Direct method**

(*a*) Gauss elimination method

(*b*) Gauss-Jordan method

(*c*) Crouts method (Factorisation method)

**2. Iterative method**

(*a*) Jacobi method

(*b*) Gauss-seidel method

# Gauss elimination method

In this method the unknowns of equations below are eliminated and the system is reduced to an upper triangular system. The unknowns are obtained by back substitution.

Let a system of simultaneous equations in $n$ unknowns $x_1, x_2 \ldots\ldots\ldots x_n$ be

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1 \quad \ldots (1)$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2 \quad \ldots (2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b_n \quad \ldots (n)$$

**Method to solve the above equations**

**Step 1.** We eliminate $x_1$ from 2nd, 3rd ............$n$th equation with the help of the first equation

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + \ldots\ldots\ldots\ldots + a'_{2n}x_n = b_2'$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a'_{n2}x_2 + \ldots\ldots\ldots\ldots + a'_{nn}x_n = b_n'$$

**Step 2.** We again eliminate $x_2$ from 3rd, 4th..... $n$th equation with the help of second equation.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a'_{2n}x_n = b_2'$$

$$a''_{33}x_3 + \ \ldots\ldots\ldots\ldots + a''_{3n}x_n = b_3''$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a''_{n3}x_3 + \dots\dots\dots + a''_{nn}x_n = b''_n$$

In the third step we will eliminate $x_3$ and in fourth step $x_4$ and so on.

Finally the system of equations will be of the following form.

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\, a_{1n}x_n = b_1$$

$$a_{22}x_2 + \dots\dots\dots + a'_{2n}x_n = b'_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$c_{nn}x_n = d_n$$

The given system is reduced to the above form *i.e.* triangular form.

# Problem-3

**Example 1.** *Solve the following equations by using Gauss-elimination method :*

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 - 2x_3 &= -2 \\ x_1 - x_2 + x_3 &= 6 \end{aligned}$$

**Solution.** Third equation is written as first equation, the system becomes as

$$x_1 - x_2 + x_3 = 6 \quad \dots (1)$$

$$2x_1 + 4x_2 + x_3 = 3 \quad \dots (2)$$

$$3x_1 + 2x_2 - 2x_3 = -2 \quad \dots (3)$$

**Step 1.** Subtracting 2 (1) from (2), and 3 (1) from (3), we get

$$x_1 - x_2 + x_3 = 6$$

$$6x_2 - x_3 = -9 \quad \dots (4)$$

$$5x_2 - 5x_3 = -20 \quad \dots (5)$$

**Step 2.** Operate $\frac{6}{5}$ (5) – (4)

$$x_1 - x_2 + x_3 = 6 \quad \dots (1)$$

$$6x_2 - x_3 = -9 \quad \dots (6)$$

$$-5x_3 = -15 \quad \dots (7)$$

**Step 3.** Backward substitution

From (7), $x_3 = \frac{-15}{-5} = 3$

From (6), $6x_2 - 3 = -9 \Rightarrow 6x_2 = -6 \Rightarrow x_2 = -1$

From (1), $x_1 - (-1) + 3 = 6 \Rightarrow x_1 = 6 - 3 - 1 = 2$

Hence, $x_1 = 2,\ x_2 = -1,\ x_3 = 3$ **Ans.**

# Gauss Jordan method

This is modification of the Gauss elimination method.

By this method we eliminate unknowns not only from the equations below but also from the equations above. In this way the system is reduced to a diagonal matrix.

Finally each equation consists of only one unknown and thus, we get the solution. Here, the labour of backward substitution for finding the unknowns is saved.

Gauss-Jordan method is modification of Gauss elimination method.

**Example** *Apply Gauss-Jordan method to solve the equations :*

$$\begin{aligned} x+y+z &= 9 \\ 2x-3y+4z &= 13 \\ 3x+4y+5z &= 40 \end{aligned}$$

**Solution.** The following system of linear equations can be written in matrix form:

By using Gauss Jordan method we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix} R_3 \to R_3 + \frac{1}{5} R_2 \quad \Rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{5} \\ 0 & -5 & 2 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 12 \end{bmatrix} R_1 \to R_1 + \frac{1}{5} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -15 \\ 12 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{7}{12} R_3 \\ R_2 \to R_2 - \frac{5}{6} R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{5} R_2 \\ R_3 \to \frac{5}{12} R_3 \end{matrix}$$

Hence, $x = 1$, $y = 3$, $z = 5$

# Iterative Methods

We start with an approximation to the true solution and by applying the method repeatedly we get better and better approximation till accurated solution is achieved.

There are two iterative methods for solving simultaneous equations.

(1) Jacobi's method (method of successive correction).

(2) Gauss-Seidel method (Method of successive correction).

## JACOBI'S METHOD

The method is illustrated by taking an example.

Let
$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \dots (1)$$

After division by suitable constants and transposition, the equations can be written as

$$\left.\begin{aligned} x &= c_1 - k_{12}y - k_{13}z \\ y &= c_2 - k_{21}x - k_{23}z \\ z &= c_3 - k_{31}x - k_{32}y \end{aligned}\right\} \quad \dots (2)$$

Let us assume $x = 0$, $y = 0$ and $z = 0$ as first approximation, substituting the values of $x, y, z$ on the right hand side of (2), we get $x = c_1$, $y = c_2$, $z = c_3$. This is the second approximation to the solution of the equations.

Again substituting these values of $x, y, z$ in (2) we get a third approximation.

The process is repeated till two successive approximations are equal or nearly equal.

**Note.** Condition for using the iterative methods is that the coefficients in the leading diagonal are large compared to the other. If these are not so, then on interchanging the equation we can make the leading diagonal dominant diagonal.

# Problem - 4

**Example** *Solve by Jacobi's method*

$$4x + y + 3z = 17$$
$$x + 5y + z = 14$$
$$2x - y + 8z = 12$$

**Solution.** The above equations can be written as

$$\left.\begin{aligned} x &= \frac{17}{4} - \frac{y}{4} - \frac{3z}{4} \\ y &= \frac{14}{5} - \frac{x}{5} - \frac{z}{5} \\ z &= \frac{3}{2} - \frac{x}{4} + \frac{y}{8} \end{aligned}\right\} \quad \ldots (1)$$

On substituting $x = y = z = 0$ on the right hand side of (1), we get

$$x = \frac{17}{4}, \; y = \frac{14}{5}, \; z = \frac{3}{2}$$

Again substituting these values of $x, y, z$ on R.H.S. of (1), we obtain

$$x = \frac{17}{4} - \frac{7}{10} - \frac{9}{8} = \frac{97}{40}$$

$$y = \frac{14}{5} - \frac{17}{20} - \frac{3}{10} = \frac{33}{20}$$

$$z = \frac{3}{2} - \frac{17}{16} + \frac{7}{20} = \frac{63}{80}$$

Again putting these values on R.H.S. of (1) we get next approximations.

$$x = \frac{17}{4} - \frac{33}{80} - \frac{189}{320} = \frac{1039}{320} = 3.25$$

$$y = \frac{14}{5} - \frac{97}{200} - \frac{63}{400} = \frac{863}{400} = 2.16$$

$$z = \frac{3}{2} - \frac{97}{160} + \frac{33}{160} = \frac{176}{160} = 1.1$$

Substituting, again, the values of $x, y, z$ on R.H.S. of (1), we get

$$x = \frac{17}{4} - \frac{2.16}{4} - \frac{3(1.1)}{4} = 2.885$$

$$y = \frac{14}{5} - \frac{3.25}{5} - \frac{1.1}{5} = 1.93$$

$$z = \frac{3}{2} - \frac{3.25}{4} + \frac{2.16}{8} = 0.96$$

Repeating the process for $x = 2.885$, $y = 1.93$, $z = 0.96$, we have

$$x = \frac{17}{4} - \frac{1.93}{4} - \frac{3}{4} \times 0.96 = 4.25 - 0.48 - 0.72 = 3.05$$

$$y = \frac{14}{5} - \frac{2.885}{5} - \frac{0.96}{5} = 2.8 - 0.577 - 0.192 = 2.03$$

$$z = \frac{3}{2} - \frac{2.885}{4} + \frac{1.93}{8} = 1.5 - 0.721 + 0.241 = 1.02$$

This can be written in a table

| Iterations | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $x = \frac{17}{4} - \frac{y}{4} - \frac{3z}{4}$ | 0 | $\frac{17}{4} = 4.25$ | $\frac{97}{40} = 2.425$ | $\frac{1039}{320} = 3.25$ | 2.885 | 3.05 |
| $y = \frac{14}{5} - \frac{x}{5} - \frac{z}{5}$ | 0 | $\frac{14}{5} = 2.8$ | $\frac{33}{20} = 1.65$ | $\frac{863}{400} = 2.16$ | 1.93 | 2.03 |
| $z = \frac{3}{2} - \frac{x}{4} + \frac{y}{8}$ | 0 | $\frac{3}{2} = 1.5$ | $\frac{63}{80} = 0.7875$ | $\frac{176}{160} = 1.1$ | 0.96 | 1.02 |

After 6th iteration $x = 3.05$, $y = 2.03$, $z = 1.02$

The actual values are $x = 3$, $y = 2$, $z = 1$ **Ans.**

# Gauss Seidel Method

Let
$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \quad \ldots (1)$$

After division by suitable constants and transposition, the equations can be written as

$$\left.\begin{aligned} x &= c_1 - k_{12}y - k_{13}z \\ y &= c_2 - k_{21}x - k_{23}z \\ z &= c_3 - k_{31}x - k_{32}y \end{aligned}\right\} \quad \ldots (2)$$

Gauss-Seidel method is a modification of Jacobi's method. In place of substituting the same set of values in all the three equations (2)

earlier step.

**Step 1.** First we put $y = z = 0$ in first of the equation (2) and $x = c_1$. Then in second equation we put this value of $x$ *i.e.*, $c_1$ and $z = 0$ and obtain y. In the third equation we use the values of $x$ and $y$ obtained earlier to get $z$.

**Step 2.** We repeat the above procedure. In the first equation we put the values of $y$ and $z$ obtained in step 1 and redetermine $x$. By using the new value of $x$ and value of $z$ obtained in step 1 we redetermine y and so on.

In other words, the latest values of the unknowns are used in each step.

Consider the following equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

The above equations can be rewritten as

$$x = \frac{1}{a_1}[d_1 - b_1 y - c_1 z], \quad y = \frac{1}{b_2}[d_2 - a_2 x - c_2 z], \quad z = \frac{1}{c_3}[d_3 - a_3 x - b_3 y]$$

Initial approximations

$$x = x_0, \quad y = y_0, \quad z = z_0$$

To find $x = x_1$

$$x_1 = \frac{1}{a_1}[d_1 - b_1 y_0 - c_1 z_0]$$

To find $y = y_1$; put $x = x_1, z = z_0$

$$y_1 = \frac{1}{b_2}[d_2 - a_2 x_1 - c_2 z_0]$$

To find $z = z_1$, put $x = x_1, \ y = y_1$

$$z_1 = \frac{1}{c_3}[d_3 - a_3 x_1 - b_3 y_1]$$ and so on.

**Note 1.** The convergence of Gauss Seidel method is twice as fast as in Jocobi's method.

**2.** If the absolute value of largest coefficient is greater than the sum of the absolute value of all the remaining coefficient than the method converses for any initial approximation.

# Problem - 5

**Example** *Describe a method for solving a system of linear equations. Solve the following system of linear equations using Gauss-Seidel method*

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43$$

**Solution.**

Here, we have

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43$$

Solving each equation of the given system for the unknowns with largest coefficient in terms of the remaining unknowns, we have

$$x_1 = \frac{1}{23}(29 - 13x_2 - 3x_3) \quad \text{... (1)}$$

$$x_2 = \frac{1}{23}(37 - 5x_1 - 7x_3) \quad \text{... (2)}$$

$$x_3 = \frac{1}{23}(43 - 11x_1 - x_2) \quad \text{... (3)}$$

**For first iteration**

Putting $x_2 = 0,\ x_3 = 0$ in (1), we get

$$x_1 = \frac{1}{23}[29] = 1.26087$$

Putting $x_1 = 1.26087,\ x_3 = 0$ in (2), we get

$$x_2 = \frac{1}{23}[37 - 5(1.26087) - 0] = 1.33459$$

Putting $x_1 = 1.26087$ and $x_2 = 1.33459$ in (3), we get

$$x_3 = \frac{1}{23}[43 - 11 \times (1.26087) - 1.33459]$$

$$= \frac{1}{23}[43 - 13.86957 - 1.33459] = 1.20851$$

**For the second iteration**

Putting $x_2 = 1.33459$ and $x_3 = 1.20851$ in (1), we get

$$x_1 = \frac{1}{23}\,[29 - 13 \times 1.33459 - 3 \times 1.20851]$$

$$= \frac{1}{23}\,[29 - 17.34967 - 3.62553] = 0.34890$$

Putting $x_1 = 0.34890$ and $x_3 = 1.20851$ in (2), we get

$$x_2 = \frac{1}{23}\,[37 - 5 \times 0.34890 - 7 \times 1.20851]$$

$$= \frac{1}{23}\,[37 - 1.74450 - 8.45957] = 1.16504$$

Putting $x_1 = 0.34890$ and $x_2 = 1.16504$ in (3), we get

$$x_3 = \frac{1}{23}\,[43 - 11 \times 0.34890 - 1.16504]$$

$$= \frac{1}{23}\,[43 - 3.8379 - 1.16504] = 1.65205$$

**For the third iteration**

Putting $x_2 = 1.16504$ and $x_3 = 1.65205$ in (1), we get

$$x_1 = \frac{1}{23}\,[29 - 13 \times 1.16504 - 3 \times 1.65205]$$

$$= \frac{1}{23}\,[29 - 15.1502 - 4.95615] = 0.38668$$

Putting $x_1 = 0.38668$ and $x_3 = 1.65205$ in (2), we get

$$x_2 = \frac{1}{23}\,[37 - 5 \times 0.38668 - 7 \times 1.65205]$$

$$= \frac{1}{23}\,[37 - 1.9334 - 11.56435] = 1.02184$$

Putting $x_1 = 0.38668$ and $x_2 = 1.02184$ in (3), we get

$$x_3 = \frac{1}{23}[43 - 11 \times 0.38668 - 1.02184]$$

$$= \frac{1}{23}[43 - 4.25348 - 1.02184]$$

$$= 1.640203$$

**For the fourth iteration**

Putting $x_2 = 1.02184$ and $x_3 = 1.640203$ in (1), we get

$$x_1 = \frac{1}{23}[29 - 13 \times 1.02184 - 3 \times 1.640203]$$

$$x_1 = \frac{1}{23}[29 - 13.28392 - 4.89498] = 0.46937$$

Putting $x_1 = 0.46937$ and $x_3 = 1.640203$ in (2), we get

$$x_2 = \frac{1}{23}[37 - 5 \times 0.46937 - 7 \times 1.640203]$$

$$= \frac{1}{23}[37 - 2.34685 - 11.481421] = 1.007466$$

Putting $x_1 = 0.46937$ and $x_2 = 1.007466$ in (3), we get

$$x_3 = \frac{1}{23}[43 - 11 \times 0.46937 - 1.007466]$$

$$= \frac{1}{23}[43 - 5.16307 - 1.007466] = 1.601281$$

23

**For the fifth iteration**

$$x_1 = \frac{1}{23}[29 - 13 \times 1.007466 - 3 \times 1.601281] = 0.48257$$

$$x_2 = \frac{1}{23}[37 - 5 \times 0.48257 - 7 \times 1.601281] = 1.016443$$

$$x_3 = \frac{1}{23}[43 - 11 \times 0.48257 - 1.016443] = 1.594578$$

The following table shows all the iterations

| | | | | | |
|---|---|---|---|---|---|
| $x_1$ | 1.26087 | 0.34890 | 0.38668 | 0.46977 | 0.48257 |
| $x_2$ | 1.33459 | 1.16504 | 1.02184 | 1.007466 | 1.016443 |
| $x_3$ | 1.20851 | 1.65205 | 1.640203 | 1.601167 | 1.594578 |

$x_1 = 0.48257$, $x_2 = 1.016443$, $x_3 = 1.594578$ **Ans.**

# Jacobi's method for symmetric matrices

The Jacobi method is suitable for finding the eigenvalues of a real symmetric matrices. A real symmetric matrix is systematically reduced to a diagonal matrix by Jacobi method. This method use the similarity transformed matrix which is simpler but has the same eigenvalues as the given matrix. The transformation matrices which are used are orthogonal matrices. The advantage of using orthogonal matrices is that it minimizes errors in the process. Jacobi method can be used to find all eigenvalues simultaneously of any real symmetric matrix $A$. We know from matrix theory that, the eigenvalues of a real symmetric matrix $A$ are real. This method reduces the given matrix to a diagonal form, where the diagonal elements are the eigenvalues of the given matrix. In this method, the given matrix $A$ is transformed to a new matrix $A_1$ by the scheme

$$A_1 = P_1^{-1} A P_1 \tag{6}$$

Where $P_1$ is an orthogonal matrix. Therefore, $P_1^{-1} = P_1^T$. This transformation introduces a zero at a non-diagonal position of $A$. Then another matrix $A_2$ is produced by the equation

$$A_2 = P_2^{-1} A_1 P_2 = P_2^{-1} P_1^{-1} A P_1 P_2 \qquad \text{[by (6)]}$$

in which a new non-diagonal element is reduced to zero. Continuing this process of reducing the non-diagonal elements to zero one by one, we finally obtain a matrix

$$A_k = P_k^{-1} P_{k-1}^{-1} \dots P_1^{-1} A P_1 P_2 \dots P_{k-1} P_k \tag{7}$$

Which is a diagonal matrix. The eigenvalues are the diagonal elements of $A_k$. The non-diagonal element need not be reduced exactly to zero but must be less than a specified small quantity. The orthogonal matrices $P_i$ used above are extensions of a rotation matrix in a two-dimensional system. $P_i$'s are chosen as follows. Suppose a non-diagonal element, say $a_{ij}$, has to be reduced to zero. If $A$ is an $n \times n$ matrix, then $P$ is also an $n \times n$ matrix, where the sub matrix

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

consisting of the ith and jth rows and columns is replaced by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

All the other diagonal elements of $P$ are equal to unity. The other non-diagonal elements are taken as zero. For example, if $A$ is a $4 \times 4$ matrix and a non-diagonal element, say $a_{23}$, has to be reduced to zero. Then, we take

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Note the second and third rows and columns in (8). Now, let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{9}$$

is a given symmetric matrix. The transformation $P_1^T A P_1$ gives

$$A_1 = P_1^T A P_1 \tag{10}$$

The element equated to zero in the $(2, 3)$ position of $A_1$ gives the equation

$$-a_{22} \sin\theta \cos\theta + a_{23} \cos^2\theta - a_{23} \sin^2\theta + a_{33} \sin\theta \cos\theta = 0$$

This equation yields,

$$\tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left[\frac{2a_{23}}{a_{22} - a_{33}}\right]$$

Solving this trigonometric equation we get four values of $\theta$. If $\theta$ has to be small, we take $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Substituting for $\theta$ in equations (8) and (10), we get the values of $P_1$ and $A_1$ respectively. Next, we work with $A_1$ to annihilate some other non-diagonal element to zero. The process is truncated when all the non-diagonal elements are numerically less than the desired accuracy. The eigenvectors are obtained as the corresponding columns of

$$P = P_1 P_2 \dots P_k \tag{11}$$

Each step of reduction in the above method is called a rotation. The pair $(i, j)$ is called the plane of rotation and $\theta$ is the angle of rotation. The sequence in which the elements are reduced to zero is $a_{12}, a_{13}, \dots, a_{1n};\ a_{23}, a_{24}, \dots, a_{2n}$ and so on. If $a_{ij} (i \neq j)$ is reduced to zero, the element $a_{ji}$ also gets reduced to zero automatically by symmetry. In Jacobi method, the number of iterations increase if the matrix is large. If $A$ is an $n \times n$ matrix, the minimum number of rotations required to reduce $A$ into a diagonal form may be $\frac{n(n-1)}{2}$.

# Problem-1

**Example** *Let us now consider the real symmetric matrix*

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

*to find the eigenvalues and the corresponding eigenvectors by Jacobi method.*

***Solution.***

The given matrix is real and symmetric. The largest off-diagonal element is $a_{13} = a_{31} = 2$. The other two elements in this $2 \times 2$ sub matrix are $a_{11} = 1$ and $a_{33} = 1$. Now, we compute $\tan 2\theta = \frac{2a_{ij}}{a_{ii}-a_{jj}}$, where $|a_{ij}|$ be numerically the largest off-diagonal element of $A$. Therfore

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{2 \times 2}{1-1} = \infty \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Therefore

$$S_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The first rotation gives,

$$D_1 = S_1^{-1} A S_1 = S_1^T A S_1$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We may observe that the elements $d_{13}$ and $d_{31}$ got annihilated. To make sure that our calculations are correct up to this step, we may also observe that the sum of the diagonal elements of $D_1$ is same as the sum of the diagonal elements of the original matrix $A$. As a second step, we choose the largest off-diagonal element of $D_1$ and is found to be $d_{12} = d_{21} = 2$. The other elements are $d_{11} = 3, d_{22} = 3$. Now, we compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{2 \times 2}{3 - 3} = \frac{4}{0} = \infty \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

Thus, we construct the second rotation matrix as

$$S_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At the end of second rotation, we get

$$D_2 = S_2^{-1} D_1 S_2 = S_2^T D_1 S_2$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{12}$$

Which turned out to be a diagonal matrix and therefore we stop the computation. From equation (12), we notice that the eigenvalues of the given matrix are $5, 1$ and $-1$. The eigenvectors are the column vectors of $S = S_1 S_2$. Therefore,

$$S = S_1 S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence the eigenvectors corresponding to $5, 1$ and $-1$ are respectively $\left[\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right]^T$, $\left[-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right]^T$ and $\left[-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]^T$. □

# Eigenvalues of a matrix by Power method

## POWER METHOD FOR APPROXIMATING EIGENVALUES

we saw that the eigenvalues of an $n \times n$ matrix $A$ are obtained by solving its characteristic equation

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_0 = 0.$$

For large values of $n$, polynomial equations like this one are difficult and time-consuming to solve. Moreover, numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors. In this section we look at an alternative method for approximating eigenvalues. As presented here, the method can be used only to find the eigenvalue of $A$ that is largest in absolute value—we call this eigenvalue the **dominant eigenvalue** of $A$. Although this restriction may seem severe, dominant eigenvalues are of primary interest in many physical applications.

**Definition of Dominant Eigenvalue and Dominant Eigenvector**

Let $\lambda_1, \lambda_2, \ldots,$ and $\lambda_n$ be the eigenvalues of an $n \times n$ matrix $A$. $\lambda_1$ is called the **dominant eigenvalue** of $A$ if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \ldots, n.$$

The eigenvectors corresponding to $\lambda_1$ are called **dominant eigenvectors** of $A$.

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = -1$) has no dominant eigenvalue. Similarly, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 1$) has no dominant eigenvalue.

**EXAMPLE 1** ***Finding a Dominant Eigenvalue***

Find the dominant eigenvalue and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

**Solution** From Example 4 of Section 7.1 we know that the characteristic polynomial of $A$ is $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$. Therefore the eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = -2$, of which the dominant one is $\lambda_2 = -2$. From the same example we know that the dominant eigenvectors of $A$ (those corresponding to $\lambda_2 = -2$) are of the form

$$\mathbf{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

## The Power Method

Like the Jacobi and Gauss-Seidel methods, the power method for approximating eigenvalues is iterative. First we assume that the matrix $A$ has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation $\mathbf{x}_0$ of one of the dominant eigen*vectors* of $A$. This initial approximation must be a *nonzero* vector in $R^n$. Finally we form the sequence given by

$$\begin{aligned}
\mathbf{x}_1 &= A\mathbf{x}_0 \\
\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) &= A^2\mathbf{x}_0 \\
\mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) &= A^3\mathbf{x}_0 \\
&\vdots \\
\mathbf{x}_k = A\mathbf{x}_{k-1} = A(A^{k-1}\mathbf{x}_0) &= A^k\mathbf{x}_0.
\end{aligned}$$

For large powers of $k$, and by properly scaling this sequence, we will see that we obtain a good approximation of the dominant eigenvector of $A$.

EXAMPLE **_Approximating a Dominant Eigenvector by the Power Method_**

Complete six iterations of the power method to approximate a dominant eigenvector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

**Solution** We begin with an initial nonzero approximation of

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We then obtain the following approximations.

| *Iteration* | | *Approximation* |
|---|---|---|
| $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix}$ | → | $-4\begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$ |
| $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix}$ | → | $10\begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$ |
| $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix}$ | → | $-22\begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$ |
| $\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix}$ | → | $46\begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$ |
| $\mathbf{x}_5 = A\mathbf{x}_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix}$ | → | $-94\begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$ |
| $\mathbf{x}_6 = A\mathbf{x}_5 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix}$ | → | $190\begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$ |

Note that the approximations in Example 2 appear to be approaching scalar multiples of

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

which we know from Example 1 is a dominant eigenvector of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

# Given's method for symmetric matrices

The Givens method leads to a tridiagonal matrix. The eigenvalues and eigenvectors of the original matrix are to determined from those of the tridiagonal matrix. Let $A$ be a real symmetric matrix. The Givens method consists of the following steps: Step 1. Reduce $A$ to a tridiagonal symmetric matrix using plane rotations. The reduction to a tridiagonal form is achieved by using the orthogonal transformations as in the Jacobi method. However, in this case we start with the subspace containing the elements $a_{22}$, $a_{23}$, $a_{32}$, $a_{33}$. Perform the plane rotation $S_1^{-1}AS_1$ using the orthogonal matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now, let us consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{13}$$

and let the orthogonal rotation matrix $S_1$ in the plane $(2,3)$ be

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \therefore S_1^{-1}AS_1 = S_1^T AS_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12}\cos\theta + a_{13}\sin\theta & -a_{12}\sin\theta + a_{13}\cos\theta \\ a_{12}\cos\theta + a_{13}\sin\theta & a_{23}\sin 2\theta + a_{22}\cos^2\theta + a_{33}\sin^2\theta & a_{23}\cos 2\theta - a_{22}\sin\theta\cos\theta + a_{33}\sin\theta\cos\theta \\ -a_{12}\sin\theta + a_{13}\cos\theta & a_{23}\cos 2\theta - a_{22}\sin\theta\cos\theta + a_{33}\sin\theta\cos\theta & -a_{23}\sin 2\theta + a_{22}\sin^2\theta + a_{33}\cos^2\theta \end{bmatrix}$$

Then in the resulting matrix, equating the element in the $(1,3)$ position to zero for reducing $S_1^{-1}AS_1$ to tridiagonal matrix, we get

$$-a_{12}\sin\theta + a_{13}\cos\theta = 0 \Rightarrow \tan\theta = \frac{a_{13}}{a_{12}} \Rightarrow \theta = \tan^{-1}\left(\frac{a_{13}}{a_{12}}\right) \tag{14}$$

By this value of $\theta$, the above transformation gives zeros in $(1,3)$ and $(3,1)$ positions. Let us further perform rotation in the plane $(2,4)$ and put the resulting element $(1,4)=0$. This would not affect the zeros obtained earlier. Then the transformations are applied to the matrix in turn so as to annihilate the elements $(1,3),(1,4),(1,5),\ldots,(1,n)$; $(2,4),(2,5),\ldots,(2,n)$ and finally we arrive at the tridiagonal matrix

$$P = \begin{bmatrix} p_1 & q_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ q_1 & p_2 & q_2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_2 & p_3 & q_3 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q_{n-2} & p_{n-1} & q_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & q_{n-1} & p_n \end{bmatrix}$$

Step 2. To obtain the eigenvalues of the tridiagonal matrix. Let the resulting tridiagonal matrix after first transformation be obtained as

$$S_1^{-1}AS_1 = S_1^TAS_1 = B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{23} & \alpha_{33} \end{bmatrix} \tag{15}$$

$$\det(B-\lambda I)=0 \Rightarrow \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & 0 \\ \alpha_{12} & \alpha_{22}-\lambda & \alpha_{23} \\ 0 & \alpha_{23} & \alpha_{33}-\lambda \end{vmatrix} = 0$$

Say $f_3(\lambda)=0$. Then we have,

$$f_0(\lambda)=1,\ f_1(\lambda)=\alpha_{11}-\lambda=\alpha_{11}-\lambda f_0(\lambda)$$

and

$$f_2(\lambda)=\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{12} & \alpha_{22}-\lambda \end{vmatrix} = (\alpha_{22}-\lambda)f_1(\lambda)-\alpha_{12}^2 f_0(\lambda)$$

Now expanding $f_3(\lambda)$ in terms of the third row, we immediately obtain

$$f_3(\lambda)=(\alpha_{33}-\lambda)\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{12} & \alpha_{22}-\lambda \end{vmatrix} - \alpha_{23}\begin{vmatrix} \alpha_{11}-\lambda & 0 \\ \alpha_{12} & \alpha_{23} \end{vmatrix}$$

$$\Rightarrow f_3(\lambda)=(\alpha_{33}-\lambda)f_2(\lambda)-\alpha_{23}^2 f_1(\lambda)$$

The recurrence formula in general is,

$$f_k(\lambda)=(\alpha_{kk}-\lambda)f_{k-1}(\lambda)-\left(\alpha_{(k-1)k}\right)^2 f_{k-2}(\lambda),\ 2\le k\le n \tag{16}$$

Above is the characteristic equation which can be solved by any standard method. Thus the roots of (16) will be the eigenvalues of the given real symmetric matrix. If none of the $\alpha_{ij}$ $(i\neq j)$ vanish then this equation generate a sequence

$\{f_k(\lambda) : k = 0, 1, \ldots, n\}$, which is called the Sturm sequence. A table of the sequence for various $\lambda$ is prepared and the number of changes in sign of the Sturm sequence is noted, the difference between the number of changes of sign for consecutive values of $\lambda$ gives an approximate location of the eigenvalues. Knowing the location of the eigenvalues, their exact values can be obtained by any iterative method. That is, if $V(x)$ denotes the number of changes in sign in the sequence for a given number $x$, then the number of zeros of $f_n$ in $(a, b)$ is $|V(a) - V(b)|$ provided $a$ or $b$ is not a zero of $f_n$. In this way, we can approximately compute the eigenvalues and by repeated bisections, we can improve these estimates.

Step 3. To obtain the eigenvectors of the tridiagonal matrix. Let $Y$ be the eigenvector of the tridiagonal matrix $B$ and let $S_1, S_2, \ldots, S_j$ be the orthogonal matrices employed in reducing the given real symmetric matrix $A$ to the tridiagonal form $B$, then the corresponding eigenvector $X$ of $A$ is given by $X = SY$, where $S = S_1 S_2 \ldots S_j$ is the product of the orthogonal matrices used in the plane rotations. The number of rotations needed for Givens method are equivalent to the number of non-tridiagonal elements of the matrix. For a $3 \times 3$ matrix, only one rotation is required; whereas for a $4 \times 4$ matrix, three rotations are required etc. That is, the total number of plane rotations required to bring a matrix of order $n$ to its tridiagonal form is $\frac{(n-1)(n-2)}{2}$.

**Example 1.** *Let us now consider the real symmetric matrix*

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

*to find the eigenvalues and the corresponding eigenvectors by Givens method.*

***Solution.***

There is only one non-tridiagonal element $a_{13} = 2$. This is to be reduced to zero, hence one rotation is required. Now, to annihilate $a_{13}$, we define the orthogonal matrix in the plane $(2,3)$ as:

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta$ is obtained by $\tan\theta = \frac{a_{13}}{a_{12}} = \frac{2}{\sqrt{2}} \Rightarrow \sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$. Therefore

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

THerefore

$$A_1 = O^{-1}AO = O^TAO = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{6} & \frac{5}{\sqrt{3}} & 2\sqrt{\frac{2}{3}} \\ 0 & -\sqrt{6} + \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix},$$

which is a tridiagonal matrix. Now, to find the eigenvalues of $A_1$, we proceed as follows:

The characteristic equation of $A_1$ is

$$\begin{bmatrix} 1-\lambda & \sqrt{6} & 0 \\ \sqrt{6} & 3-\lambda & -\sqrt{2} \\ 0 & -\sqrt{2} & 1-\lambda \end{bmatrix} = 0$$

The Sturm sequence, i.e., the leading minors of order $0, 1, 2, 3$ are given by $f_0(\lambda) = 1$, $f_1(\lambda) = 1 - \lambda$, $f_2(\lambda) = (3 - \lambda)f_1(\lambda) - 6f_0(\lambda)$ and $f_3(\lambda) = (1 - \lambda)f_2(\lambda) - 2f_1(\lambda)$. Let us now consider the changes of sign in the Sturm sequence as

| $\lambda$ | $f_0(\lambda)$ | $f_1(\lambda)$ | $f_2(\lambda)$ | $f_3(\lambda)$ | $N(\lambda)$ |
|---|---|---|---|---|---|
| $-2$ | 1 | 3 | 9 | 21 | 0 |
| 0 | 1 | 1 | $-3$ | $-5$ | 1 |
| 2 | 1 | $-1$ | $-7$ | 9 | 2 |
| 3 | 1 | $-2$ | $-6$ | 16 | 2 |
| 4 | 1 | $-3$ | $-3$ | 15 | 2 |
| 6 | 1 | $-5$ | 9 | $-35$ | 3 |

Above table shows that there is an eigenvalue in the intervals $(-2, 0)$, $(0, 2)$ and $(4, 6)$. We now find better estimates of the eigenvalues by repeated bisections. First, we shall find the eigenvalue in the interval $(-2, 0)$ by bisecting it at $-1$.

| $\lambda$ | $f_0(\lambda)$ | $f_1(\lambda)$ | $f_2(\lambda)$ | $f_3(\lambda)$ | $N(\lambda)$ |
|---|---|---|---|---|---|
| $-2$ | 1 | 3 | 9 | 21 | 0 |
| $-1$ | 1 | 2 | 2 | 0 | ... |

Note that $f_3(-1) = 0$, so that $\lambda = -1$ is an eigenvalue. Now, we shall find the eigenvalue in the interval $(0, 2)$ by bisecting it at 1.

| $\lambda$ | $f_0(\lambda)$ | $f_1(\lambda)$ | $f_2(\lambda)$ | $f_3(\lambda)$ | $N(\lambda)$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | $-3$ | $-5$ | 1 |
| 1 | 1 | 0 | $-6$ | 0 | ... |

Since $f_3(1) = 0$, so $\lambda = 1$ is an eigenvalue. Next, we shall find the eigenvalue in the interval $(4, 6)$ by bisecting it at 5.

| $\lambda$ | $f_0(\lambda)$ | $f_1(\lambda)$ | $f_2(\lambda)$ | $f_3(\lambda)$ | $N(\lambda)$ |
|---|---|---|---|---|---|
| 5 | 1 | $-4$ | 2 | 0 | ... |
| 6 | 1 | $-5$ | 9 | $-35$ | 3 |

Again, since $f_3(5) = 0$, so $\lambda = 5$ is an eigenvalue. Therefore, the eigenvalues of $A_1$ are $5, 1$ and $-1$ and hence the eigenvalues of $A$ are also $5, 1$ and $-1$. Now, to find the eigenvectors of $A_1$ for each of the eigenvalues, we proceed as follows:

For $\lambda = 5$, let the eigenvector of $A_1$ be $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then we have,

$$A_1 Y = \lambda Y \Rightarrow \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 5 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Which gives the equations,

$$y_1 + \sqrt{6}y_2 = 5y_1 \tag{17}$$

$$\sqrt{6}y_1 + 3y_2 - \sqrt{2}y_3 = 5y_2$$

$$\text{and} \quad -\sqrt{2}y_2 + y_3 = 5y_3 \tag{18}$$

Equation (17) gives,

$$4y_1 = \sqrt{6}y_2 \Rightarrow \frac{y_1}{\sqrt{6}} = \frac{y_2}{4} \Rightarrow \frac{y_1}{\frac{1}{2}} = \frac{y_2}{\sqrt{\frac{2}{3}}}$$

Equation (18) gives,

$$-\sqrt{2}y_2 = 4y_3 \Rightarrow \frac{y_2}{4} = \frac{y_3}{-\sqrt{2}} \Rightarrow \frac{y_2}{\sqrt{\frac{2}{3}}} = \frac{y_3}{-\frac{1}{2\sqrt{3}}}$$

Therefore, the eigenvector of $A_1$ for $\lambda = 5$ is $Y = \left[\frac{1}{2}, \sqrt{\frac{2}{3}}, -\frac{1}{2\sqrt{3}}\right]^T$. Therefore, the eigenvector $X$ of $A$ for $\lambda = 5$ is given by

$$X = OY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{2\sqrt{3}} \end{bmatrix}$$
$$= \left[\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right]^T$$

where $O$ is the orthogonal matrix used in the plane rotation. For $\lambda = -1$, let the eigenvector of $A_1$ be $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then we have,

$$A_1Y = \lambda Y \Rightarrow \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 3 & -\sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -1 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Which gives the equations,

$$y_1 + \sqrt{6}y_2 = -y_1 \tag{19}$$

$$\sqrt{6}y_1 + 3y_2 - \sqrt{2}y_3 = -y_2$$

$$\text{and } -\sqrt{2}y_2 + y_3 = -y_3 \tag{20}$$

Equation (19) gives,

$$2y_1 = -\sqrt{6}y_2 \Rightarrow \frac{y_1}{-\sqrt{6}} = \frac{y_2}{2} \Rightarrow \frac{y_1}{-\frac{1}{\sqrt{2}}} = \frac{y_2}{\frac{1}{\sqrt{3}}}$$

Equation (20) gives,

$$\sqrt{2}y_2 = 2y_3 \Rightarrow \frac{y_2}{2} = \frac{y_3}{\sqrt{2}} \Rightarrow \frac{y_2}{\frac{1}{\sqrt{3}}} = \frac{y_3}{\frac{1}{\sqrt{6}}}$$

Therefore, the eigenvector of $A_1$ for $\lambda = -1$ is $Y = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right]^T$. Therefore, the eigenvector $X$ of $A$ for $\lambda = -1$ is given by

$$X = OY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \left[-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]^T$$

Similarly, the eigenvector of $A_1$ for $\lambda = 1$ is $Y = \left[-\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right]^T$. Therefore, the eigenvector $X$ of $A$ for $\lambda = 1$ is given by

$$X = OY = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \left[-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right]^T$$

**Ans.**

□

# THANK YOU