<u>Mechanics of Soilds</u>
(BME-14)

by

Dr. Jeeoot Singh/ Mr. Ankit Saxena

Mechnical Engineering Department

Madan Mohan Malaviya University of Technology

Gorakhpur

BME-14	MECHANICS OF SOLIDS	
Course category	:	Department Core (DC)
Pre-requisite Subject	:	NIL
Contact hours/week	:	Lecture: 3, Tutorial:1 , Practical: 0
Number of Credits	:	4
Course Assessment methods	:	Continuous assessment through tutorials, attendance, home assignments, quizzes and one Minor test and One Major Theory Examination
Course Outcomes	:	The students are expected to be able to demonstrate the following knowledge, skills and attitudes after completing this course

Course Outcomes

1. Ability to determine stresses in solid members under different conditions.
2. The ability to calculate deflections in beams under different support conditions, deflection in helical and leaf springs under different loading conditions.
3. The ability to determine stresses in thin and thick cylindrical and thin spherical shells and buckling loads in column under different loading conditions.
4. Ability to understand advanced topics of mechanics of solids for further research and industry applications.

Syllabus

UNIT-I
Stress and strain, elastic constants, Poisson's ratio, Principal planes and principal stresses, Mohr's circle for plane stress and plane strain; Bending and torsion and its combination, Strain energy due to principal stresses, Energy of distortion and dilatation. thermal stresses; strain gauges and rosettes.
UNIT-II
Beams: Review of SFD BMD, Pure bending, combined direct and bending stresses, shear stresses in beams, combined bending and torsion of solid and hollow circular shafts, Deflection of beams, Equation of elastic curve, Mecaulay's method, Area moment method, Fixed beam carrying point load and uniformly distributed load, continuous beams, Castgliano's theorem Introduction to Springs, Helical springs under axial loads and axial twist, Deflection of spring by energy method, Open and closed coil helical springs under axial and twist loadings.

UNIT-III
Thin cylindrical and spherical shells: Hoop and Longitudinal stresses and strain, Cylindrical shell with hemispherical ends, Volumetric strain, Wire wound cylinders, spherical shell. **Thick cylindrical shell:** Stresses in thick cylinders subjected to internal or external pressures, Compound cylinders, Stresses due to interference fits. **Columns and Struts:** Classification, Euler's theory for long column for different end conditions, Limitations, Rankine formulae for struts/columns. Introduction to other theories.
UNIT-IV
Generalised Hooks Law. Introduction to 3D stresses and Mohr's circle. Elastic stabilities and Theories of Failure. Determination of shear centre for I-section and channel section.

Books & References

1. Introduction of Mechanics of Materials – I.H. Shames
2. Strength of Materials-S. Ramamurtham (Dhanpat Rai Publishing Co.)
3. Strength of Materials-R. K. Rajput (S. Chand)
4. Strength of Materials-Ryder (Mcmillan Publishers India Limited)
5. Strength of Materials-Timoshenko and Young (Tata McGraw Hill)
6. Advanced Mechanics of Solids-L S Srinath (Tata McGraw Hill)
7. Mechanics of Solids – Egor P. Popov (Pearson)
8. Mechanics of materials-Pytel (CL Engineering)

Unit-1

1. Stresses and strain
2. Elastic constant
3. Poisson's ratio
4. Principal planes and principal stresses
5. Mohr's circle for plane stress and plane strain
6. Bending, torsional and its combinations
7. Strain energy due to principal stresses
8. Energy of distortions
9. Thermal stresses
10. Strain gauges and rosettes

Few basic concepts of engineering mechanics

1. Static equilibrium equations

2. FBD (free body diagram)

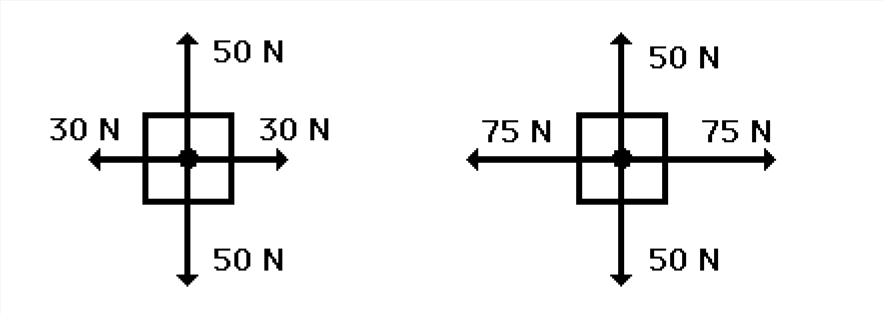
3. Centroid and its calculations

4. Moment of inertia (area moment of inertia)

5. Parallel & perpendicular axis theorem

Static equilibrium equations ?

Static equilibrium means that the net force acting on the object is zero or object is not moving.

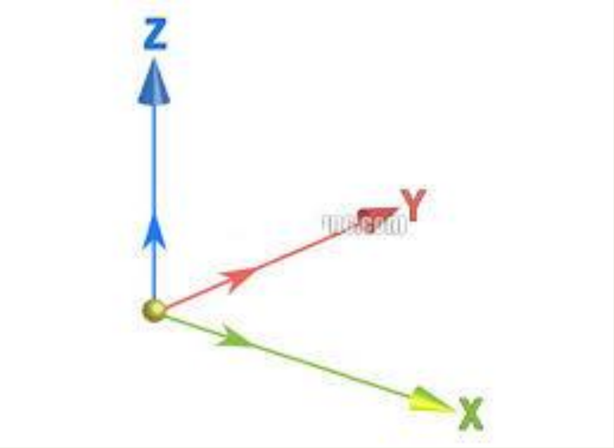


in 2D space:

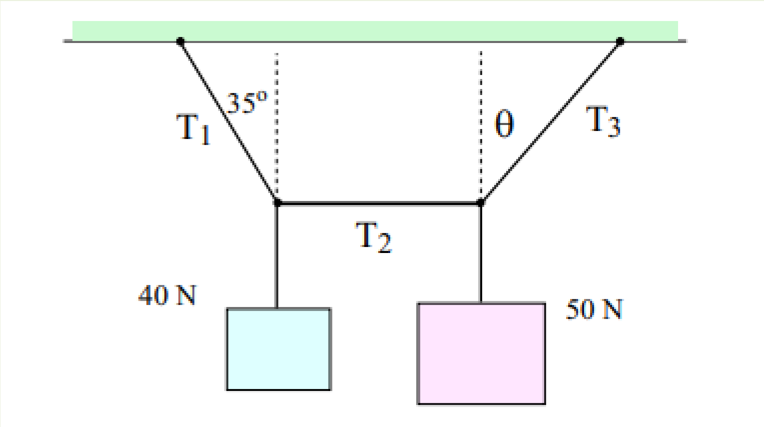
1. $\sum H = 0$ (Submission of horizontal forces should be zero)
2. $\sum V = 0$ (Submission of vertical forces should be zero
3. $\sum M = 0$ (submission of all forces moment about any axis should be zero)

In 3D space : We have 6 static equilibrium equations

1. $\sum Fz = 0$
2. $\sum Mx = 0$
3. $\sum My = 0$
4. $\sum Mz = 0$



Q1. The system as shown in figure is in equilibrium with the string in the center exactly horizontal. Find (a) tension T1, (b) tension T2, (c) tension T3 and (d) angle θ.



Hint: Free body Diagram (FBD)

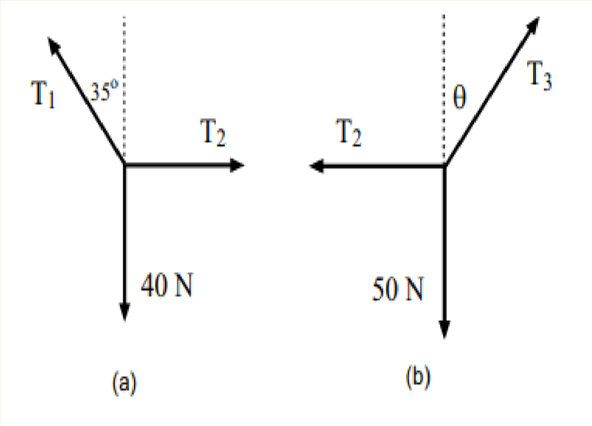
Answer:

T1 = 48.8 N

T2 = 28 N

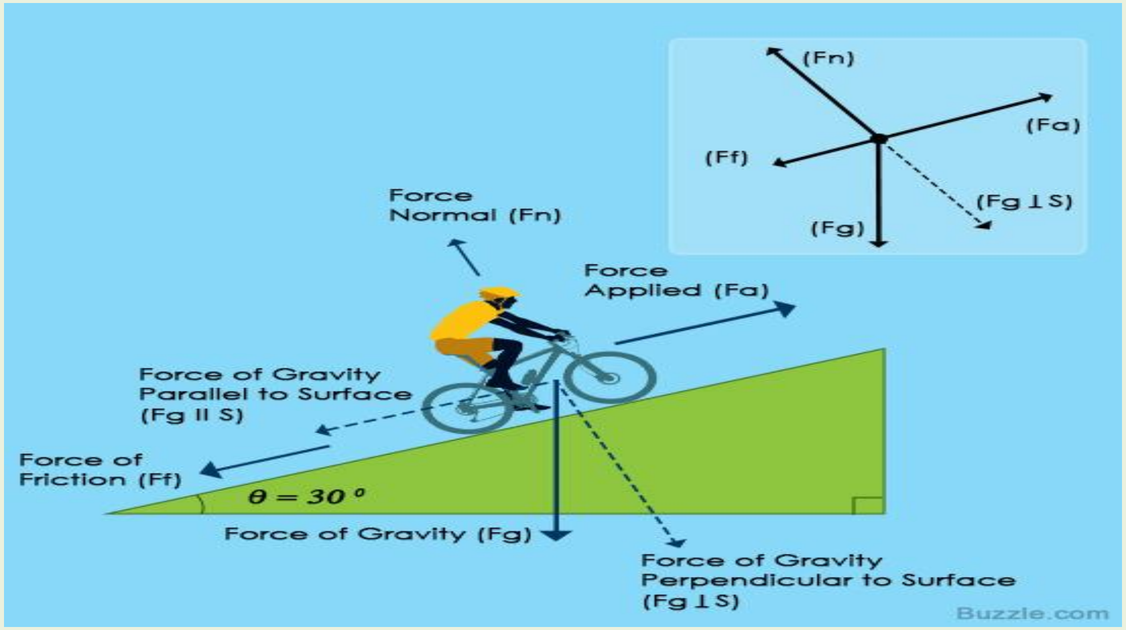
T3 = 57.3 N

θ = 29.3◦



<u>Free Body Diagram</u>
(FBD)

It is used to show the relative magnitude and direction of all forces acting upon an object in a given situation.



Centroid & Centre of gravity

For 2D object

Like : Triangle, rectangle, circle etc.

For 3D object

Like : Book, cup, Ball etc.

- Centroid & centre of gravity is a point where the entire mass of the body is assumed to be act.

Example → Centroid calculations for Rectangular area.

let $G \rightarrow$ Rectangular area centroid $(\bar{x}, \bar{y})$

b → base
h → height

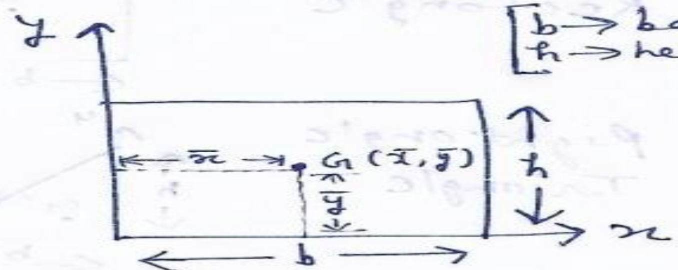


fig 1

$$\bar{x} = \frac{\int x\, dA}{A} \qquad \bar{y} = \frac{\int y \cdot dA}{A}$$

$dA \rightarrow$ Small element strip area.

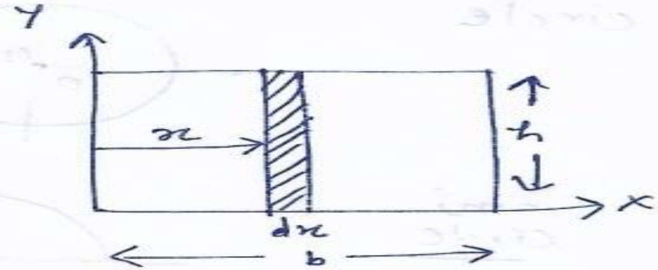


fig 2

for $\bar{x}$ calculation →

$$\bar{x} = \frac{\int x \cdot dA}{A}$$

$$dA = dx \cdot h$$
$$A = b \times h$$

so $$\bar{x} = \int_0^b \frac{x \cancel{h} \cdot dx}{b \times \cancel{h}}$$

$$\boxed{\bar{x} = \frac{\int_0^b x\, dx}{b} = \frac{\left(\frac{b^2}{2}\right)}{b} = \frac{b}{2}}$$

In the same way →

for $\bar{y}$ calculation

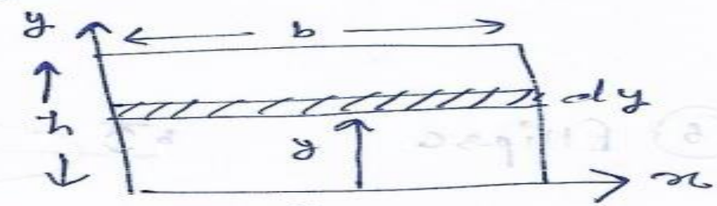


fig. 3

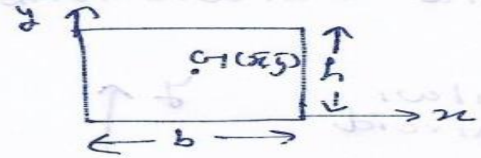
$$\bar{y} = \frac{\int_0^h y \cdot b \cdot dy}{bh}$$

$$\boxed{\bar{y} = h/2}$$

ANKIT SAXENA (saxena01ankit@gmail.com)

few common shape →

1) Rectangle

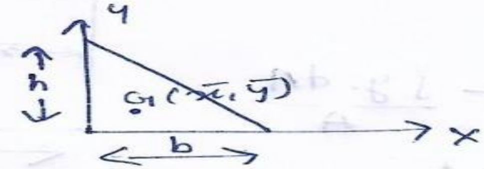


$\text{Area} = b \times h$

$\bar{x} = b/2$

$\bar{y} = h/2$

(2) Right angle Triangle

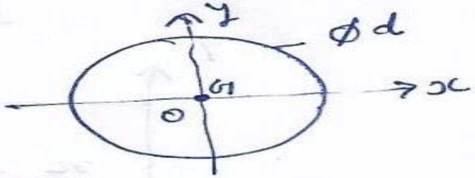


$\text{Area} = \frac{bh}{2}$

$\bar{x} = b/3$

$\bar{y} = h/3$

(3) circle

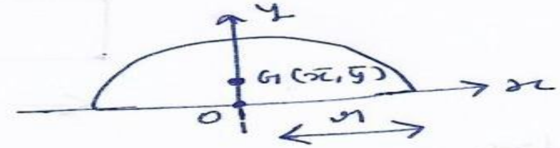


$\text{Area} = \frac{\pi}{4} d^2$

$\bar{x} = 0$

$\bar{y} = 0$

(4) Semi circle

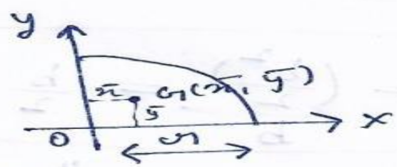


$\text{Area} = \frac{\pi r^2}{2}$

$\bar{x} = 0$

$\bar{y} = \frac{4r}{3\pi}$

(5) Quarter circle

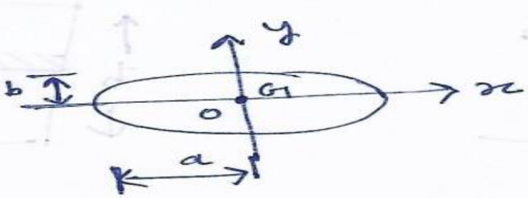


$\text{Area} = \frac{\pi r^2}{4}$

$\bar{x} = \frac{4r}{3\pi}$

$\bar{y} = \frac{4r}{3\pi}$

(6) Ellipse



$\text{area} = \pi ab$

$\bar{x} = 0$

$\bar{y} = 0$

Centroid calculation for composite shape

The plane geometrical figures (such as T-section, I-section, L-section etc.) have only areas but no mass, the centre of area of such figures is known as centroid.

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity with respect to some axis of reference,

Then

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots\dots}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \dots\dots}{a_1 + a_2 + a_3 + \dots}$$

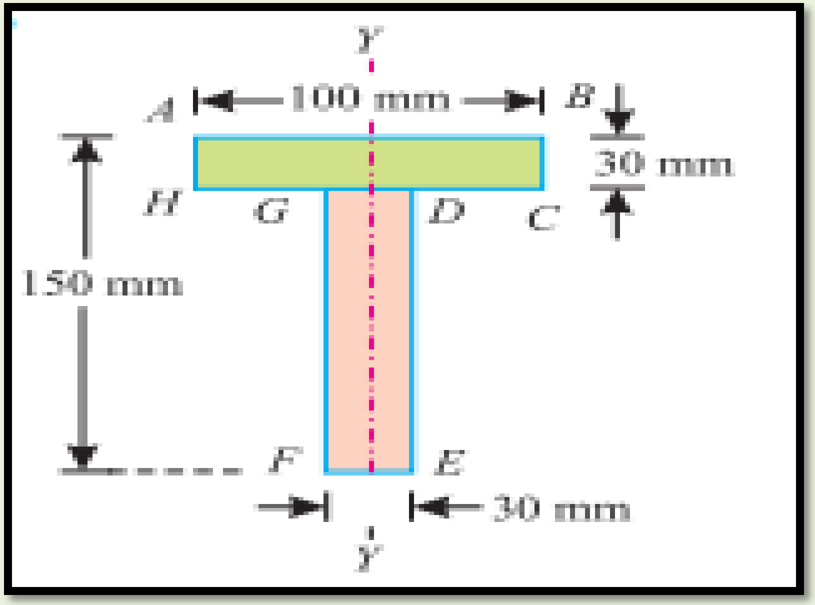
<u>Centroid calculation for composite shape</u> …

- where a1, a2, a3…….. etc., are the areas into which the whole figure is divided x1, x2, x3 ….. etc., are the respective co-ordinates of the areas a1, a2, a3……. on X-X axis with respect to same axis of reference.
- y1, y2, y3……. etc., are the respective co-ordinates of the areas a1, a2, a3……. on Y-Y axis with respect to same axis of the reference.

<u>Note:</u>

1. While using the above formula, x1, x2, x3 ….. or y1, y2, y3 or x and y must be measured from the same axis of reference (or point of reference) and on the same side of it.
2. If the figure is on both sides of the axis of reference, then the distances in one direction are taken as positive and those in the opposite directions must be taken as negative

Q 2. Find the centre of gravity of a 100 mm * 150 mm * 30 mm T-section.



Note : If section is having symmetry about any axis, its centroid will lie on the symmetrical axis.

Moment of inertia (Area moment of inertia)

It is the second moment of area

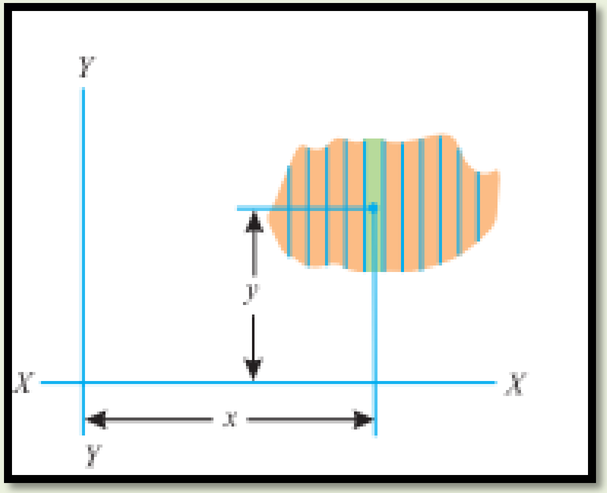
Consider a plane figure, whose moment of inertia is required to be found out about X-X axis and Y-Y axis.

Let us divide the whole area into a no. of strips. Consider one of these strips.

Let dA = Area of the strip

x = Distance of the centre of gravity of the strip on X-X axis and

y = Distance of the centre of gravity of the strip on Y-Y axis.



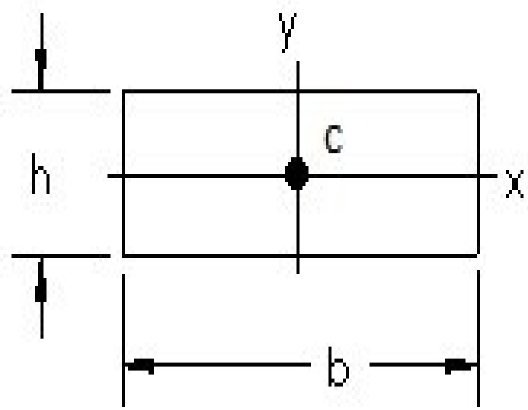
$$I_{YY} = \sum dA \cdot x^2$$

$$I_{XX} = \sum dA \cdot y^2$$

Moment of inertia (Area moment of inertia)

few common shapes

- Rectangular



$$A = bh$$

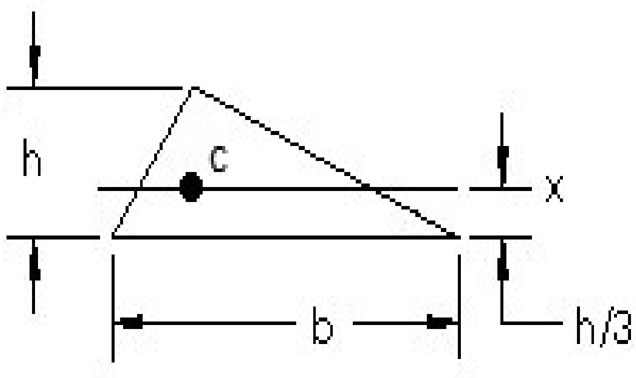
$$I_x = \frac{1}{12} bh^3$$

$$I_y = \frac{1}{12} hb^3$$

Moment of inertia (Area moment of inertia)

few common shapes

- Triangle



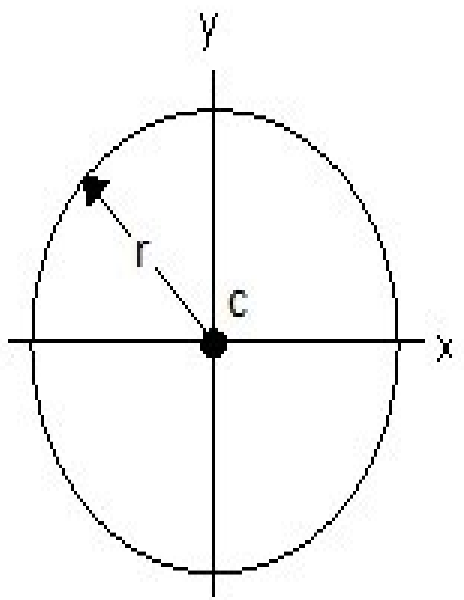
$$A = \frac{1}{2} bh$$

$$I_x = \frac{1}{36} bh^3$$

Moment of inertia (Area moment of inertia)

few common shapes

- Circle



$$A = \pi r^2$$

$$I_x = \frac{1}{4}\pi r^4$$

$$I_y = \frac{1}{4}\pi r^4$$

ANKIT SAXENA (saxena01ankit@gmail.com)

Parallel axis theorem

The moment of inertia about an axis parallel to the centroid axis is equal to the sum of moment of inertia about the centroid axis and product of area and square of distance between the these two axes.

$$I = I_c + Mh^2$$

I = Mmoment of inertia about any axis

Ic = Moment of inertia about the centroid axis

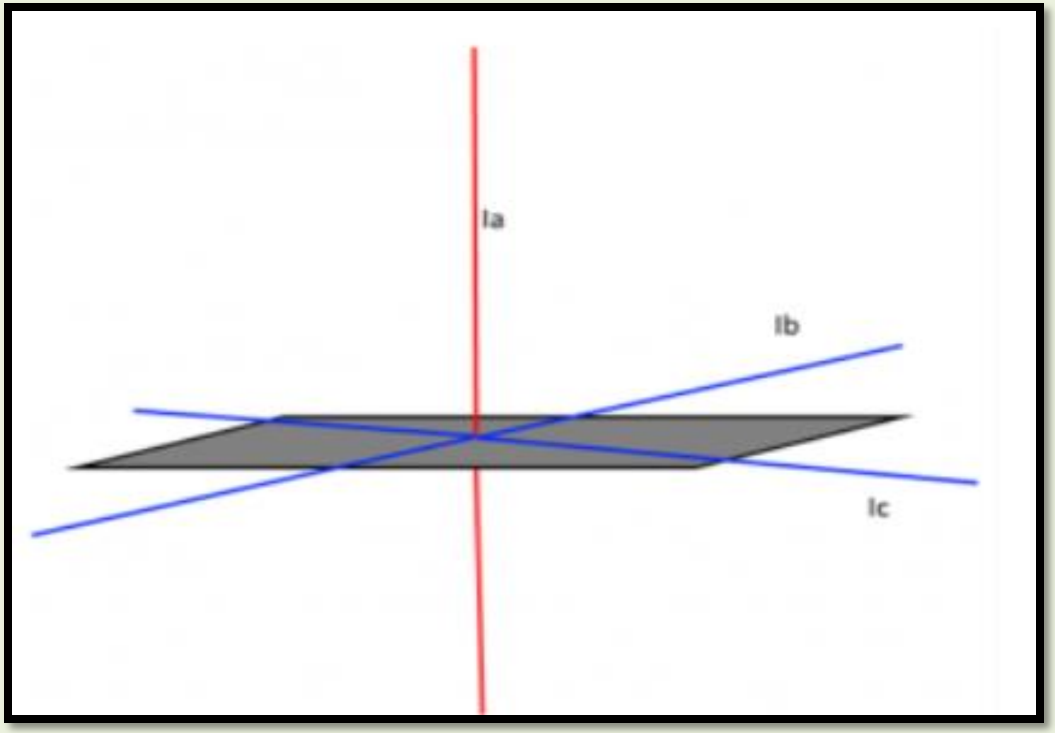
M = Area

h2 = Square of distance between the these two axes

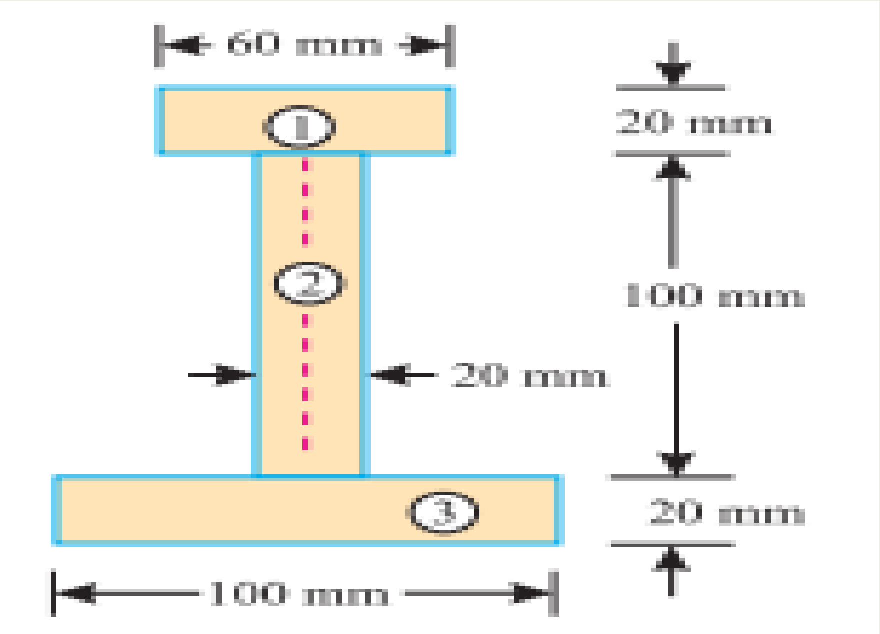
Perpendicular axis theorem

Perpendicular axis theorem is used when the body is symmetric in shape about two out of the three axes, if moment of inertia about two of the axes are known the moment of inertia about the third axis can be found using the expression:

$$I_a = I_b + I_c$$



Q 3. An I- section is made up of 3 rectangles. Find the moment of inertia of the section about the horizontal axis passing through the centre of gravity of the section.



MECHANICS OF SOLIDS

What is the difference between engineering mechanics and mechanics of solids?

Engineering mechanics : we study the external forces and their effects on rigid body.

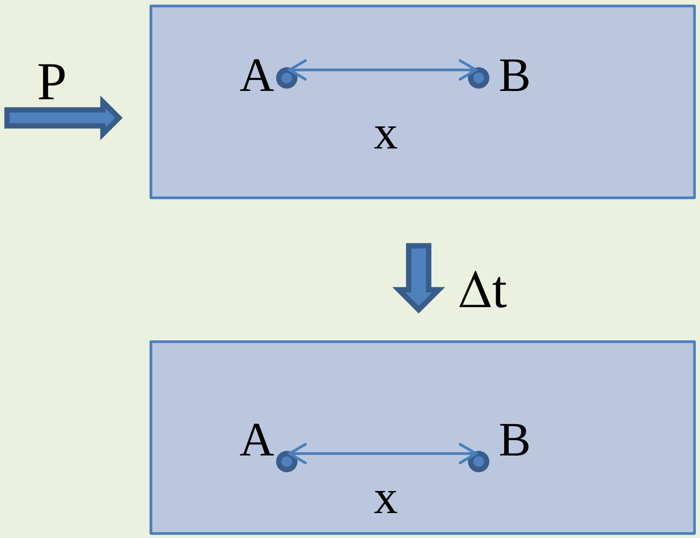
Mechanics of solids/ Strength of material/ Mechanics of materials :

We study the internal resisting force (I.R.F) which are developed during the elastic deformation of a body under the action of external forces.

In engineering mechanics we always deals with the rigid body. But in mechanics of solids we deals with the deformable body.

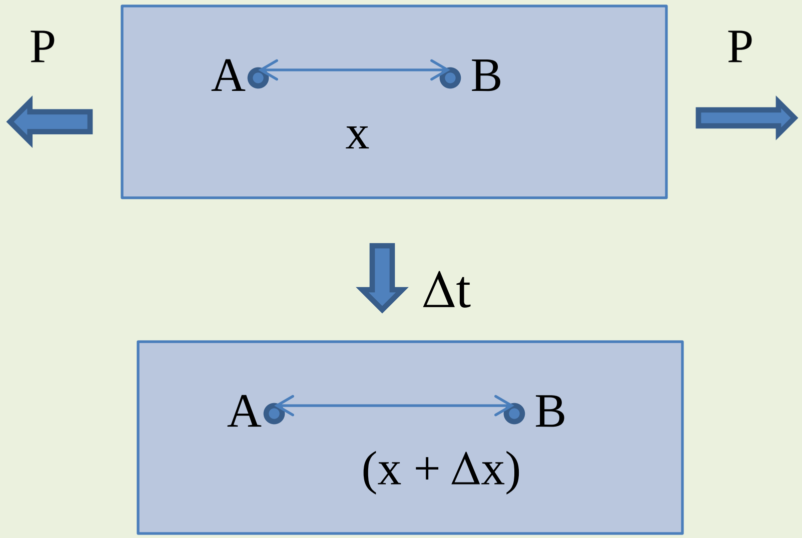
Rigid Body

- Rigid body is basically defined as a body where changes in the distance between any two of its points is negligible.



Deformable body

- Deformable body is basically defined as a body where changes in the distance between any two of its points could not be neglected.

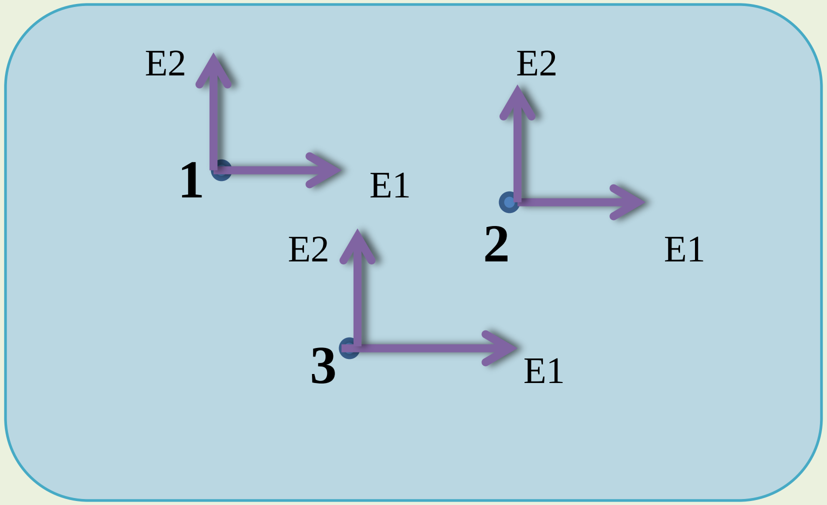


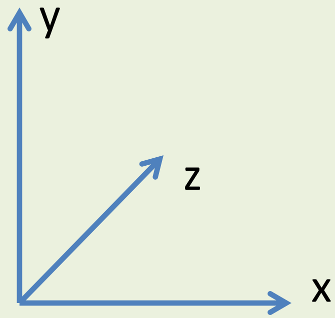
Assumptions

- Body should be homogeneous and isotropic.
- It obeys Hook's law.
- Body should be prismatic.
- Load is considered as static load.
- Self weight of the body should be neglected.

Homogeneous Material

- A material is said to be homogeneous when it exhibits same elastic properties at any point in a given direction.
- Elastic properties are independent of point.

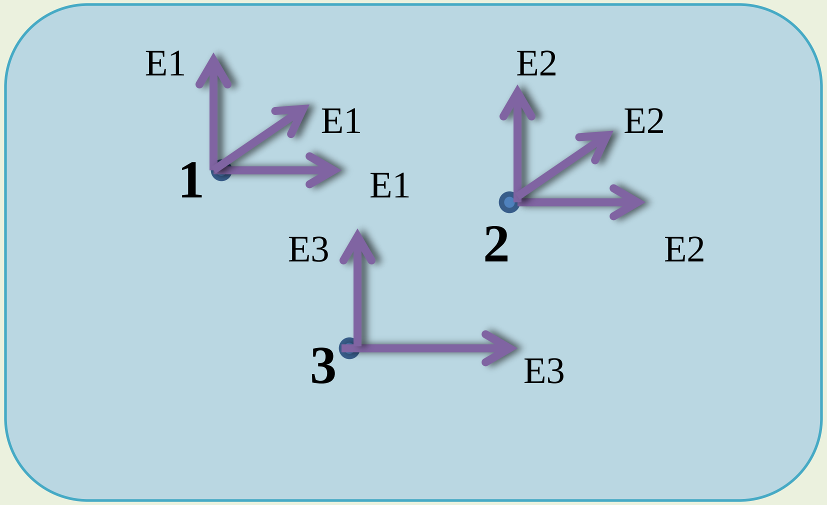


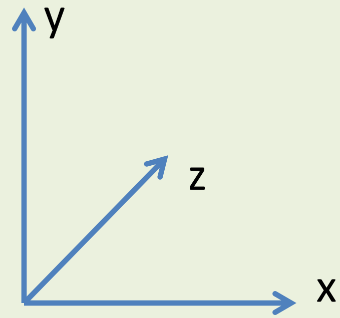


Body A

Isotropic Material

- A material is said to be isotropic when it exhibits same elastic properties in any direction at a given point.
- Elastic properties are independent of direction.

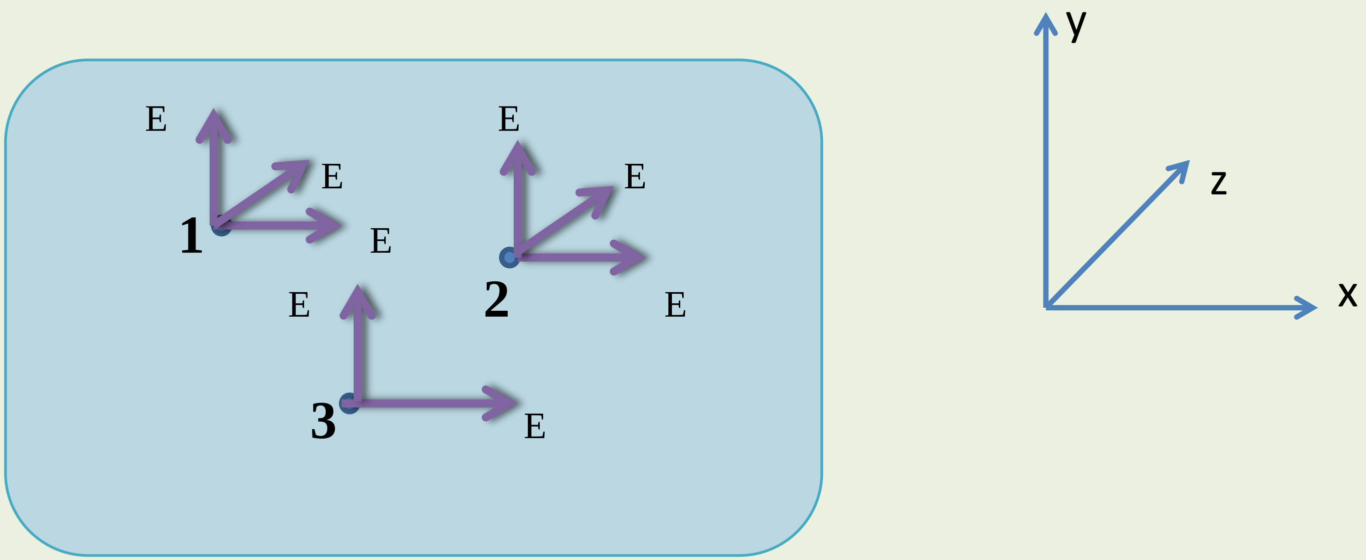




Body B

Homogeneous & Isotropic Material

- A material is said to be both homogeneous & isotropic when it exhibits same elastic properties at any point and at any direction.
- Elastic properties are independent from both direction and point.



Body C

Hook's Law

- When force is applied to a material, we know that it either stretches or compresses in response to the applied force.
- **Stress:** It is defined as the internal resisting force per unit cross-section area at a given point. It is denoted by the symbol **σ**.

$$\sigma = [\text{Force / unit cross-section area}]_{\text{at a point}}$$

$$\sigma = [F / A]$$

- **Strain:** It is the ratio of change in dimension to the original dimension. It is denoted by the symbol **ε.**

$$\varepsilon = [\text{change in dimension / Original dimension}]$$

$$\varepsilon = [\Delta l / L]$$

According to Hook's law,

"The strain of the material is proportional to the applied stress up to the proportional limit of that material."

$$\text{Stress} \propto \text{strain}$$

$$\sigma \propto \varepsilon$$

$$\sigma = E\,\varepsilon$$

E = Young's modulus of elasticity

$$[F / A] = E\,[\Delta l / L]$$

$$\Delta l = \frac{FL}{AE}$$

This is a static deflection/ deformation equation

Prismatic Body

- A body is said to be prismatic body if it has uniform cross section through out its length.

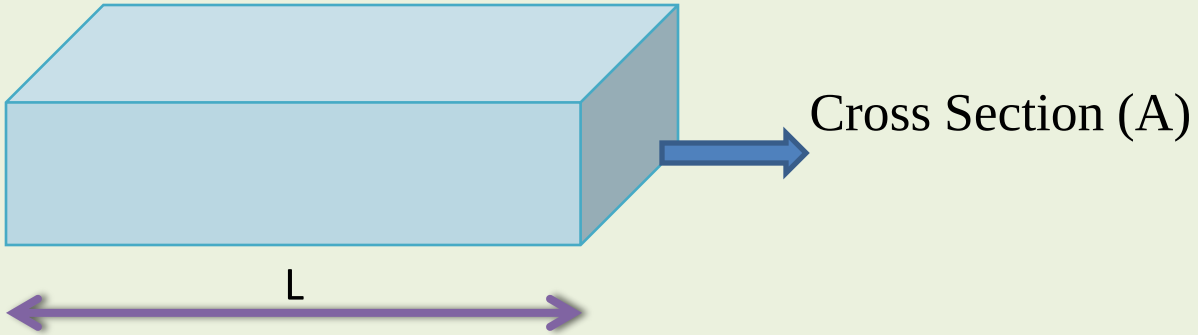
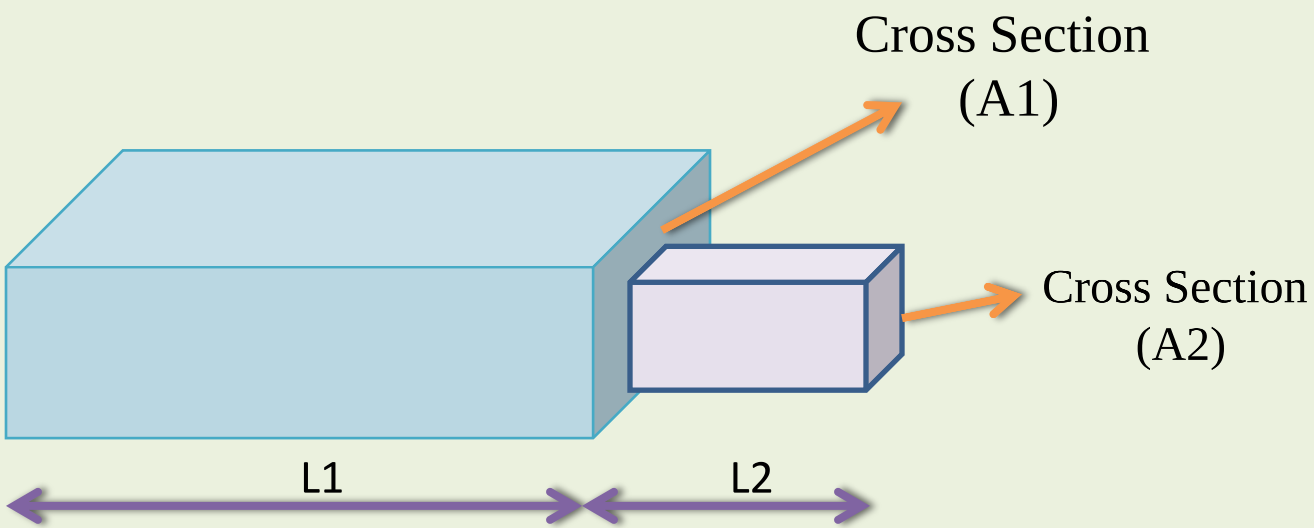


Fig. Prismatic bar having rectangular x-s/c through out it's length 'L'

Non-Prismatic Body

·



Note:

1. This is a non-prismatic bar because it is not having uniform x-s/c through out its entire length.
2. But this is consider as prismatic bar for length L1 & L2 alone with having x-s/c A1& A2 respectively.

Important Point

- Static deflection equation only can apply for a prismatic body.

$$\Delta l = \frac{FL}{AE}$$

- For non-prismatic body, we can apply the above equation only for the prismatic potion.

Static load

- Load is a vector quantity.
- Static load is defined as load whose magnitude and direction is constant throughout.

Load

- It is a external force or couple which is subjected to a object or body.
- In simple word load can also be defined as weight of one component with respect to another component.

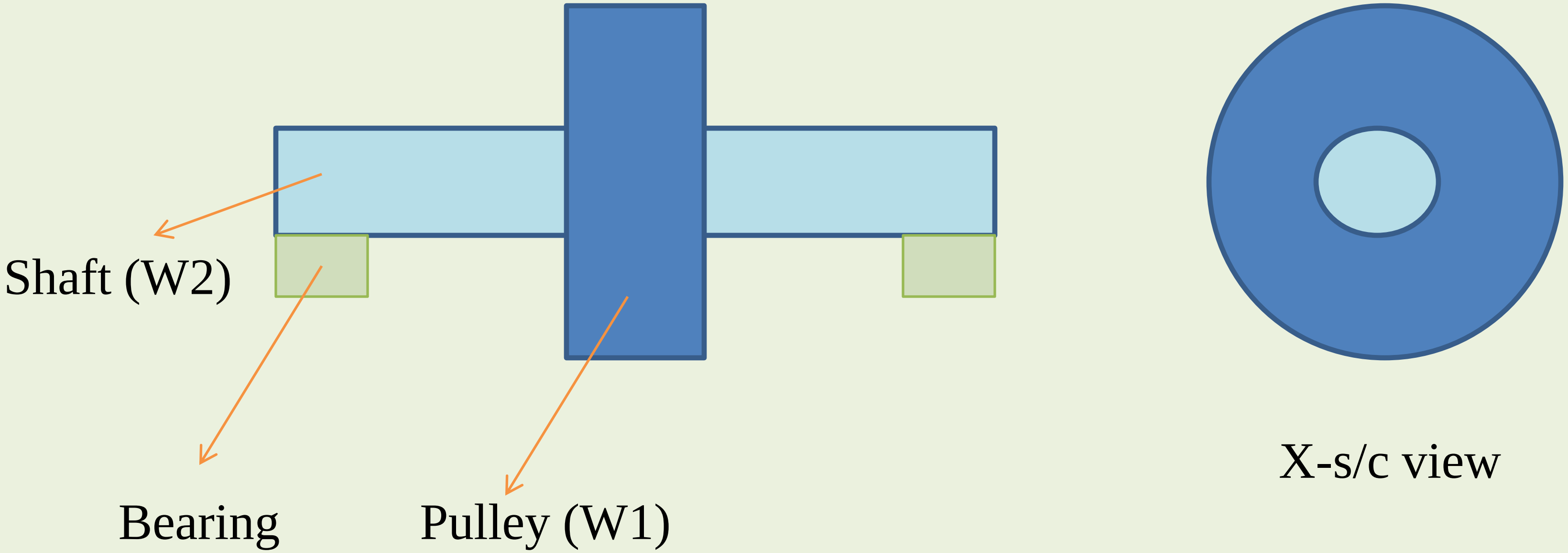


Fig. Shaft support a pulley & bearing's support shaft and pulley both

1. While designing a pulley, self weight of pulley is neglected.
2. While designing a shaft, the pulley weight is considered as load on the shaft but self weight of shaft is neglected.
3. While designing a bearings, the pulley and shaft weights are considered as load on the bearings but self weight of bearings are neglected.

Classification of loads

LOADS

- **Based on time**
- **Based on direction of load w.r.t x-s/c**
- **Based on distribution of loads**

Based on Time

Static Load

Dead Load
(refers to loads that relatively don't change over time)

Gradually Applied load
(When a load is applied in small installments)

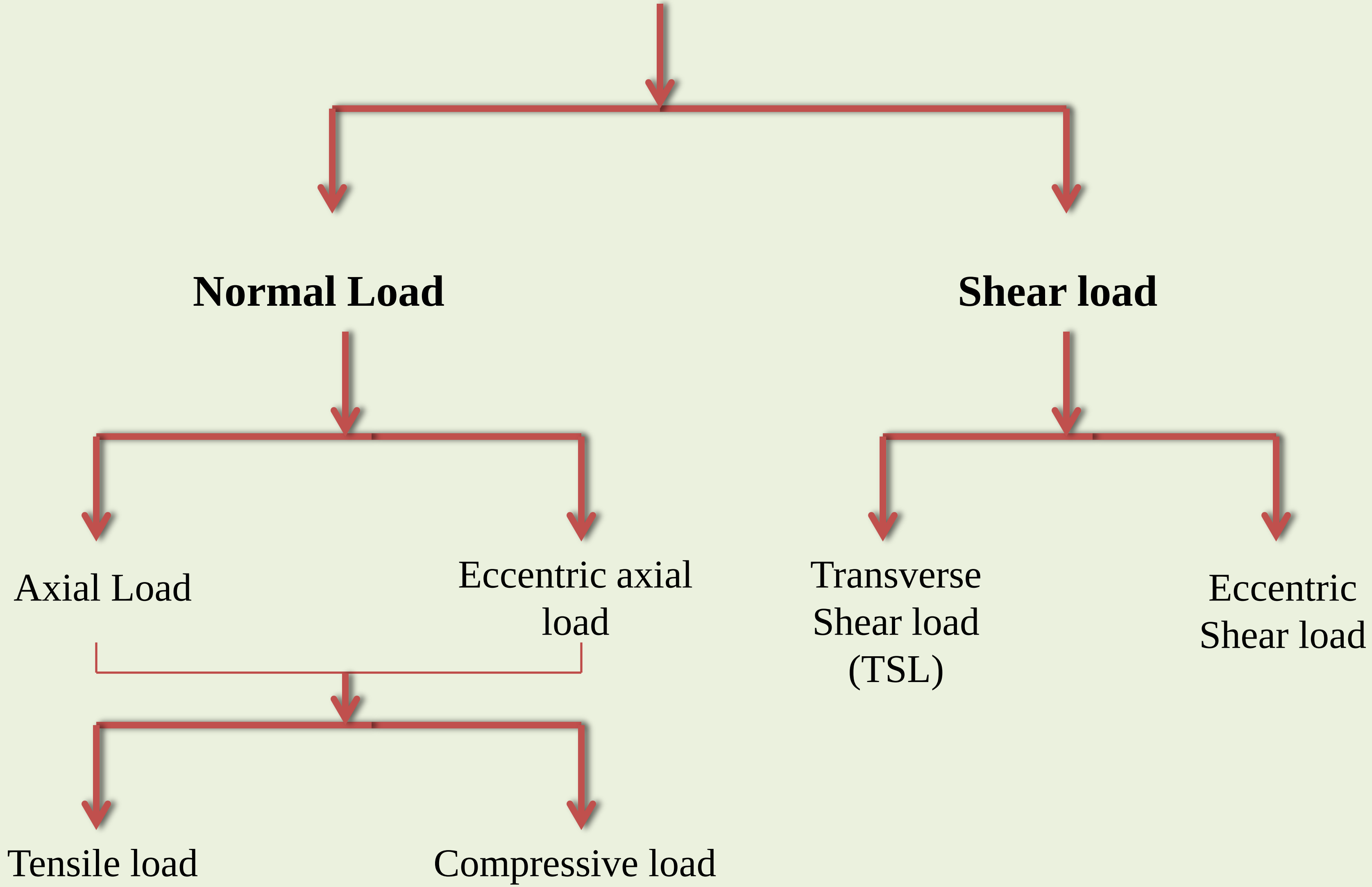
Dynamic load

Impact/ shock
Or
Suddenly applied load
(load with a sudden impact due to falling or hitting one object on another)

Fatigue
Or
Cyclic
Or
Variable loads

ANKIT SAXENA (saxena01ankit@gmail.com)

Based on direction of load w.r.t x-s/c



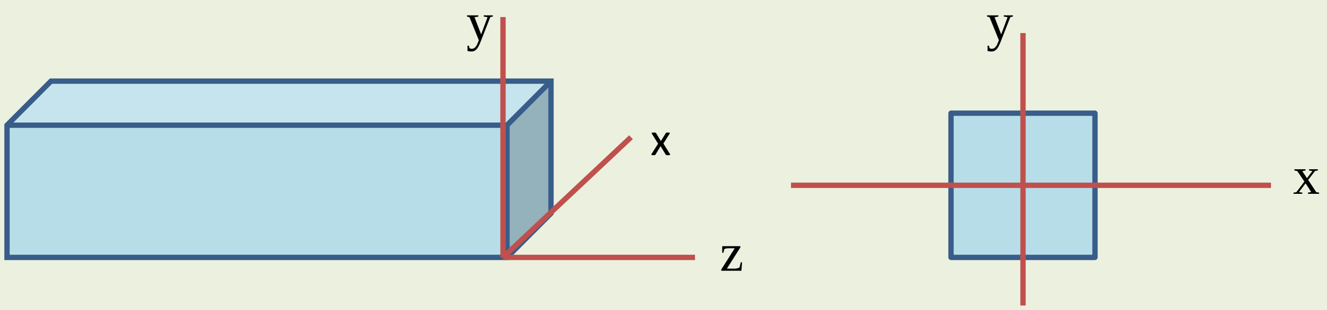
Based on distribution of loads

- **Concentrated Load**
  - Concentrated Point load
  - Concentrated Point moment
- **Distributed load**
  - Uniformly Distributed Load (UDL)
  - Uniformly Variable load (UVL)
  - Uniformly Distributed Moment (UDM)

Important terminology

Centroidal axis: The axes that pass through the centroid of the x-s/c is known as centroidal axis.

- Consider a bar having a square cross section.

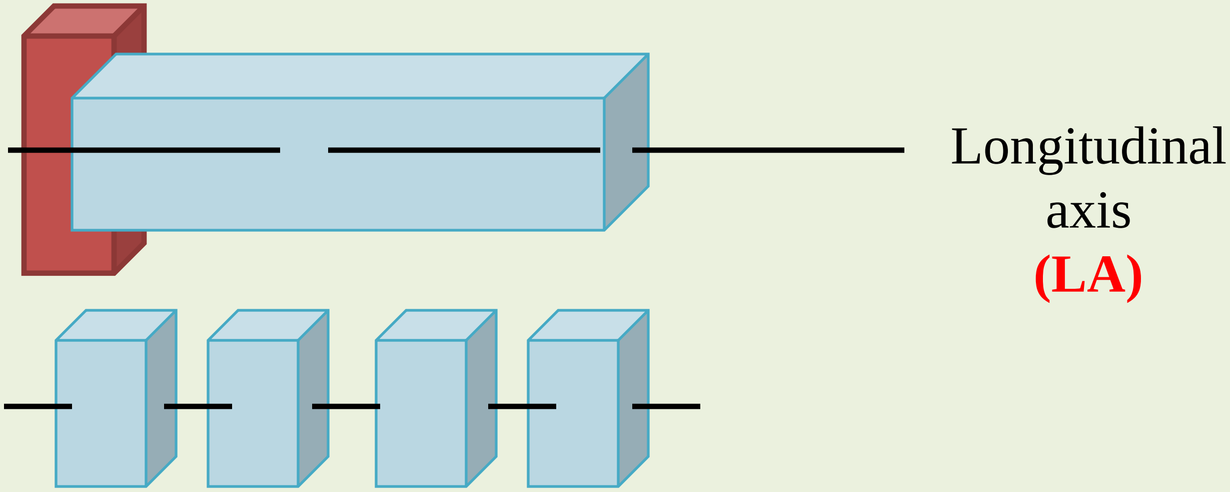


Front view **x-s/c view**

- We know that, centroid of square x-s/c lie at its the diagonal intersection.
- Here x & y axes are the centroidal axes.
- Or in simple words we can say that centroidal axes lies in a plane of cross section.

Longitudinal axis :

- Imaginary axis that passing through the centroid of each and every x-s/c is known as longitudinal axis.
- Longitudinal axes is always perpendicular to the plane of x-s/c.



Polar Axis : Any axis that is perpendicular to the plane of cross section is known as polar axis.

Normal load :

- If the line of action of force is perpendicular to the cross section area, is known as normal load.
- If the line of action of force coincide with longitudinal axis is known as axial normal load.

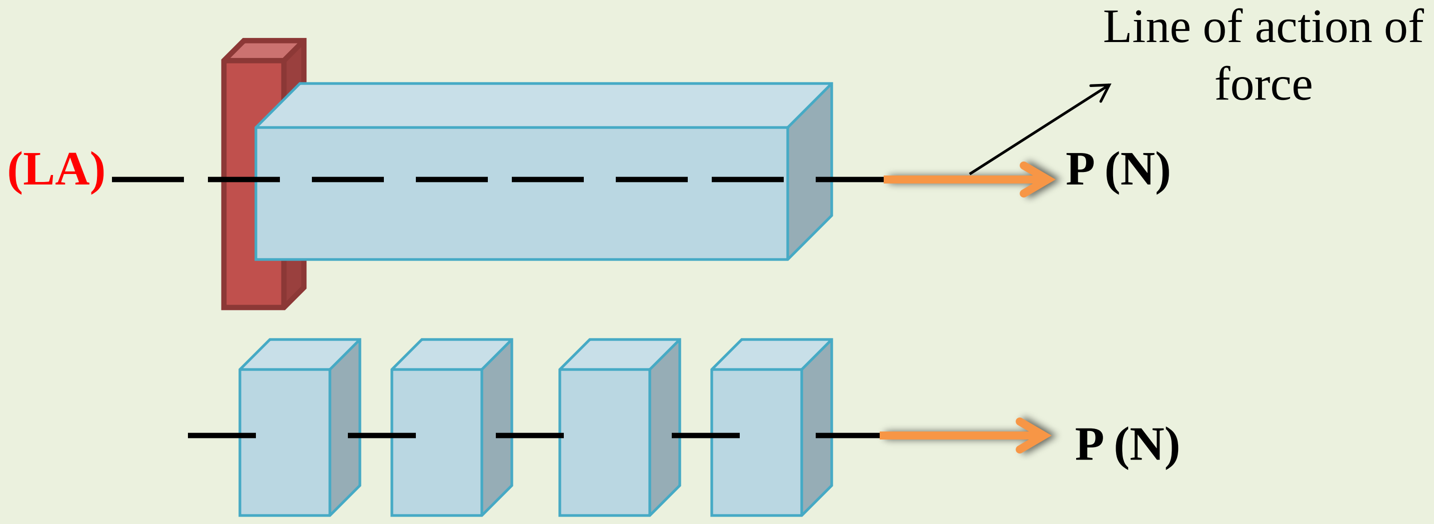


Figure: Normal axial load

- If the line of action of force having eccentricity (e) with longitudinal axis is known as eccentric axial normal load.

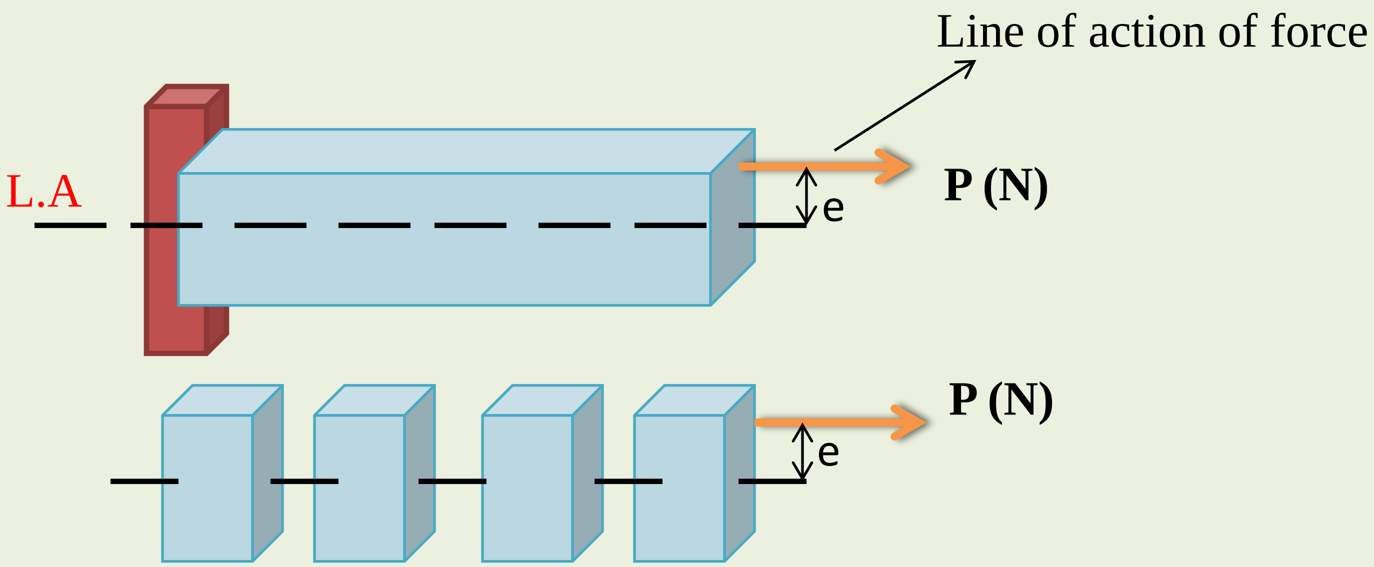
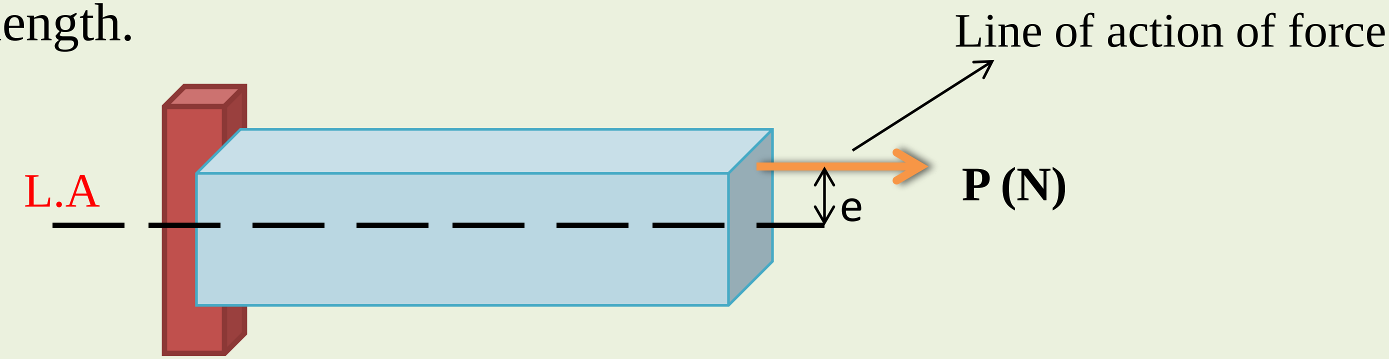
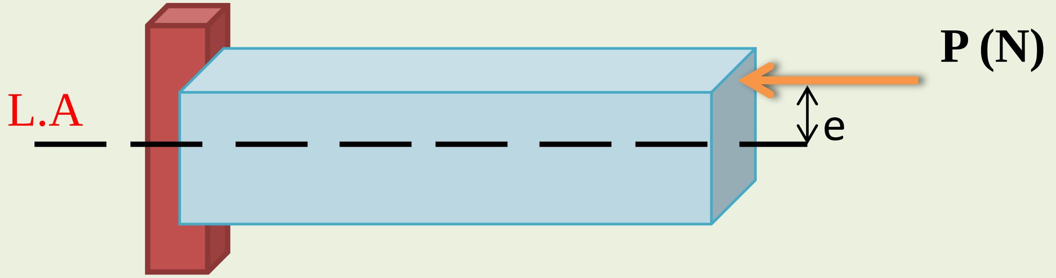


Figure: Eccentric axial normal load

Tensile load: If the line of action of force is away from the x-s/c or in outward direction is know as tensile load. Tensile load having tendency to increase the member length.

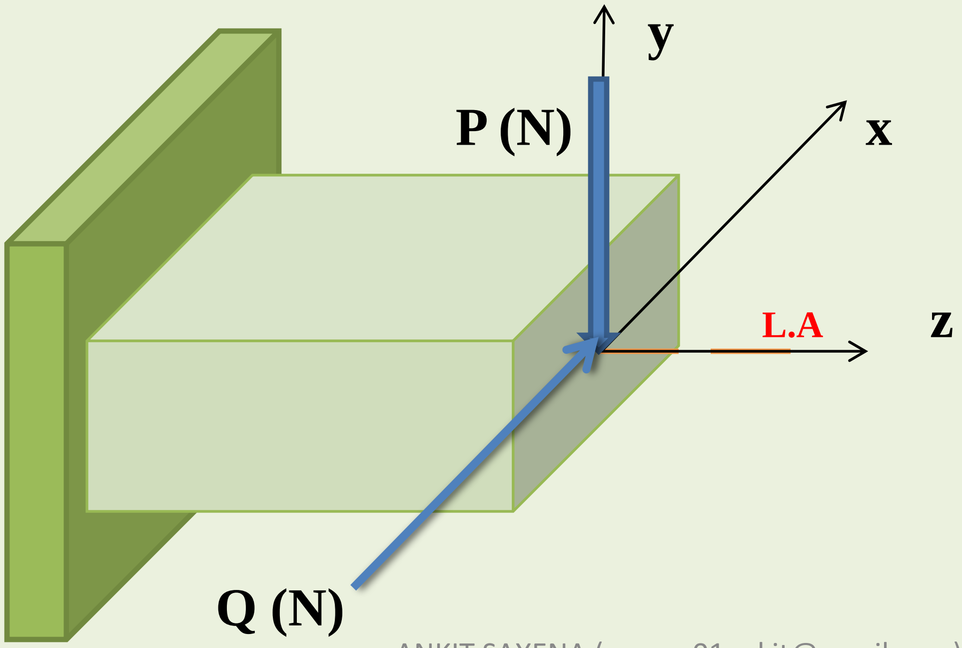


Compressive load: If the line of action of force is towards the x-s/c or in inward direction is know as compressive load. Compressive load having tendency to decrease the member length.



Shear load :

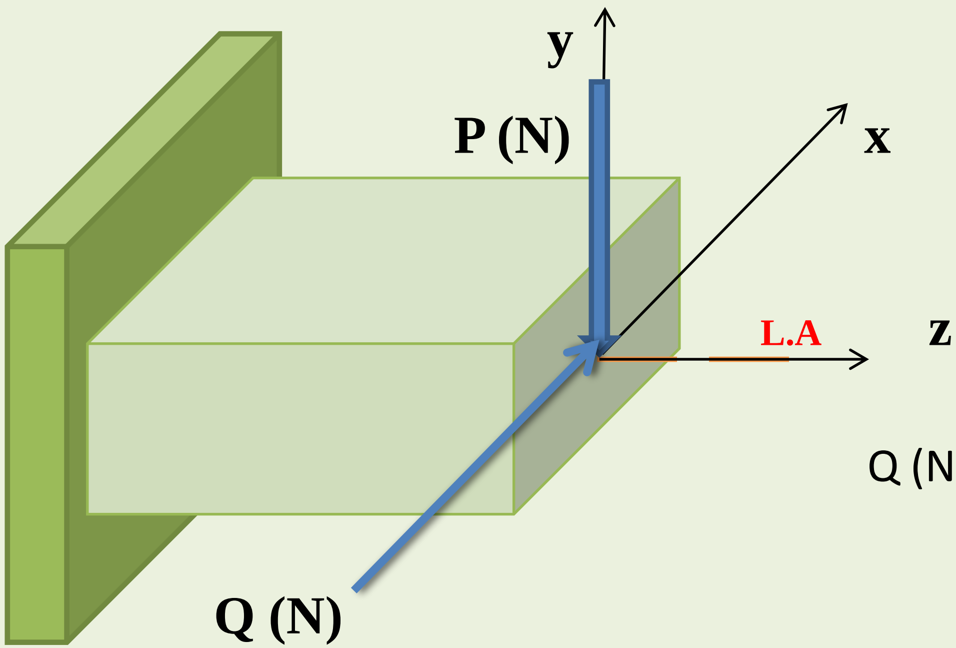
- If the line of action of force is parallel to the plane of cross-section is known as shear load.
- The line of action of force is perpendicular to the longitudinal axis.
- The line of action of force is parallel to the centroidal axis. It may or may not pass through the centroid of the x-s/c.
-

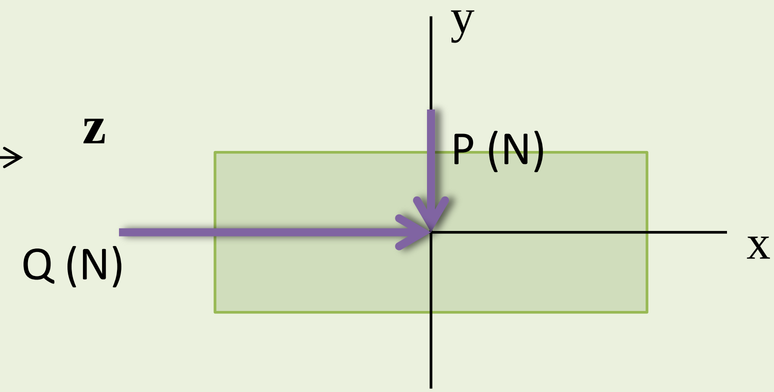


- x & y are the centroidal axes.
- z is a polar axis.

Transverse shear load (TSL):

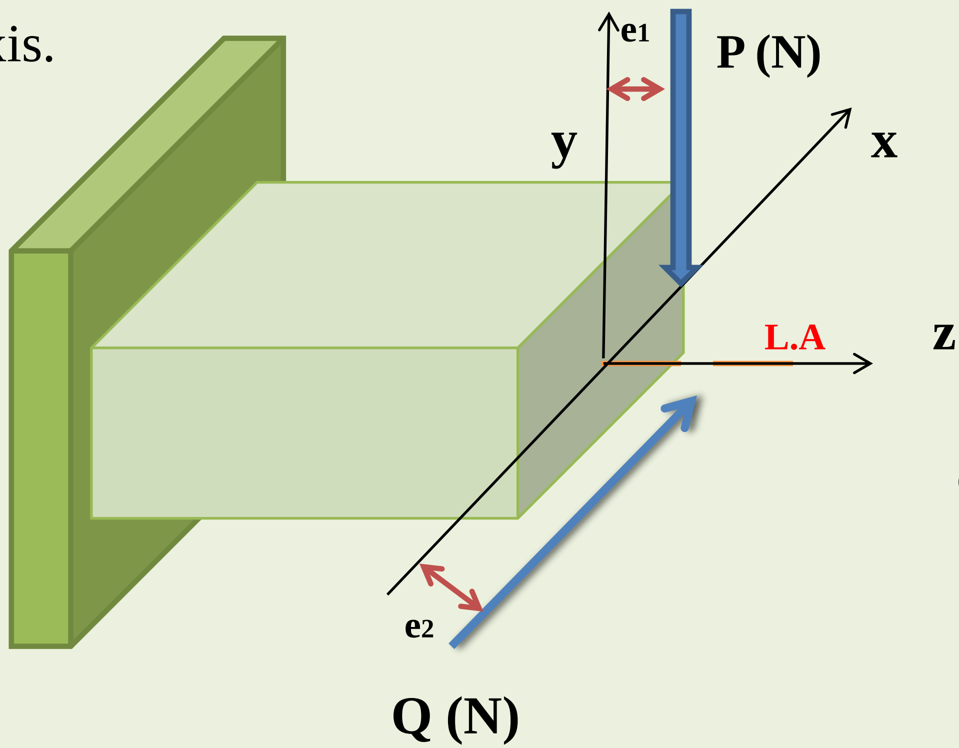
- The line of action of force is parallel to the plane of x-s/c and pass through the centroid of the x-s/c.

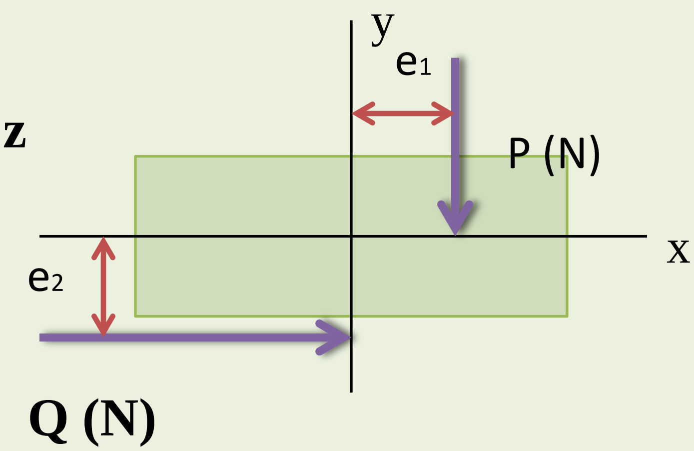




Eccentric Transverse shear load (ETSL):

- The line of action of force is parallel to the plane of x-s/c and but not pass through the centroid of the x-s/c having eccentricity (e) from the centroidal axis.





Representation of structure member in 2D

- For the simplification any structure member (like beam, column etc.) is represented by its longitudinal axis (LA) and its support condition.

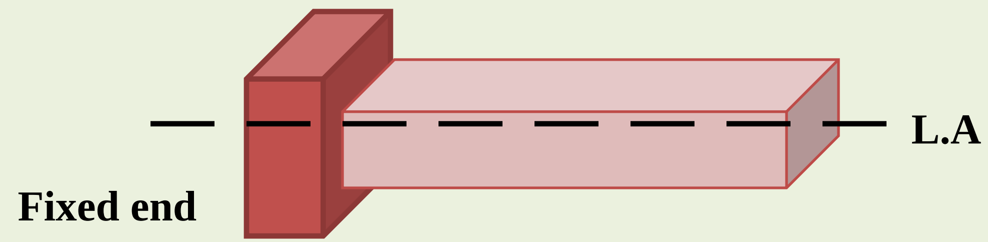
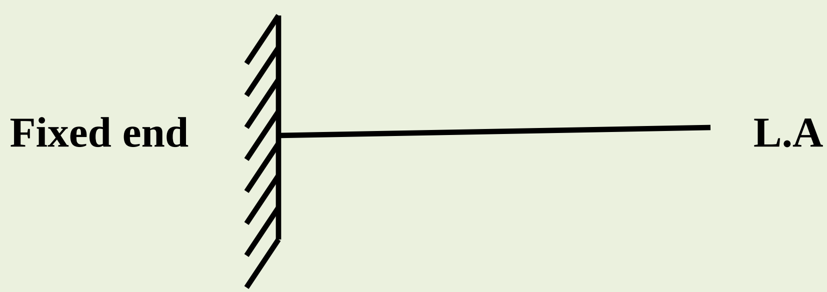


Figure- Cantilever beam (one end is fixed & one end is free)

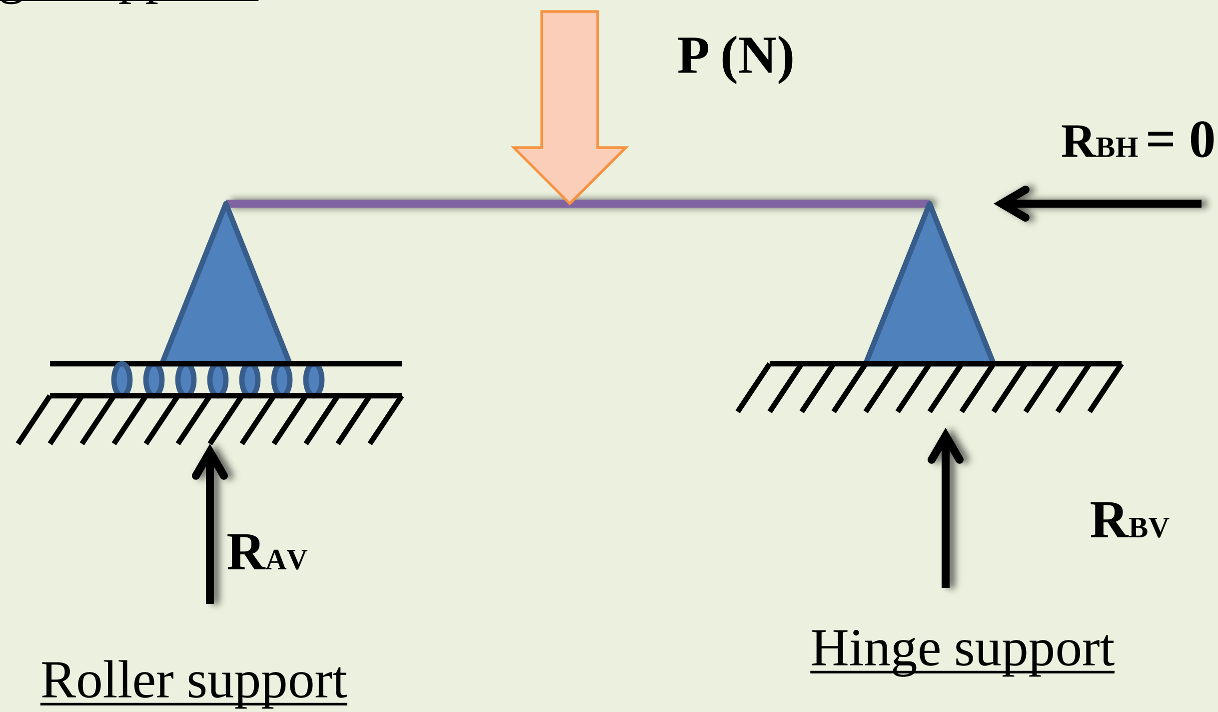


Support & its representation

Number of reactions at any support	=	Number of restricted motion by that support

- Supports
  - Simple Support
    - Roller Support
    - Hinge support
  - Fixed Support

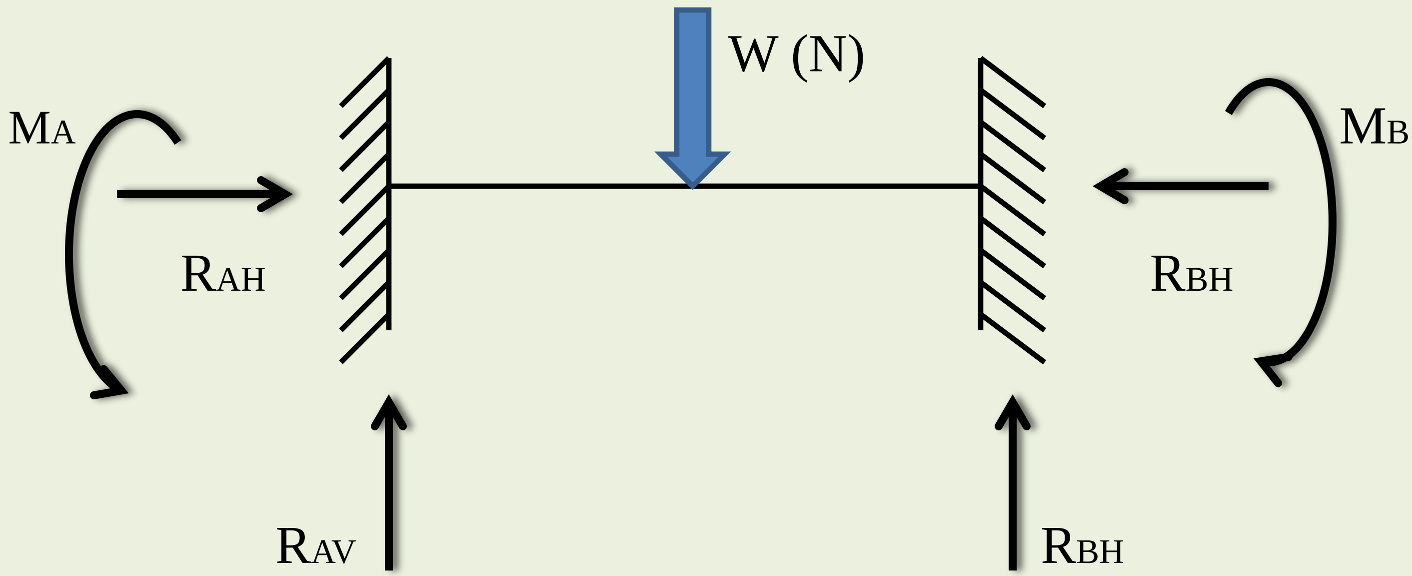
Roller & Hinge Support :



- Roller support only restrict the vertical motion.
- So only vertical reaction is present.
- R_{AV} is the vertical reaction.

- Hinge support restrict the vertical & horizontal motion, rotation is permitted.
- So only vertical (**R_{BV}**) & horizontal reaction (**R_{BH}**) is permitted.

Fixed Support:



- At fixed support vertical, horizontal and rotation motion all are restricted.
- So vertical, horizontal and moment reactions will act at the support.

Important Point

- If there is not any horizontal external loading in member, there will be no any horizontal reaction act at support.
- In simple support there will be no moment reaction act at the support or we can say that moment reaction will be zero.
- In fixed support if there is only vertical external loading, due to vertical loading both vertical & moment reactions will produced.

Beams

(It is a structure member which is subjected to transverse shear load TSL)

Beams

- Statically determinate beams
  (Number of reaction determine by using static equilibrium equation)
  - Simply supported beam
  - Over hanging beam
  - Cantilever beam
- Statically indeterminate beams
  (Total no. of reaction in the beam) > (No. of useful total static equilibrium)
  - Fixed beam
  - Propped cantilever beam
  - Continuous beam

ANKIT SAXENA
(saxena01ankit@gmail.com)

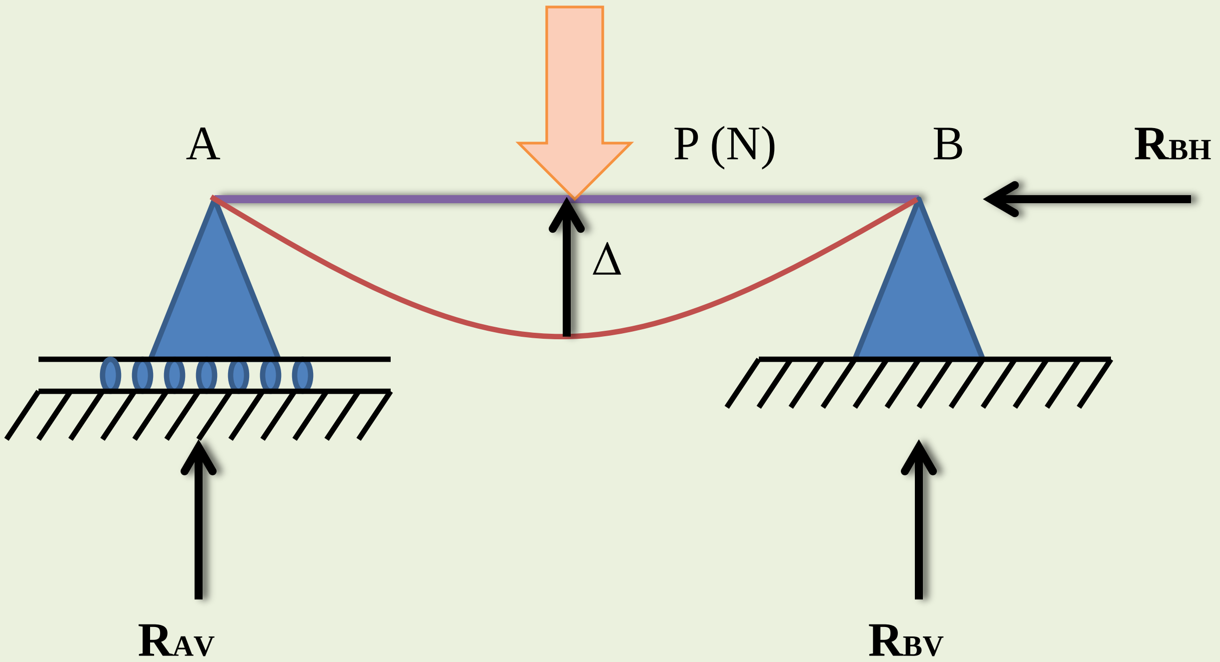
Beam representation

- Longitudinal axis
- Types of support (Roller, hinge or fixed)
- Span length
- Loads acting on it. (External loads & support reactions)

Span length: It is nothing but the total length of the beam structure.

Simply supported beam (S.S.B):

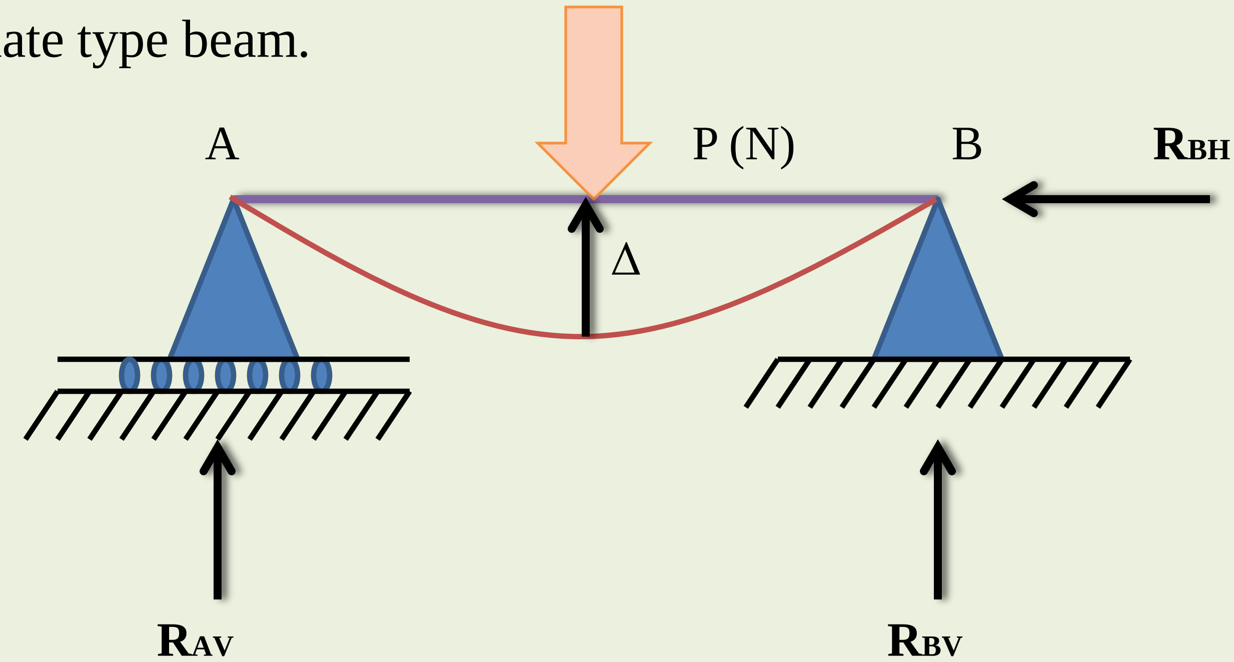
- It has only 2 support (roller or hinge type) at the end.
- The end deflection is zero i.e. deflection at the support is zero.
- Maximum deflection occurs between the support.



- If the load is applied at the mid of beam span, the maximum deflection occurs at the centre of beam.

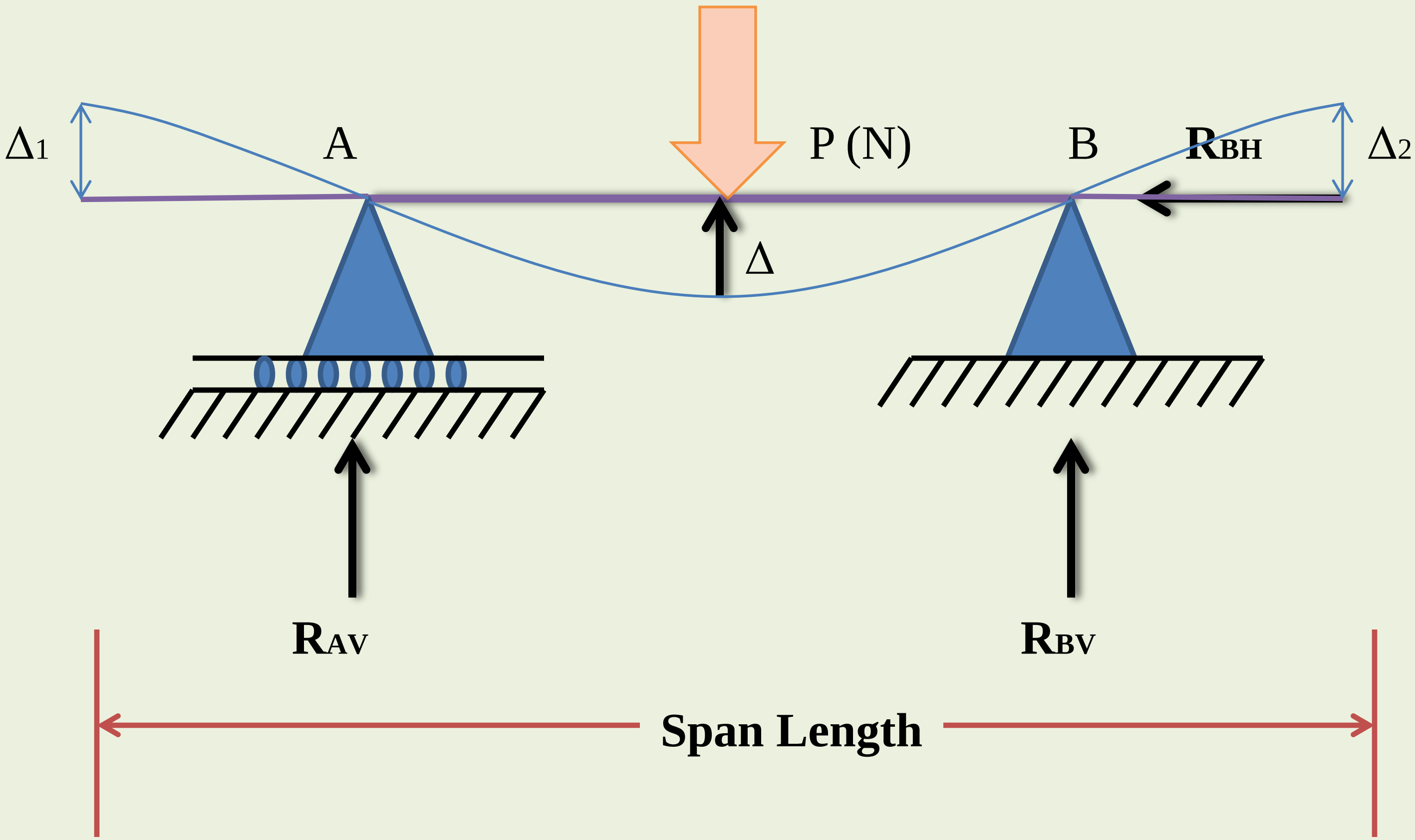
Simply supported beam (S.S.B):

- The reaction $R_{BH} = 0$, because there is not any horizontal external loading on the beam.
- The S.S.B consists of one hinge and one roller support is always statically determinate type beam.



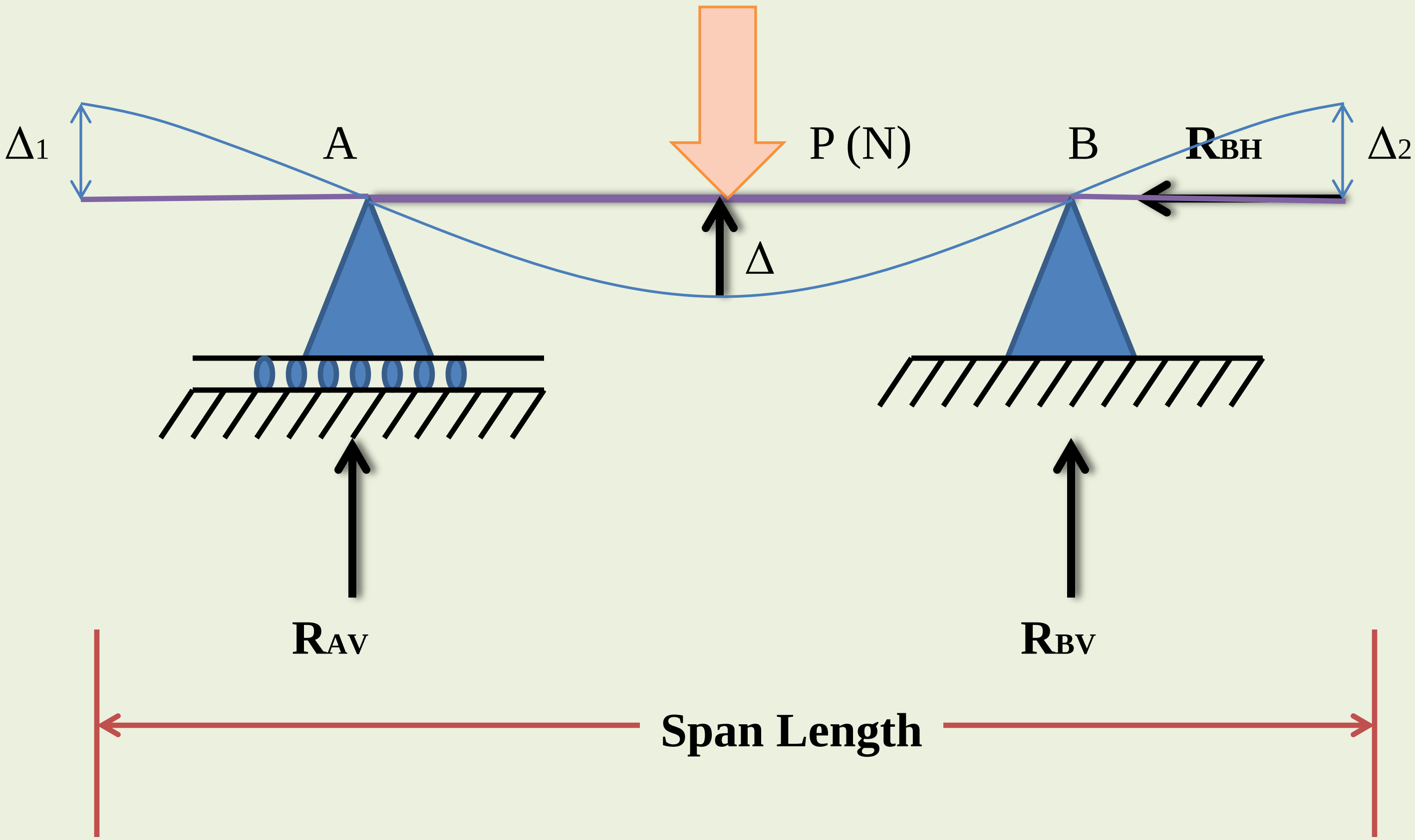
Over hanging beam :

- In over hanging beam we use only 2 support but the location of the support is not at the ends. It is modified form of S.S.B.
- Supports are used in between the ends.



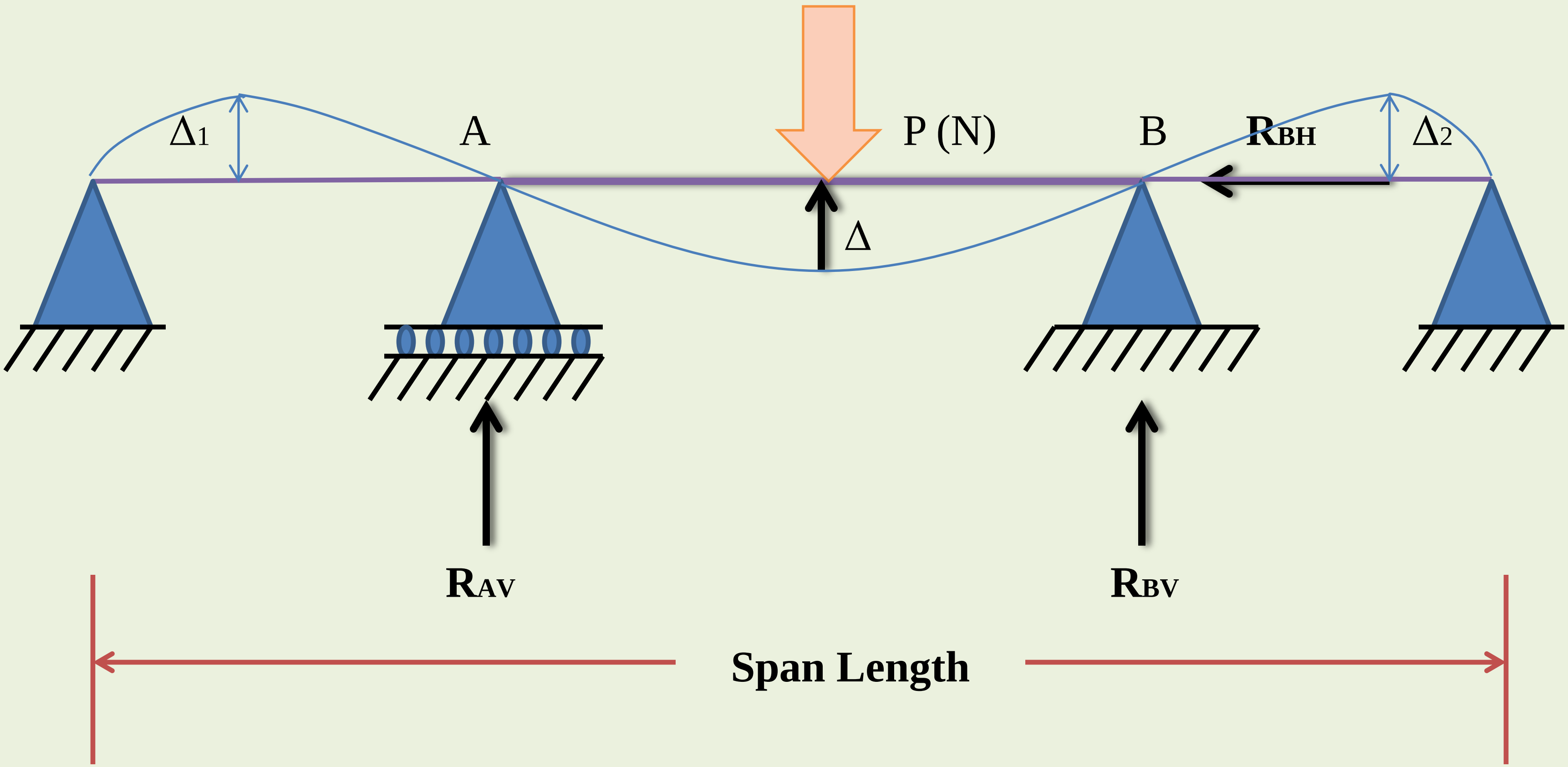
Over hanging beam :

- By the use of over hanging beam, we reduce the centre deflection.
- But now end deflection Δ_1 & Δ_2 produced.



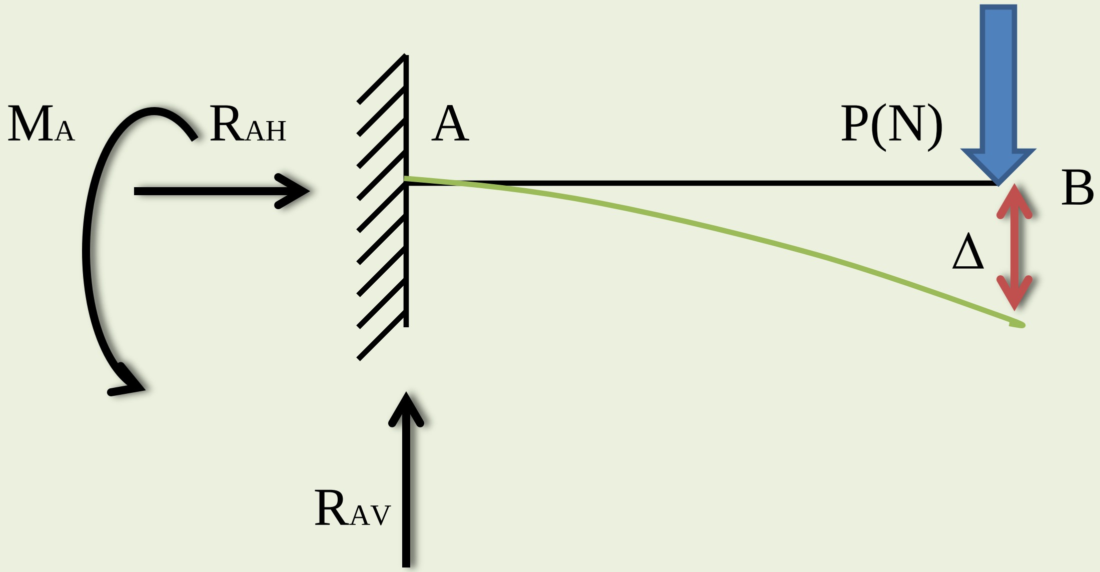
Continuous beam :

- It is the modified form of over hanging beam.
- Here we used more than 2 supports.
- It is always statically indeterminate type beam structure.



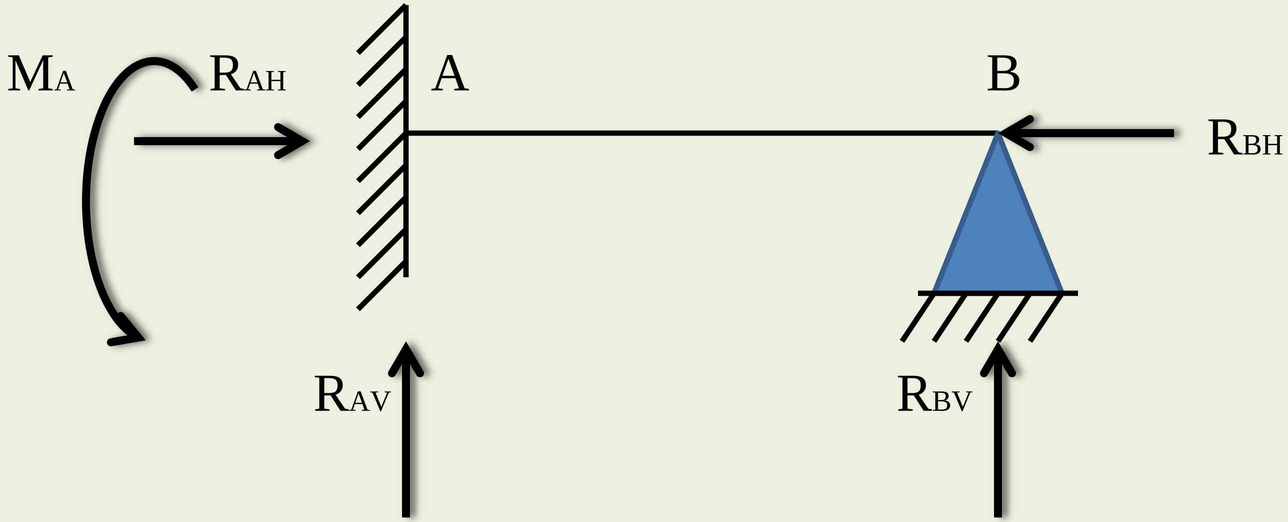
Cantilever Beam :

- This type of beam is having one end is fixed and one end is free.
- It is always statically determinate type beam.
- Deflection at the fixed end is zero and at the free end is maximum.



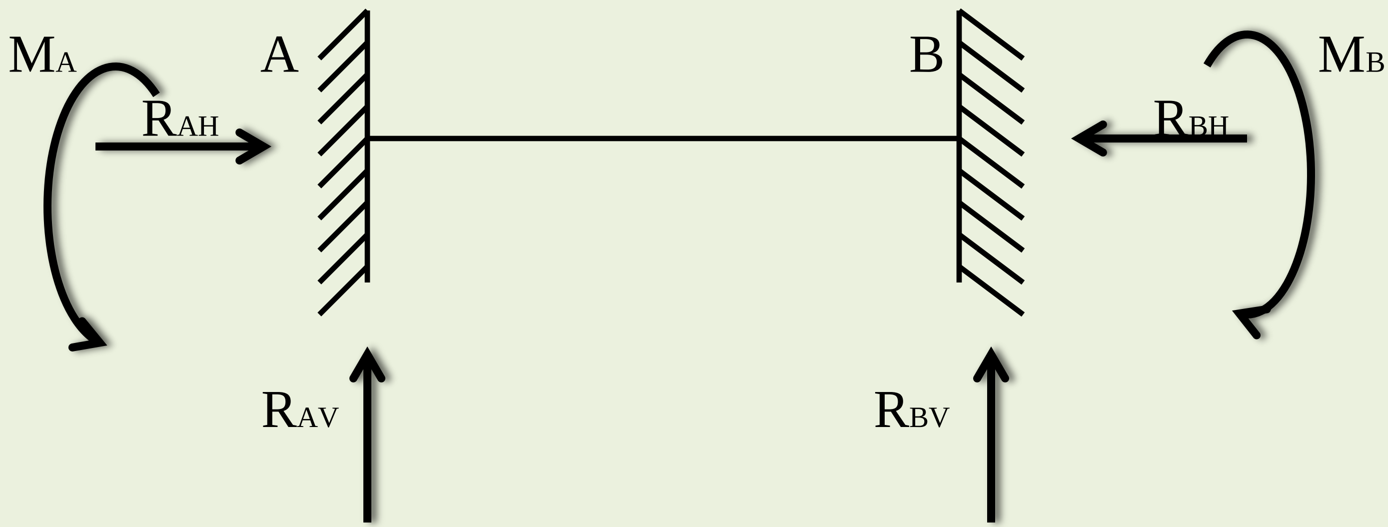
Propped Beam :

- This beam is a modified form of cantilever beam. To avoid end deflection we used simply support at the free end.
- It is always statically indeterminate type beam.



- For finding reactions in such beam we used compatibility equations.

Fixed beam:



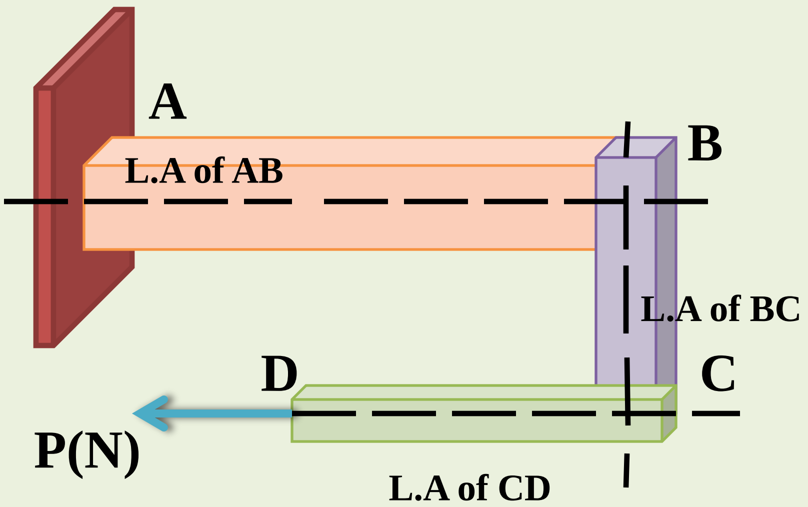
- It is always a statically indeterminate type of beam.

Load Summary

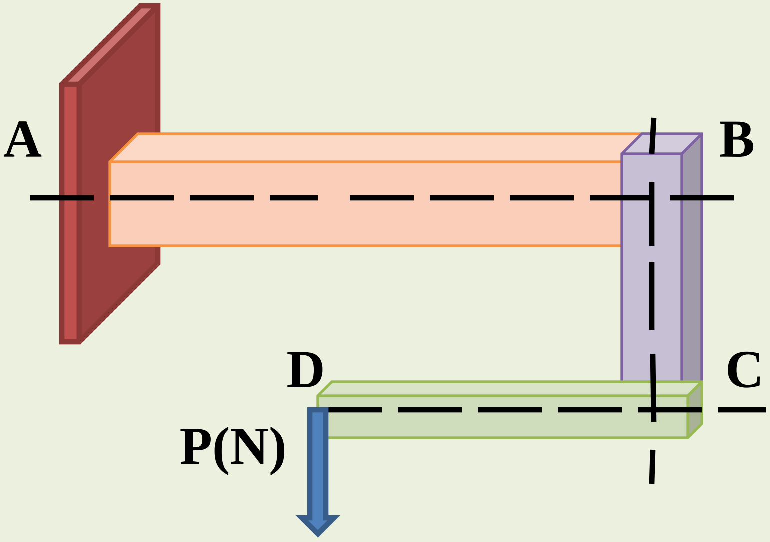
Load	W.r.t Plane of x-s/c	W.r.t. longitudinal axis
Axial Load (AL)	Line of action of force **Perpendicular** to the plane of X-S/C	Line of action of force **along** the longitudinal axis
Eccentric axial load (EAL)	Line of action of force **Perpendicular** to the plane of X-S/C	Line of action of force **parallel** to the longitudinal axis and having eccentricity
Transverse shear load (TSL)	Line of action of force **parallel** to the plane of X-S/C	• Line of action of force perpendicular to the longitudinal axis. • Line of action of force passing through the longitudinal axis.

Load	W.r.t Plane of x-s/c	W.r.t. longitudinal axis
Eccentric Transverse shear load (ETSL)	Line of action of force **parallel** to the plane of X-S/C	• Line of action of force perpendicular to the longitudinal axis. • Line of action of force will not pass through the centroid of the x-s/c. • There is a eccentricity between the line of action of force and longitudinal axis.

Q 1. For the given structure made of AB, BC & CD member. A external load P (N) is apply at the free end of member CD. Determine the which type of load (AL, TSL & ETSL) with respect to member AB, BC & CD.



Q 2. For the given structure made of AB, BC & CD member. A external load P (N) is apply at the free end of member CD. Determine the which type of load (AL, TSL & ETSL) with respect to member AB, BC & CD.



Bending couple

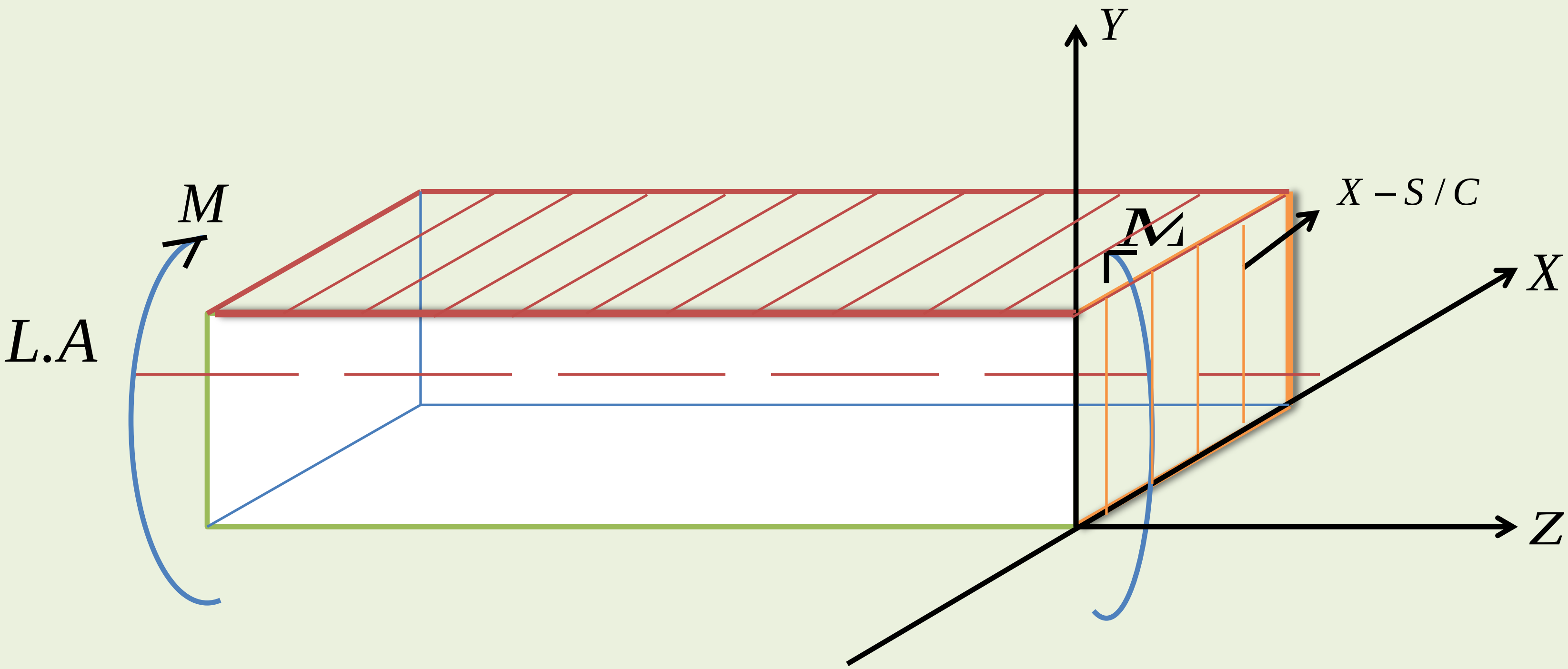
- A couple is said to be a bending couple when plane of the couple is along the longitudinal axis of the member.

(or)

- A couple is said to be a bending couple when plane of the couple is perpendicular to the plane of x-s/c of the member.

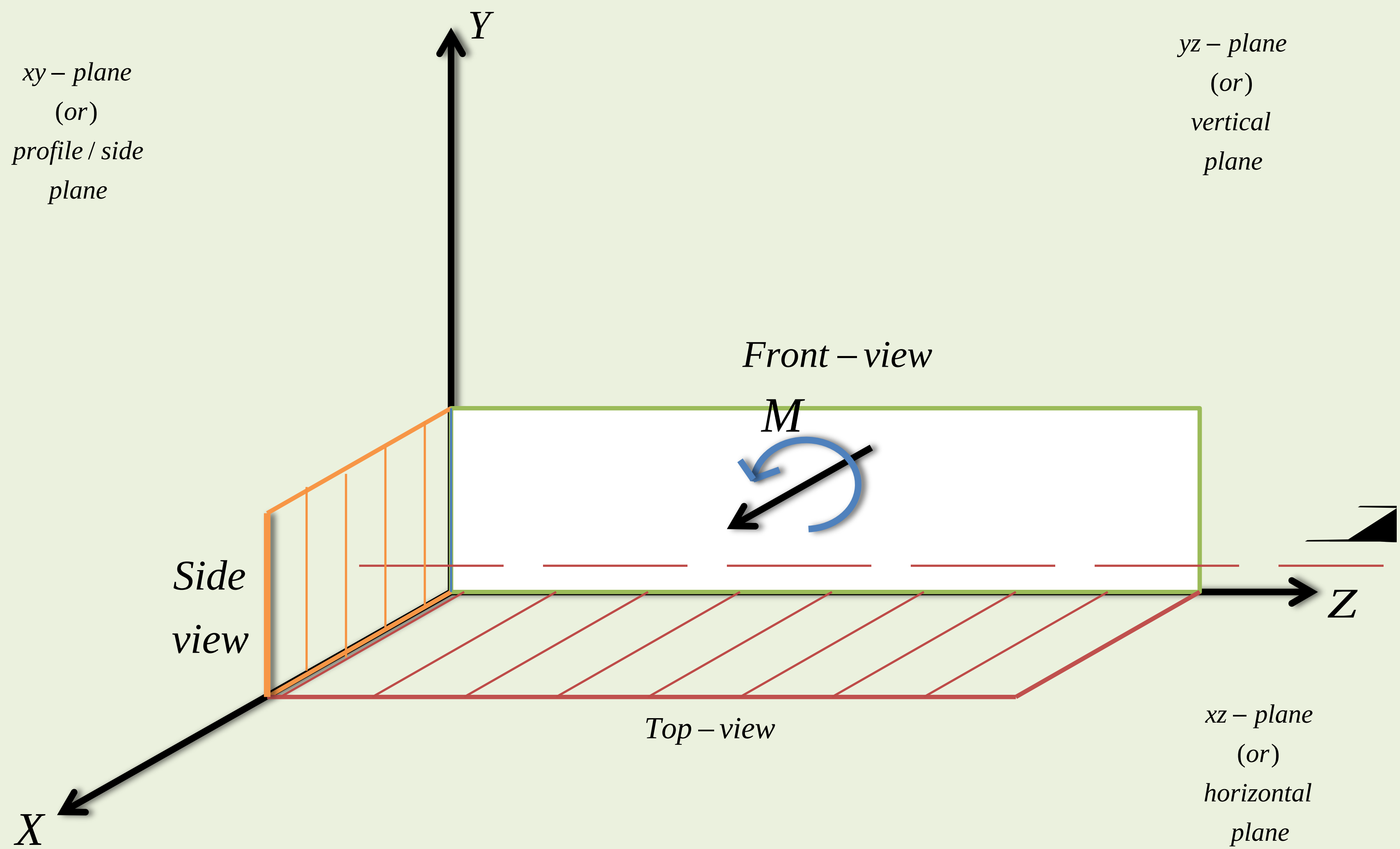
(or)

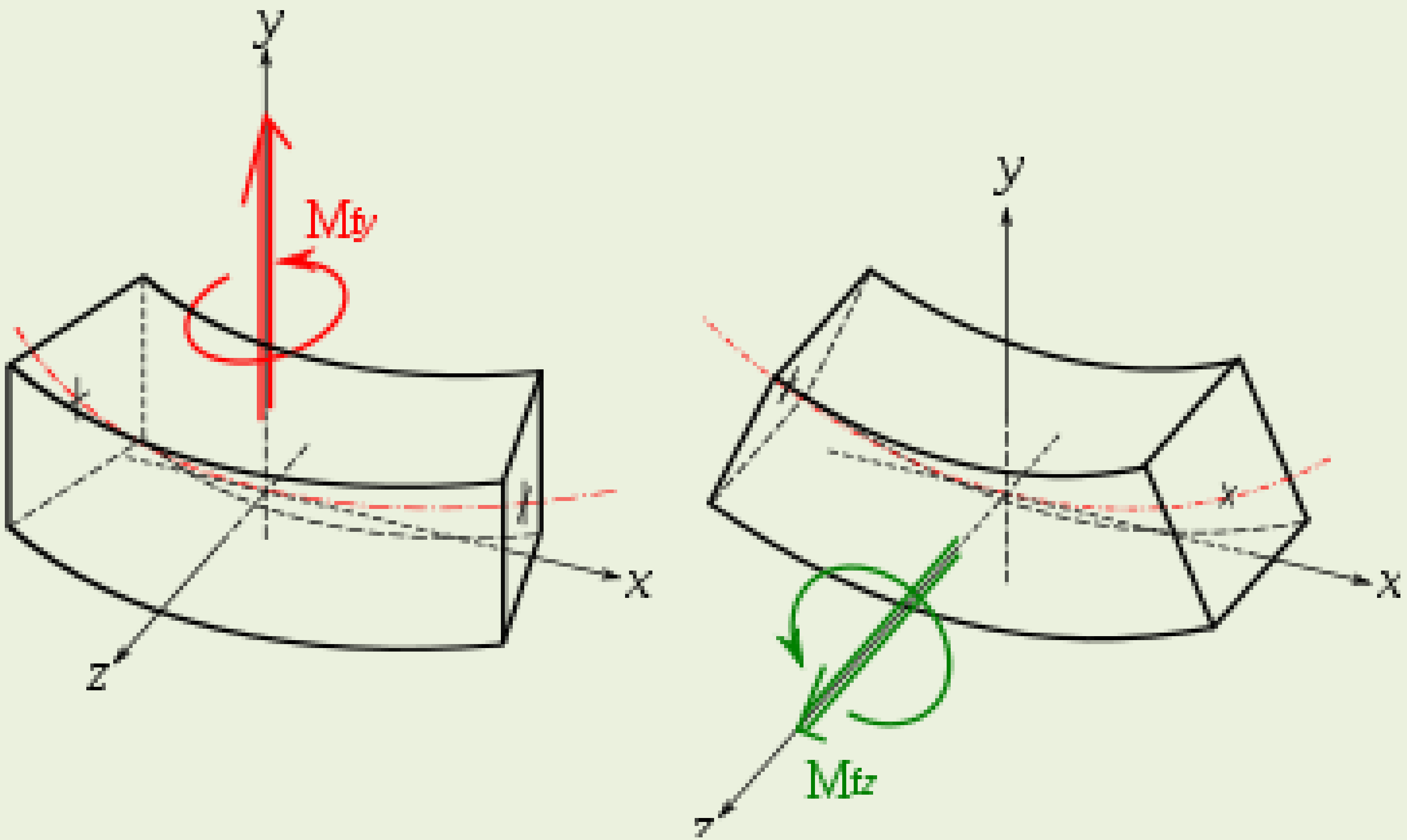
- A couple is said to be a bending couple when it acts about a centroidal axis which is in the plane of x-s/c.



Y
M
X − S / C
X
L.A
Z

Orthographic Projection




y
Mfy
x
z
y
x
Mfz
z

Important points

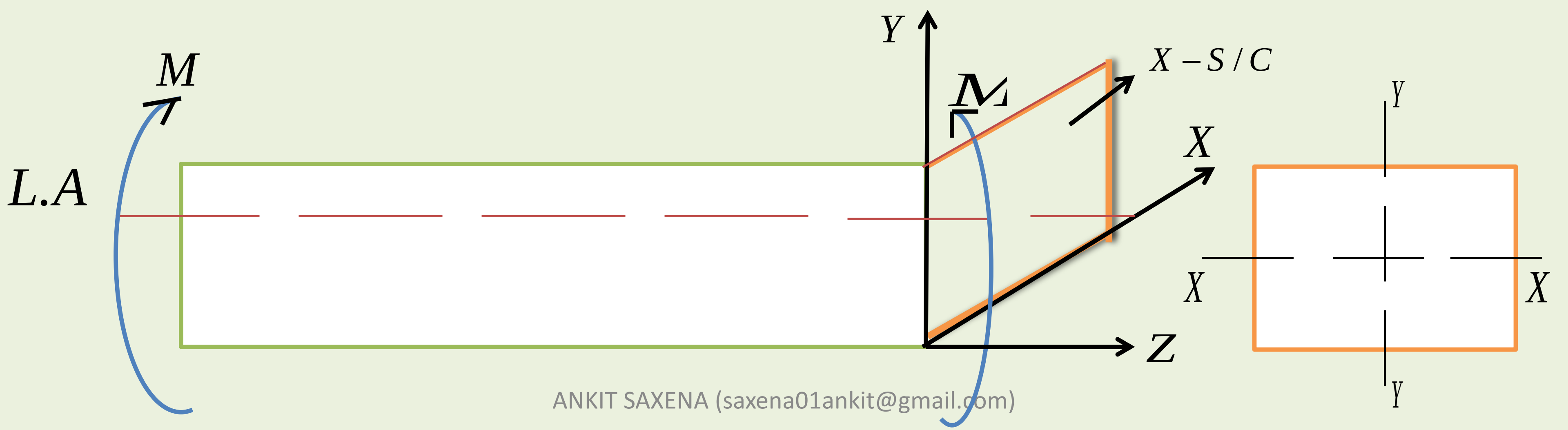
- Y-Z plane = Vertical plane
- X-Y plane = Profile or side plane
- X-Z plane = Horizontal plane
- X-section will lie is profile profile plane
- Longitudinal axes will lie is vertical plane
- Bending couple can be lie in vertical plane , horizontal plane or both the plane
- X & Y are the centroidal axes.
- Bending couple will lie along the centroidal axes.

Type of Bending

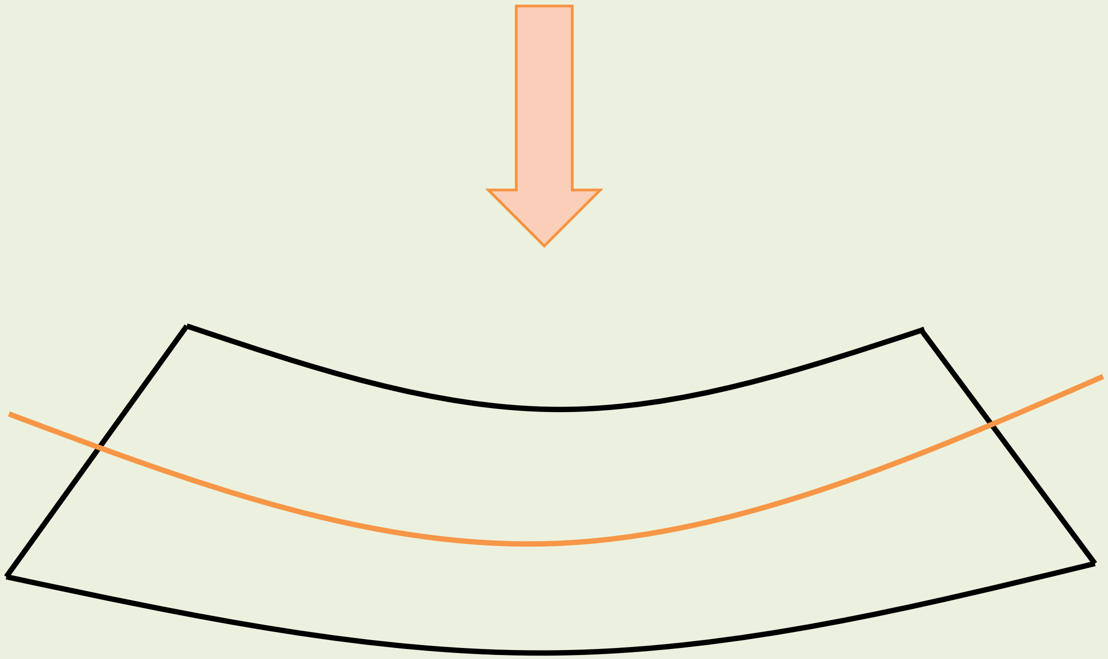
$$Bending$$

Sagging (or) $+ve$ Bending	Hogging (or) $-ve$ Bending

Consider a bar having rectangular x-section is subjected to a sagging (+ve type) bending moment.



After apply Bending moment

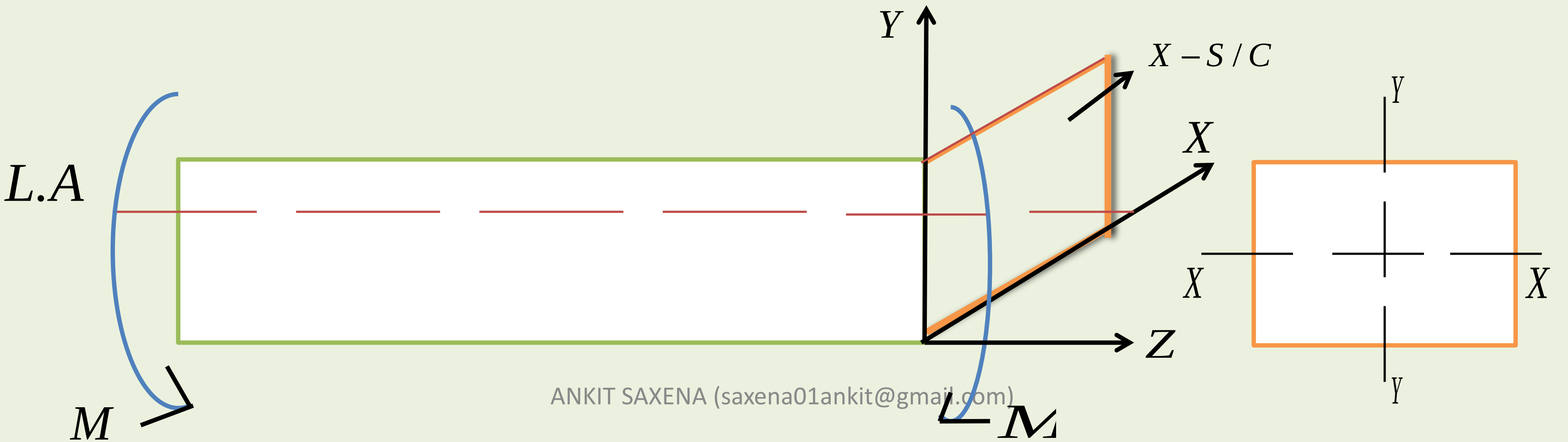


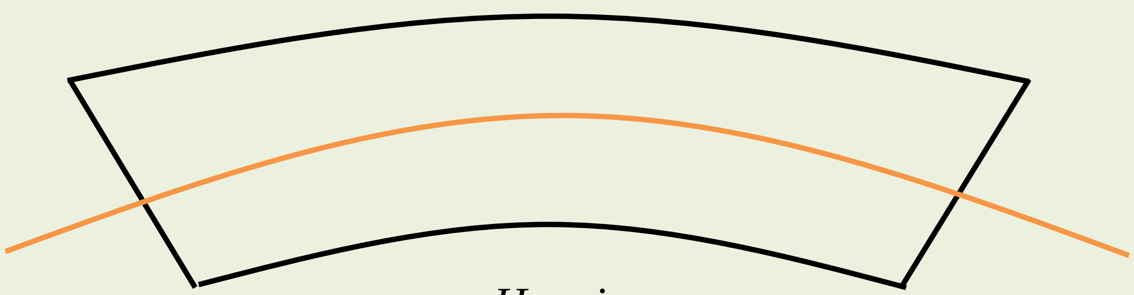
Sagging

(or)

$+ve$ *Bending*

Consider a bar having rectangular x-section is subjected to a hogging (-ve type) bending moment.





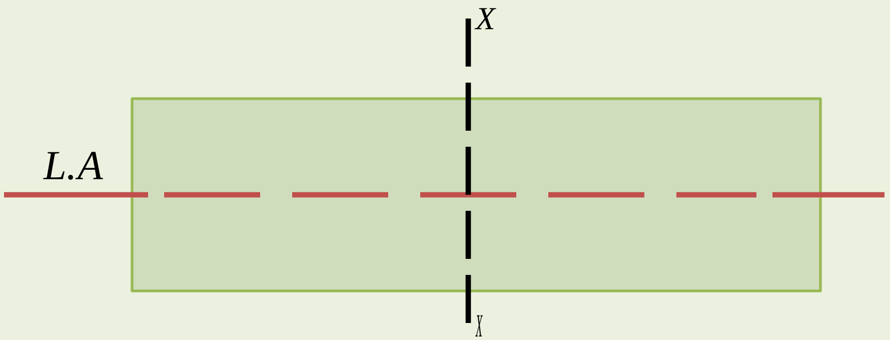
Hogging

(or)

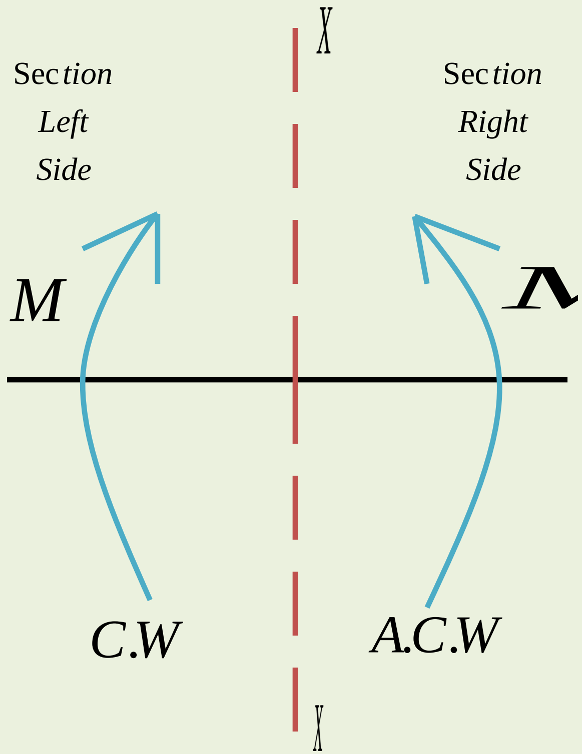
$-ve$ *Bending*

Bending Sign Convention

- Cut a section x-x in member length anyway.



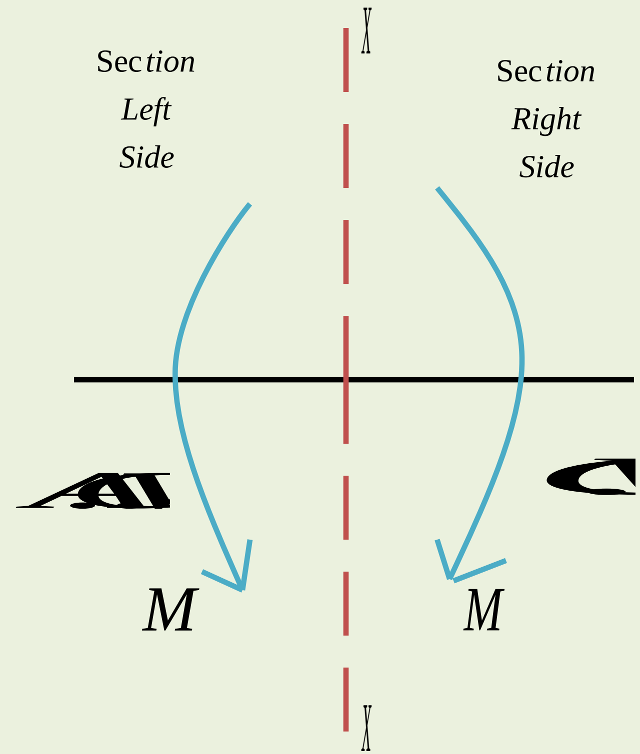
- If the moment direction right side of the section x-x is anticlockwise & moment direction left side of the section is clockwise, consider as a sagging type bending or +ve type bending.
- If the moment direction right side of the section x-x is clockwise & moment direction left side of the section is anticlockwise, consider as a hogging type bending or -ve type bending.



Sagging

(or)

+ve Bending



Hogging

(or)

−ve Bending

Torsional or twisting couple

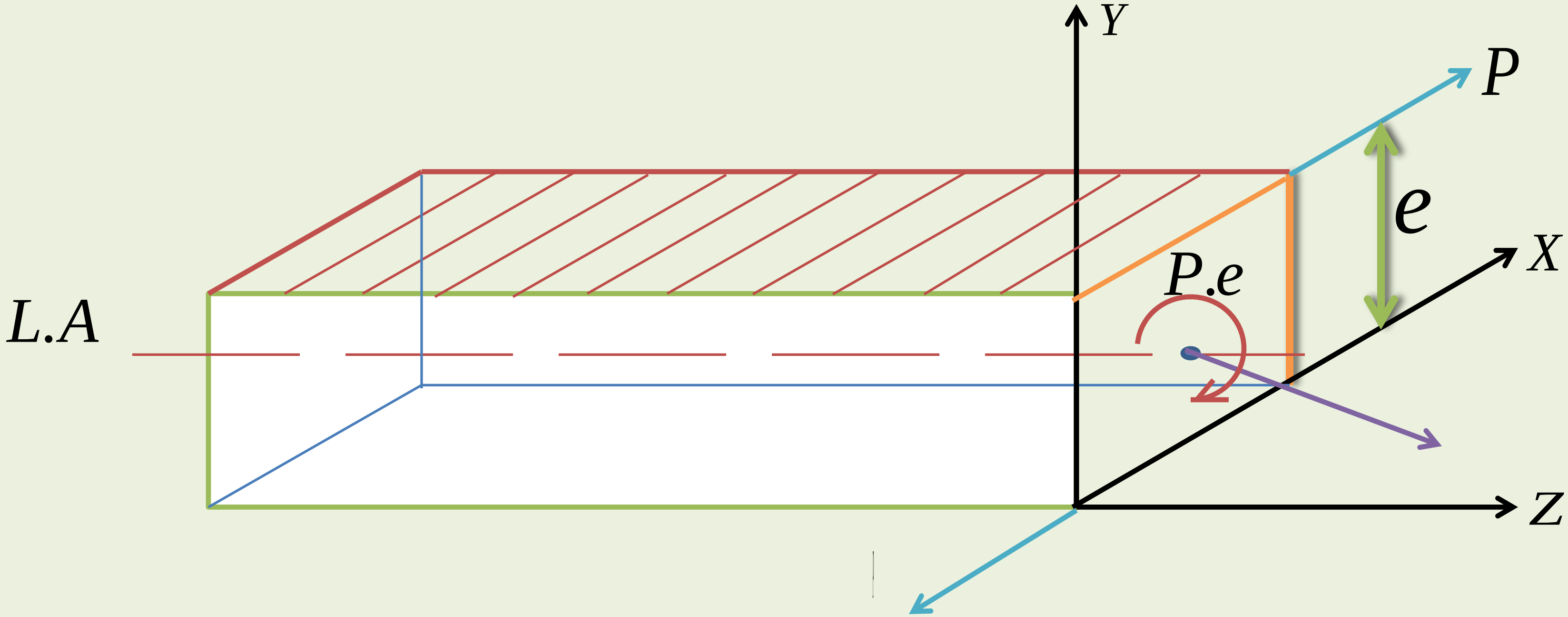
- A couple is said to be twisting couple when the plane of the couple is perpendicular to the longitudinal axis of the member.

(or)

- A couple is said to be twisting couple when the plane of the couple is parallel to the plane of cross section.

(or)

- A couple is said to be twisting couple when the plane of the couple is perpendicular to the polar axis.



P x e = Torsional or Twisting couple.

YX - Side (or) profile plane

YZ - Vertical plane

XZ - Horizontal plane

From the above figure,

Plane of couple (P x e) is in the side plane i.e. YX plane.

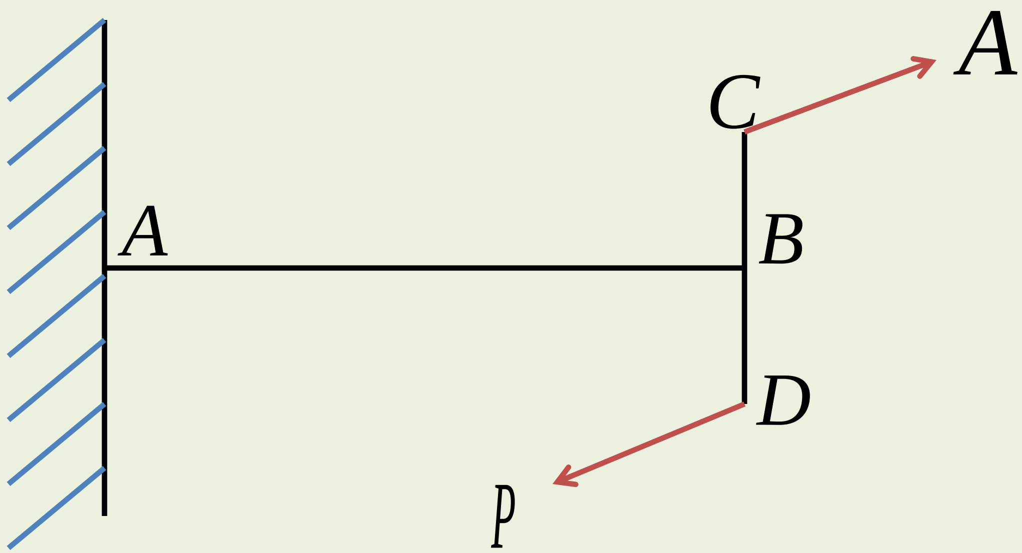
$\otimes$ ⟹ Represents line of action of force $\perp$ to the plane of paper & in inwards direction.

$\odot$ ⟹ Represents line of action of force $\perp$ to the plane of paper & in outwards direction.

Example:

$AB \rightarrow Rod$

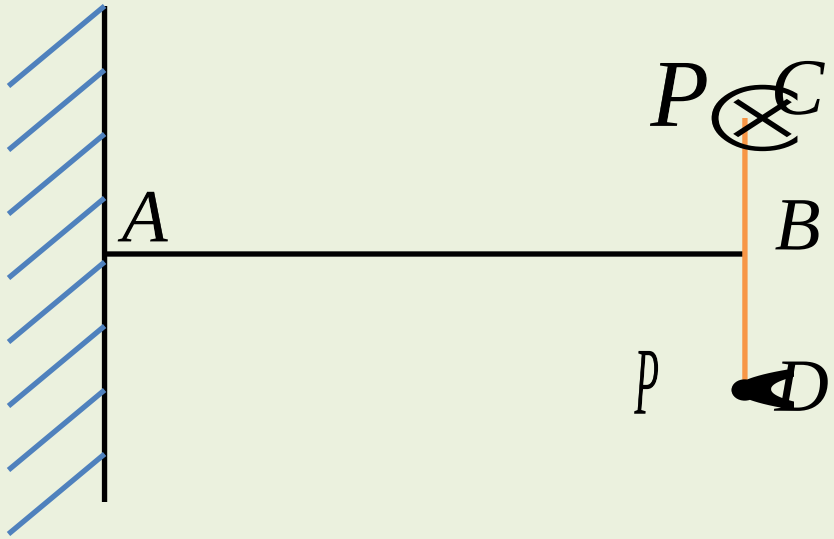
$CD \rightarrow Lever$



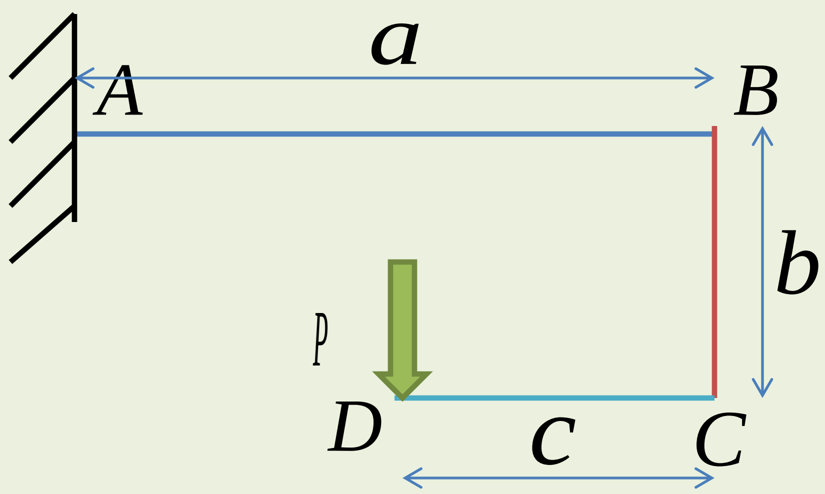
It can also be represent as-

$$AB \rightarrow Rod$$

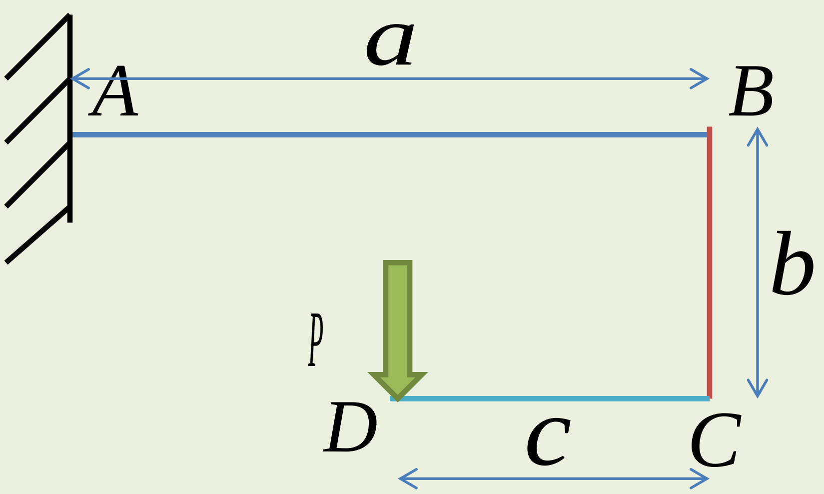
$$CD \rightarrow Lever$$



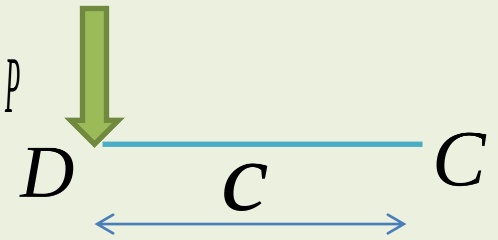
Q 3. For the given structure made of AB, BC & CD member. A external load P (N) is apply at the free end of member CD. Determine the which type of load (AL, TSL, ETSL, Bending & torsion) with respect to member AB, BC & CD.



Solution:

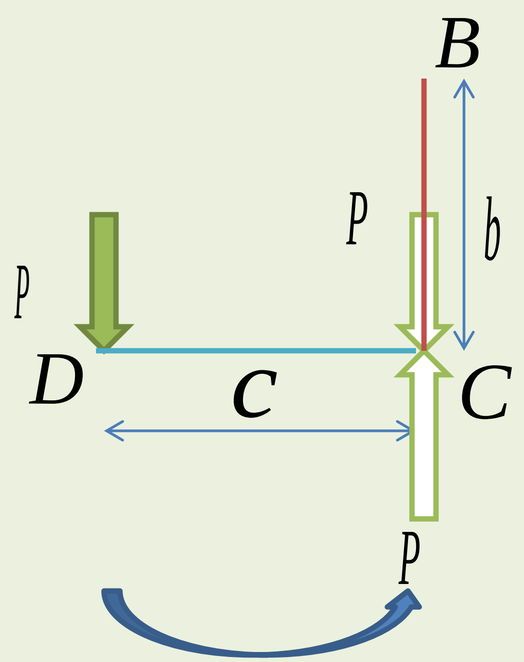


<u>For member CD</u>



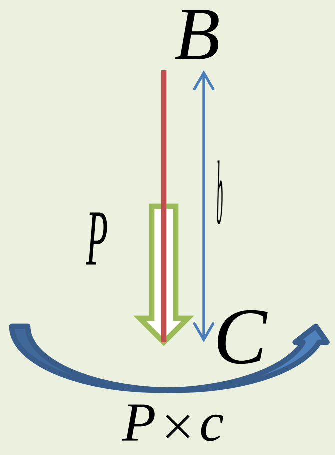
(load P act as transverse shear load)

<u>For Member BC</u>

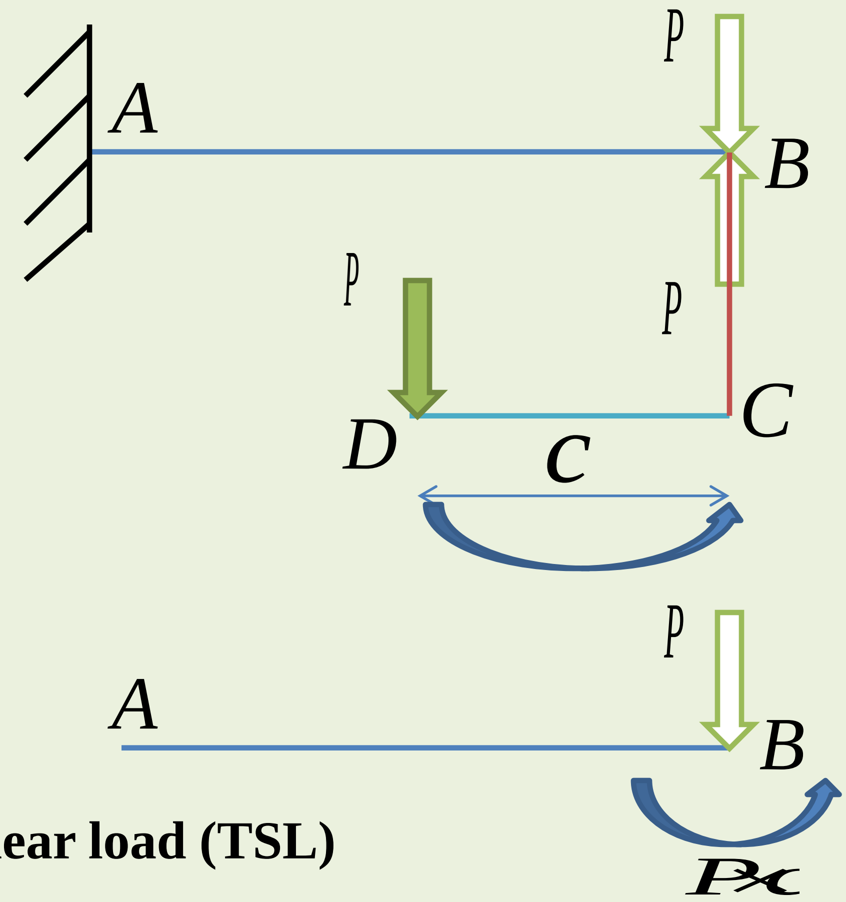


P act as axial tensile load

P x c act as bending couple



<u>For member AB</u>



P act as transverse shear load (TSL)

P x c act as bending couple.

Loading Diagram

(AXIAL LOADING)

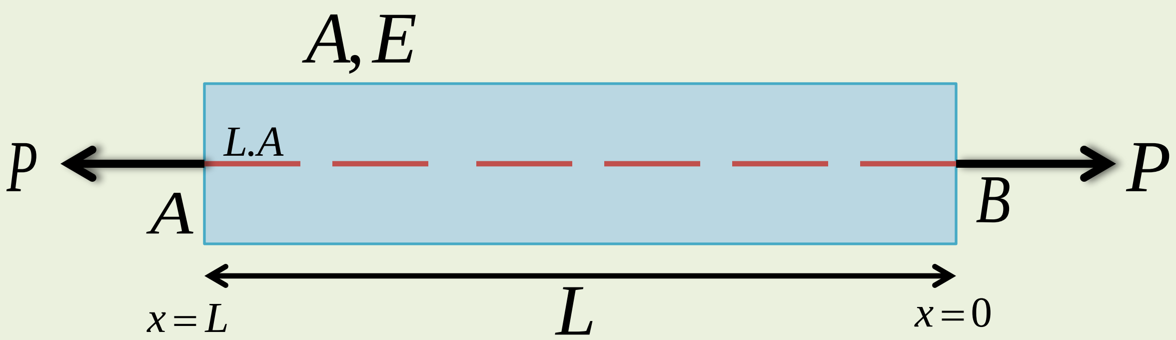
Axial Loading Diagram

Pure Axial Loading :

A bar is said to be under pure axial loading when it is subjected two equal and opposite axial loads in such a way that the magnitude and direction of axial load remains constant through out the length of the member.

Axial load = Constant

Shear Force = Bending moment = Twisting Moment = 0



Important

As per the method of section,

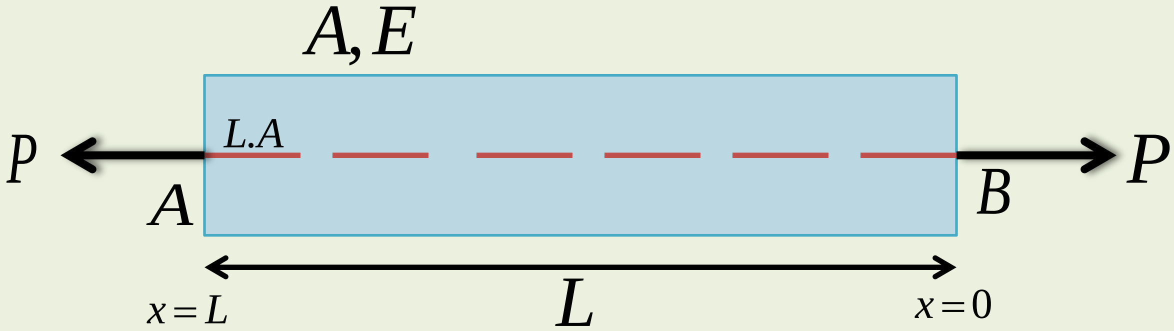
"The load acting at any x-s/c is algebraic sum of the corresponding loads acting either on the left hand side (L.H.S) of the x-s/c or on the right hand side (R.H.S) of the x-s/c. "

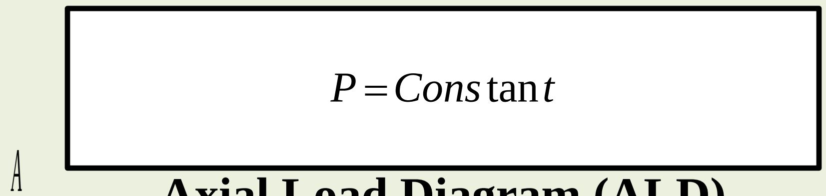
$$\sigma = \frac{P}{A}$$

$$\delta l = \frac{PL}{AE}$$

Conditions-

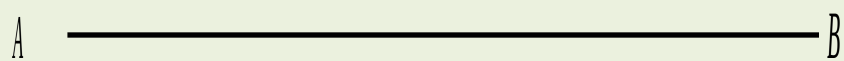
1. Bar should be prismatic.
2. Bar should be under pure axial load.
3. Bar should made of homogeneous & isotropic material.



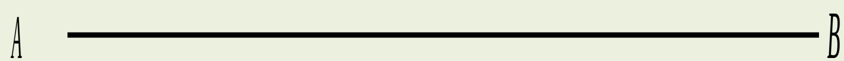


Axial Load Diagram (ALD)

$P = Cons\,t$



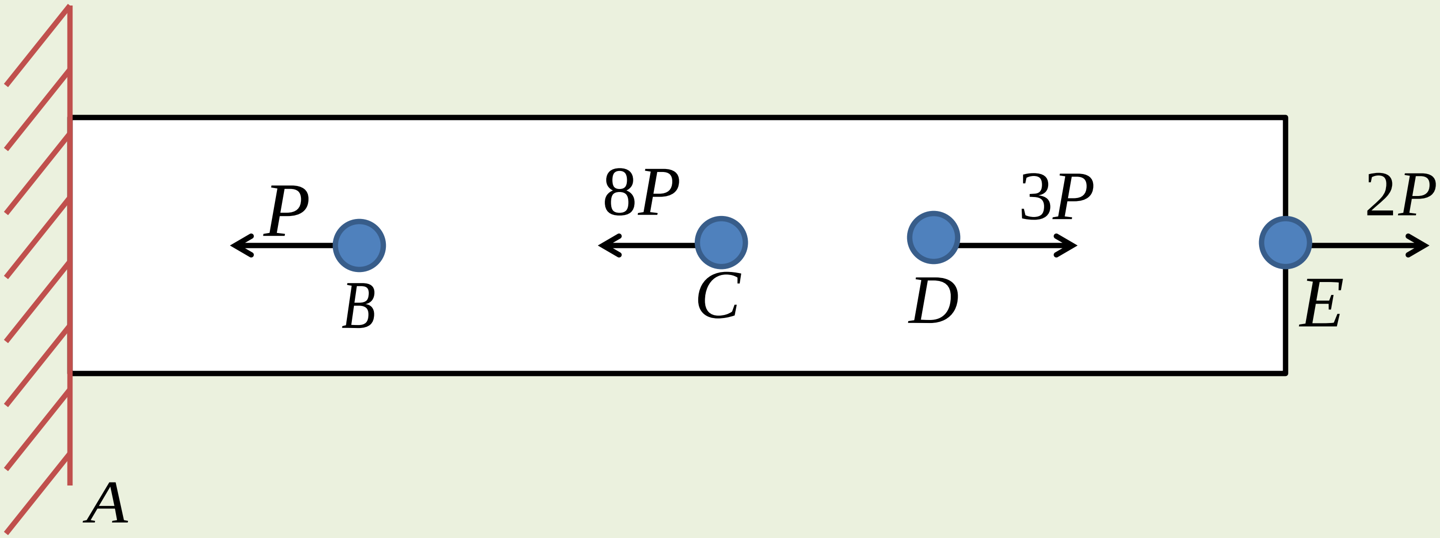
SFD



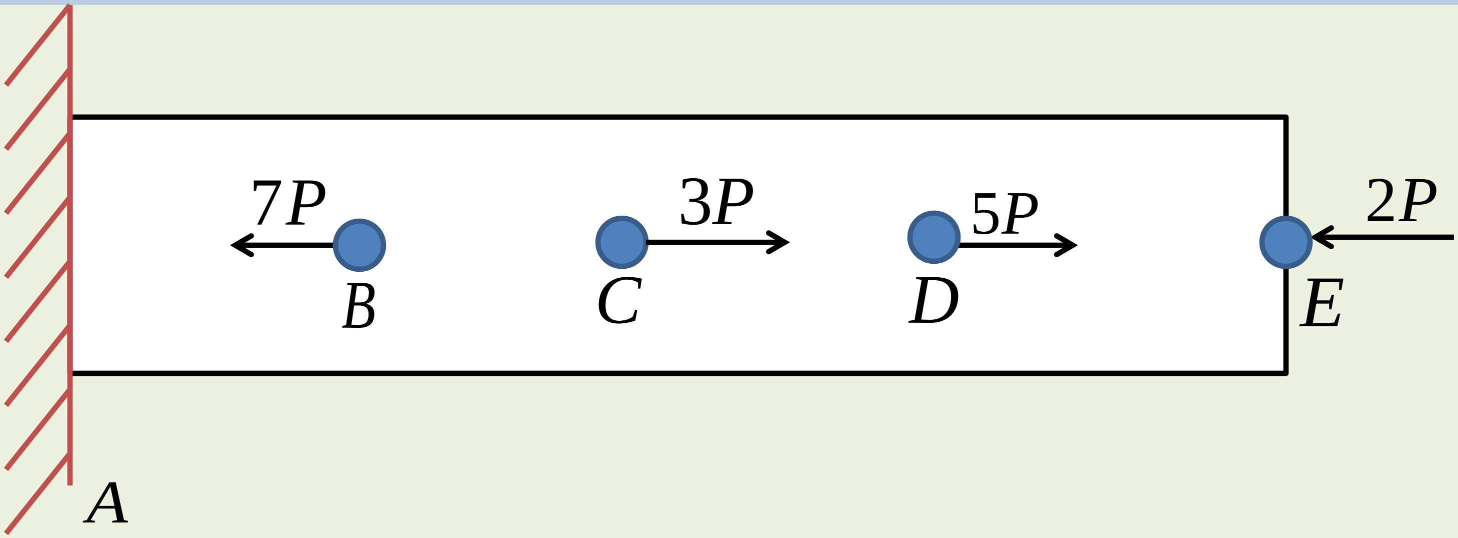
BMD

TMD

Q1. For the bar as shown in the figure determine maximum tensile load & maximum compressive load acting on the x-s/c of bar.



Q2. For the bar as shown in the figure determine the following maximum tensile load & maximum compressive load acting on the x-s/c of bar. Also determine the axial loads acting at thex-s/c A,B,C,D &E.



LOADING DIAGRAM

BENDING MOMENT DIAGRAM

(BMD)

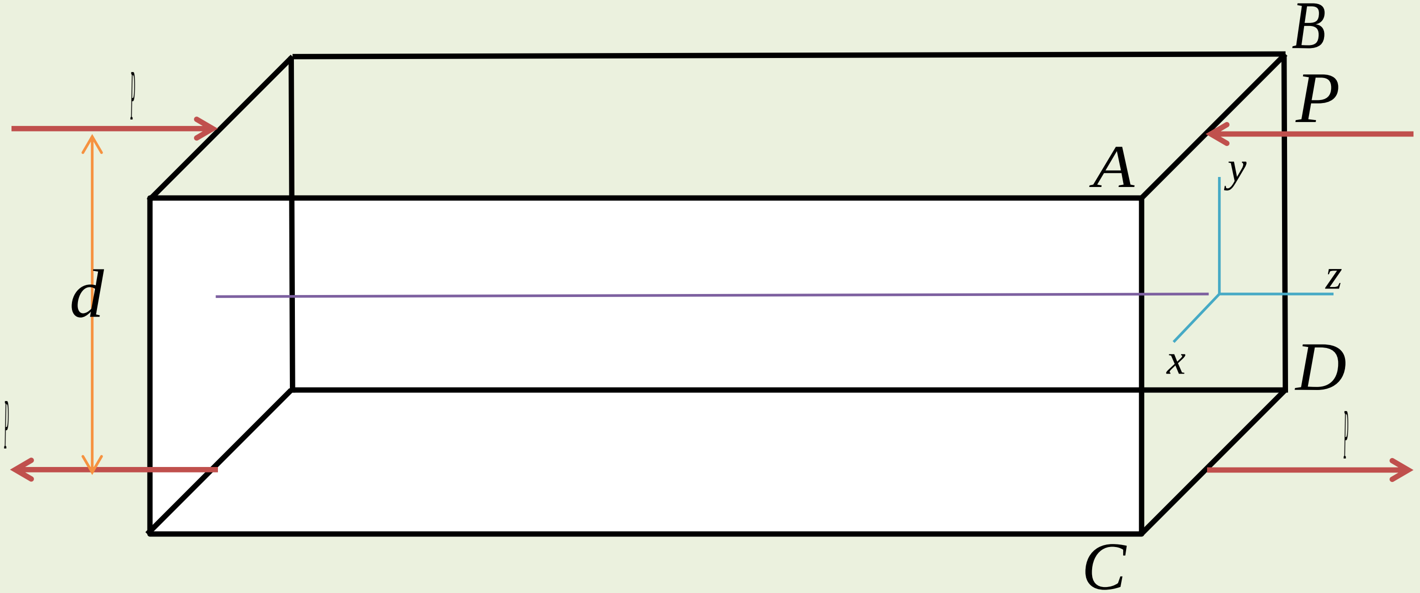
Pure Bending

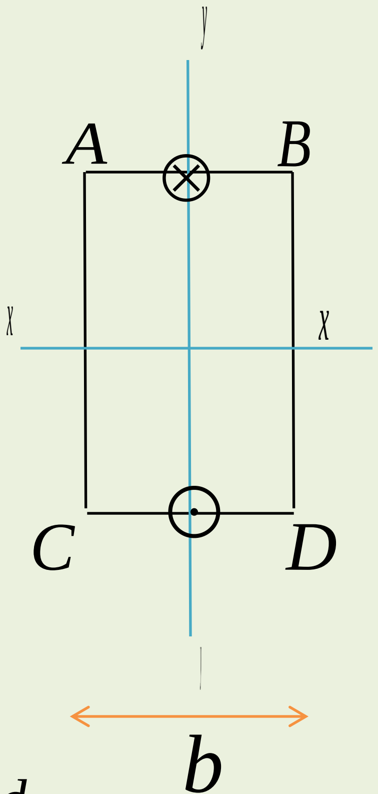
A member is said to be under pure bending when it is subjected two equal and opposite couples in a plane along the axis of the member (Longitudinal axis) in such a way that the magnitude of bending moment remain constant.

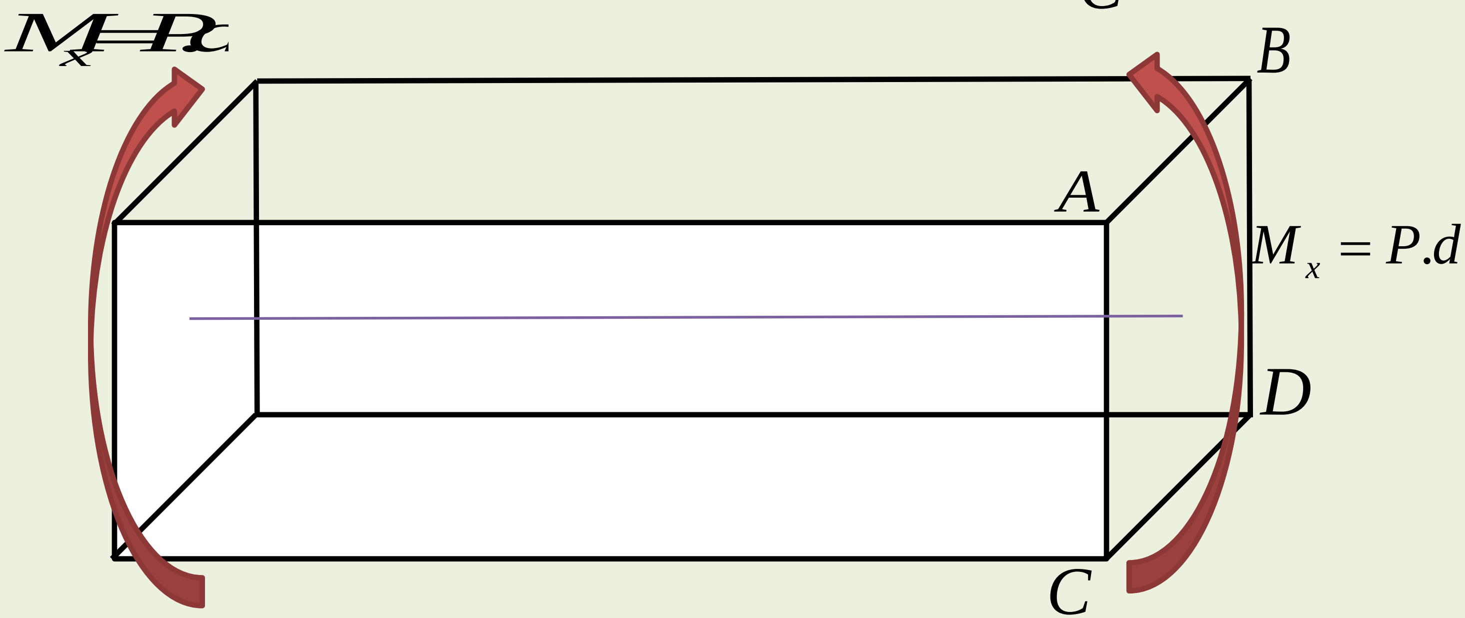
So,

Bending moment = constant

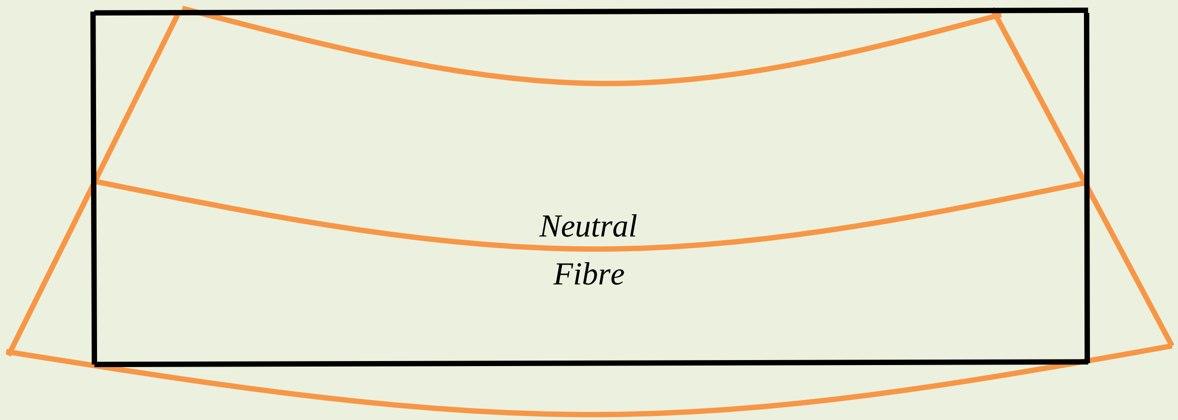
Axial force = Shear force = Twisting moment = 0







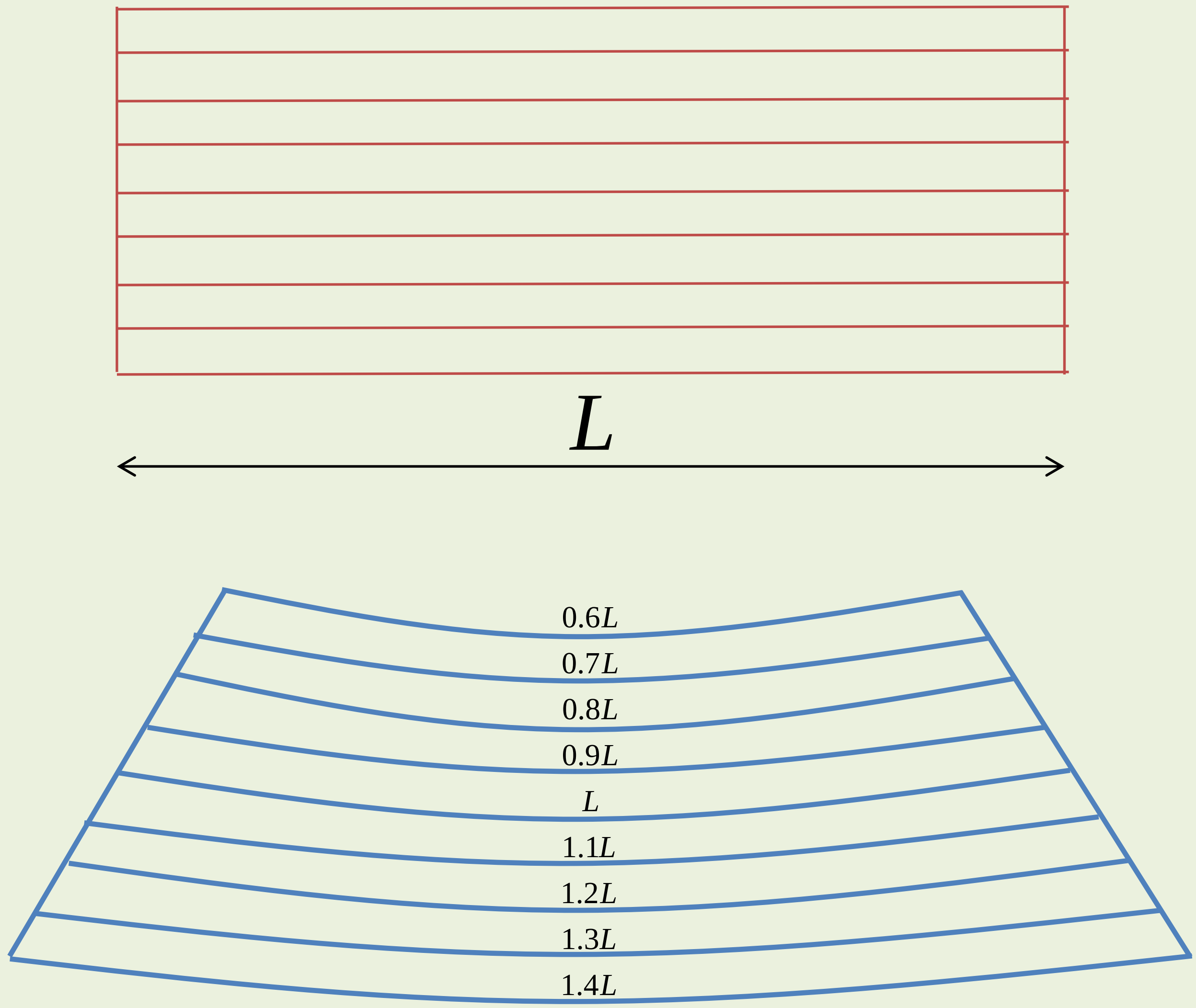
ANKIT SAXENA (saxena01ankit@gmail.com)



Sagging

(or)

+veBending



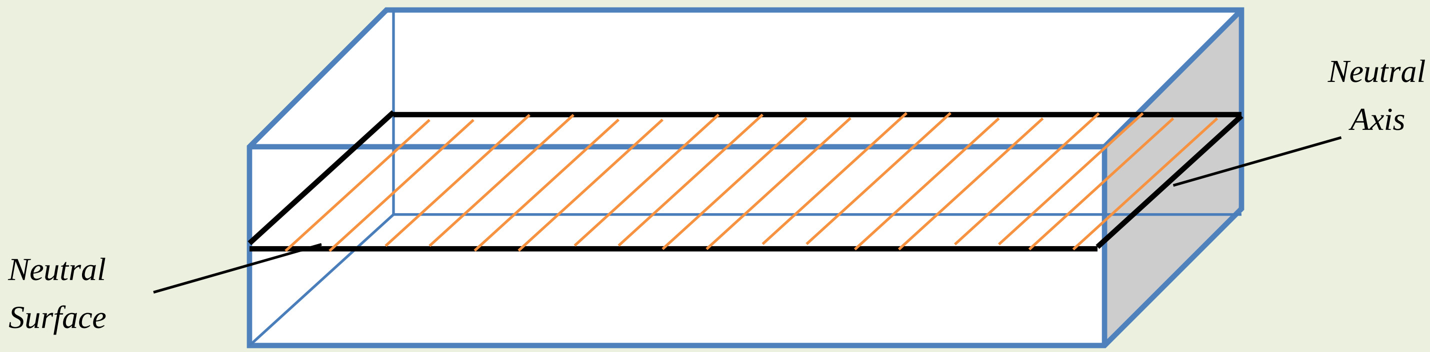
ANKIT SAXENA (saxena01ankit@gmail.com)

Neutral Fiber (Neutral Surface)

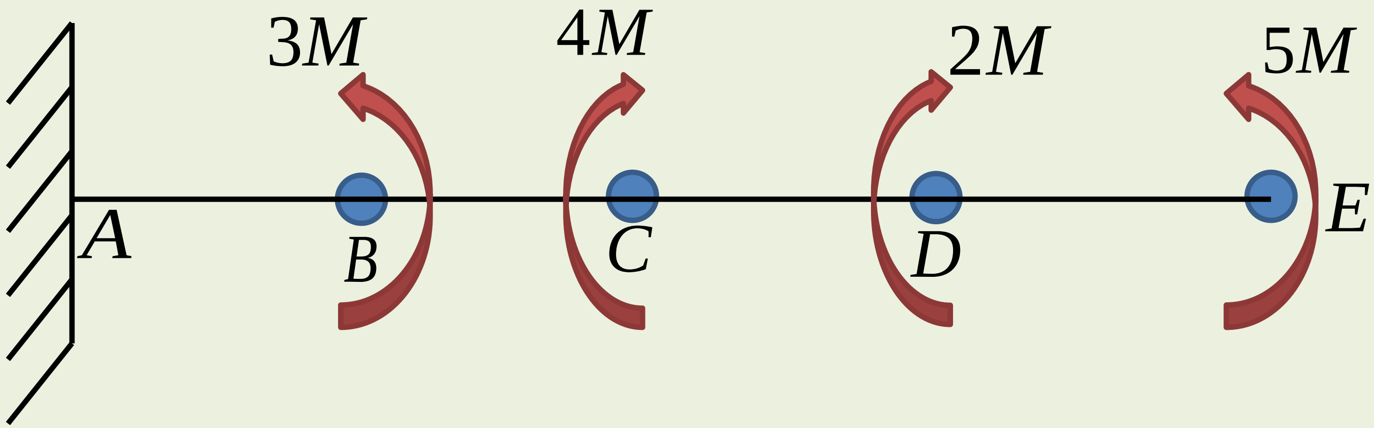
The fiber whose length before and after the bending get unchanged, is called neutral fiber or neutral surface.

Neutral Axis

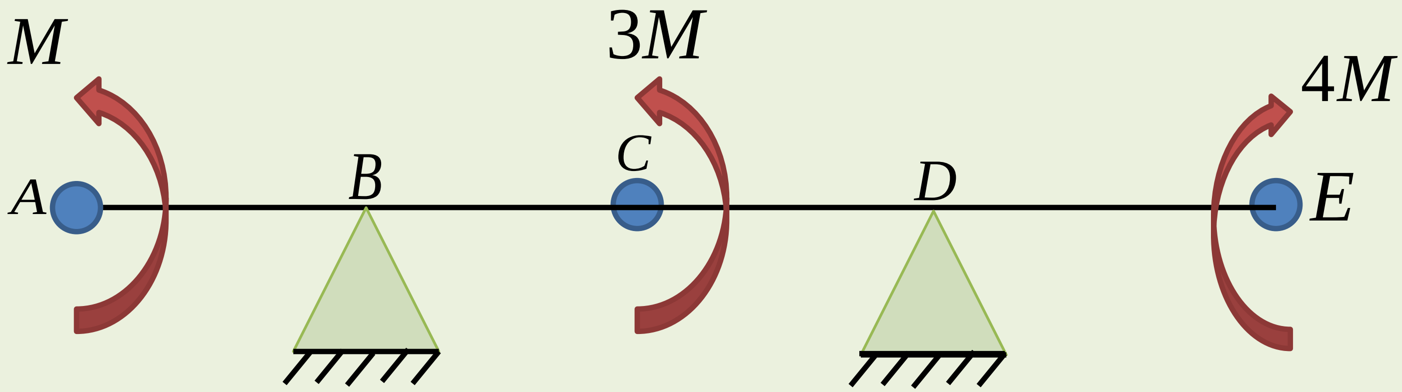
Line of intersecting of neutral surface with the cross-section is called neutral axis.



Q1. For the cantilever beam as shown in the figure draw the bending moment diagram. Determine maximum sagging and maximum hogging bending moment acting at the x-s/c of the beam.



Q2. For the S.S.B as shown in the figure draw the bending moment diagram. Determine maximum sagging and maximum hogging bending moment acting at the x-s/c of the beam.



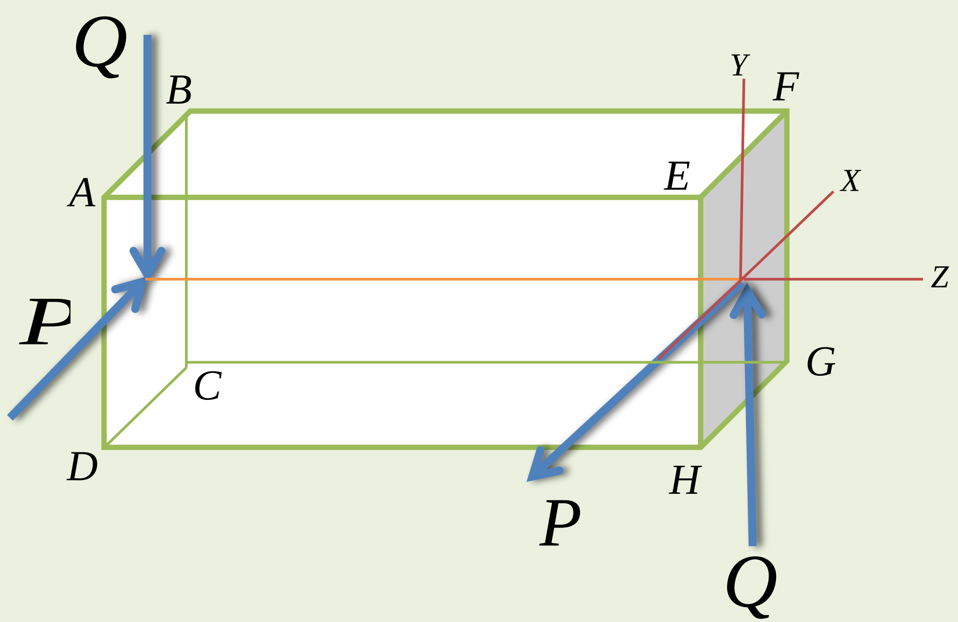
Point of Contra flexure

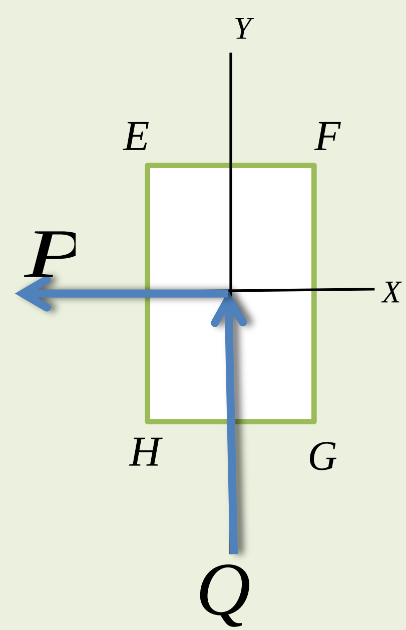
Point of contra flexure is the point where bending moment changes its sign i.e., from positive value to a negative value or Vice versa.

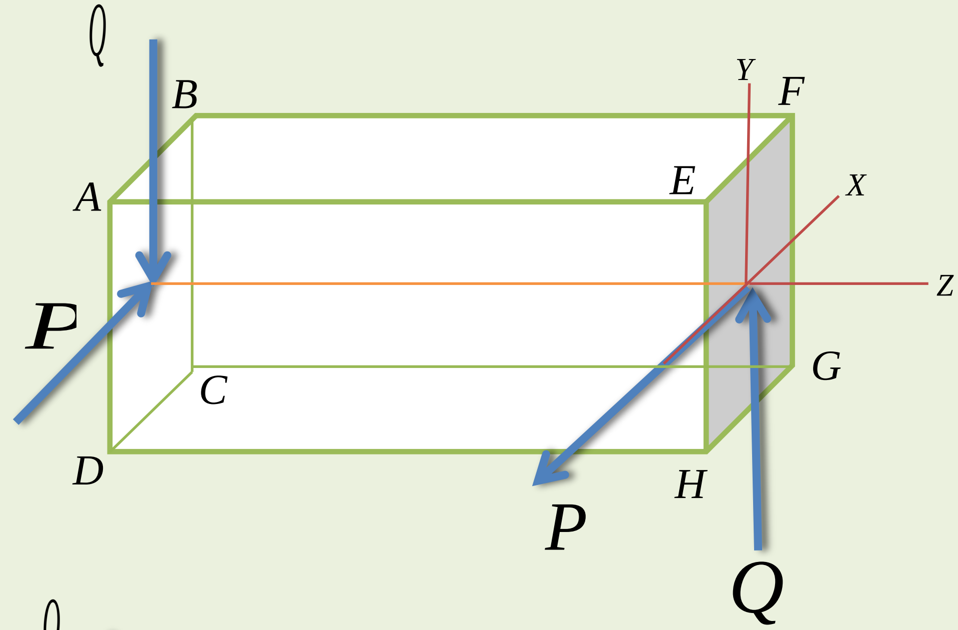
SHEAR FORCE DIAGRAM

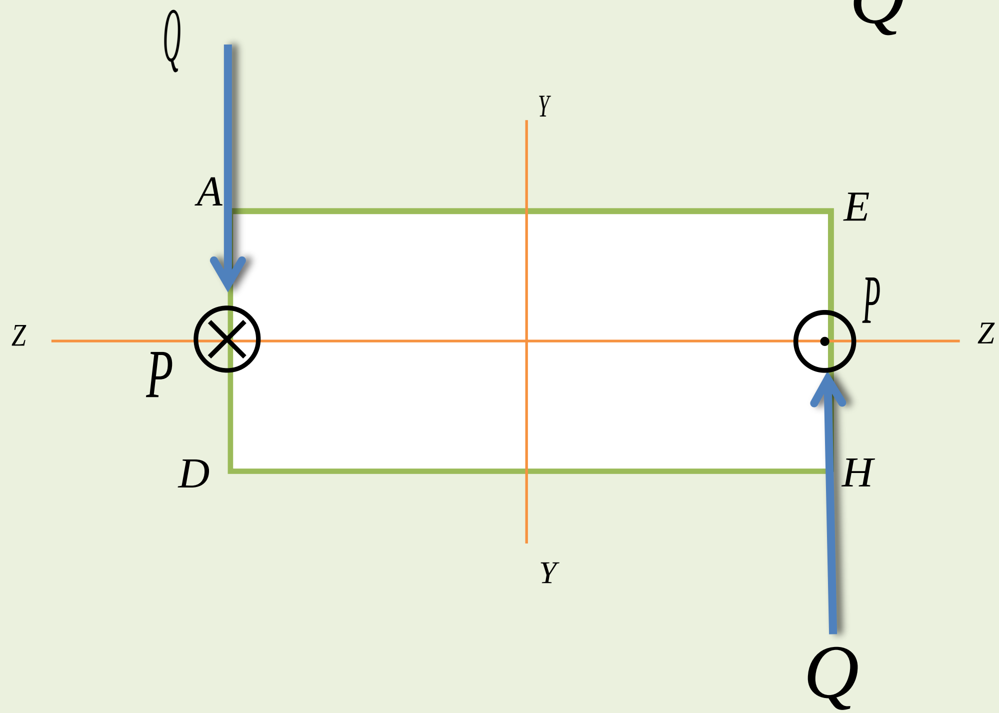
Shear Force Definition

- Shear force acting at any x-s/c of the member is equal algebraic sum of shear forces (i.e. II to the x-s/c) either on the L.H.S of the member or R.H.S of x-s/c of the member.



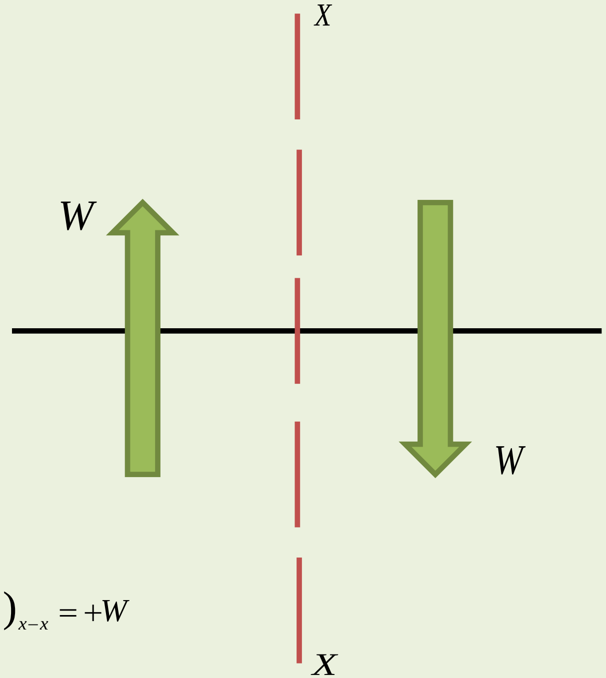



Q
B
A
E
F
Y
X
Z
P
C
G
D
H
P
Q

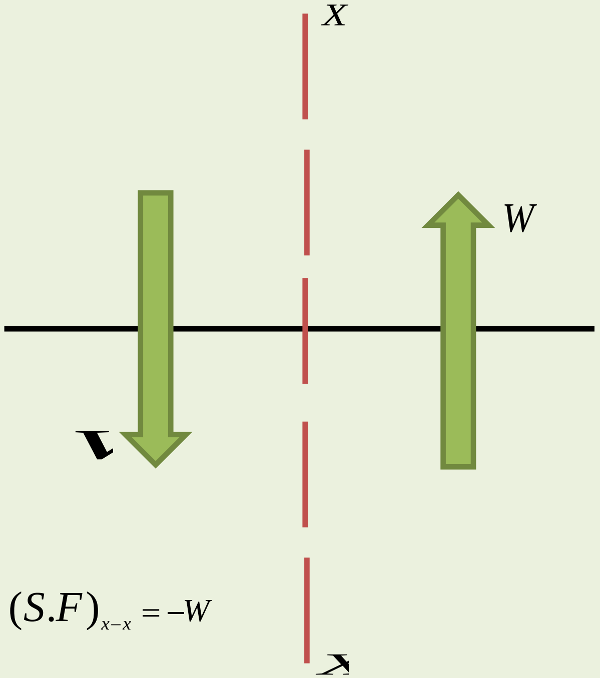

Q
Y
A
E
P
Z
P
Z
D
H
Y
Q

Sign Convention

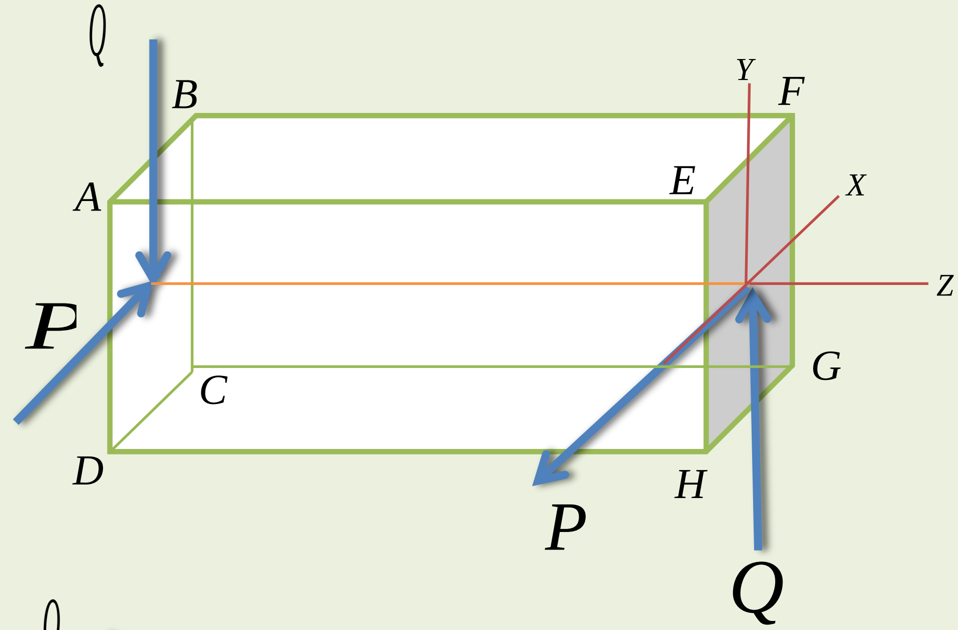
- Shear force at any section of the member is said to be positive when it acts in the upward direction on the left hand of the section & in downward direction on the right hand side of the x-s/c and vice versa.

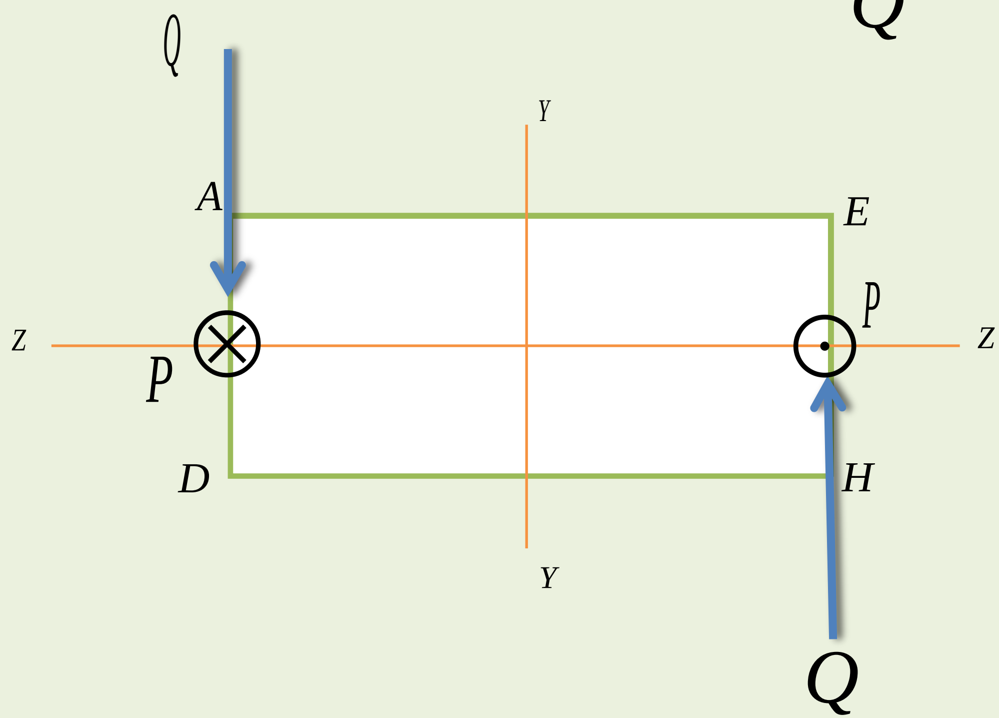


$(S.F)_{x-x} = +W$



$(S.F)_{x-x} = -W$





ANKIT SAXENA (saxena01ankit@gmail.com)

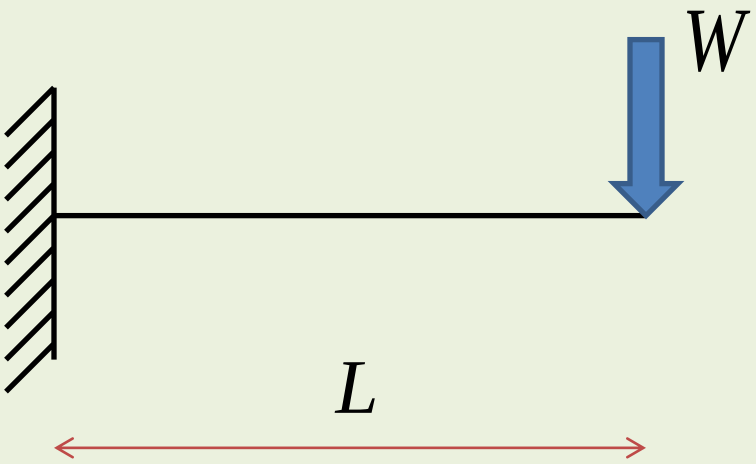
Important Points

- If a member is subjected to shear loading (either TSL or ETSL), it always experienced bending moment acting on it.
- Shear force acting at any section is the slope of bending moment diagram for that particular section. We can write as also,

$$(S.F)_{x-x} = \frac{dM_{x-x}}{dx}$$

- If B.M is constant - S.F is zero
- If B.M is variable - S.F (Shear Force) is non-zero
- If member is subjected to Eccentric transverse shear loading (ETSL), it always experienced twisting moment acting on it.

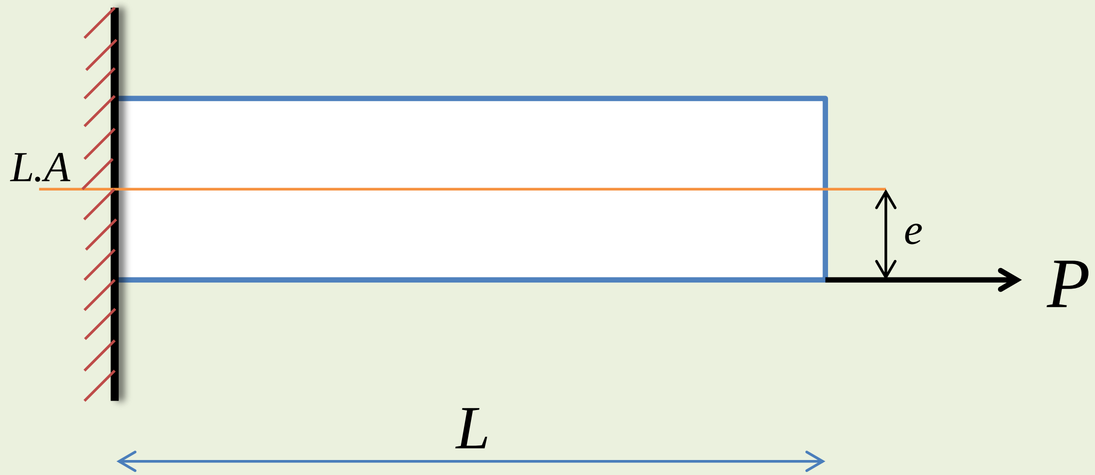
Q Draw the shear force and bending moment diagram for the beam under giving loading condition.



LOADING DIAGRAM

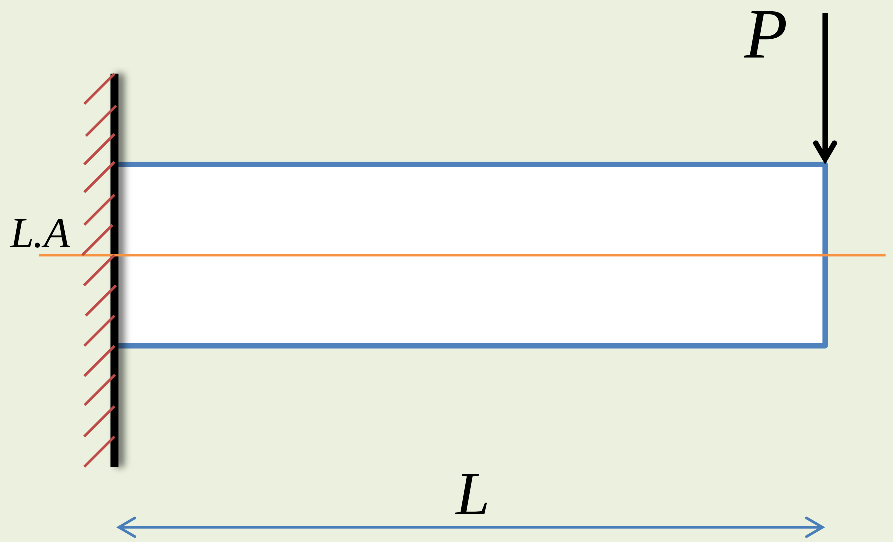
EQUIVALENT LOADING DIAGRAM

Equivalent Load Under Eccentric Axial Load



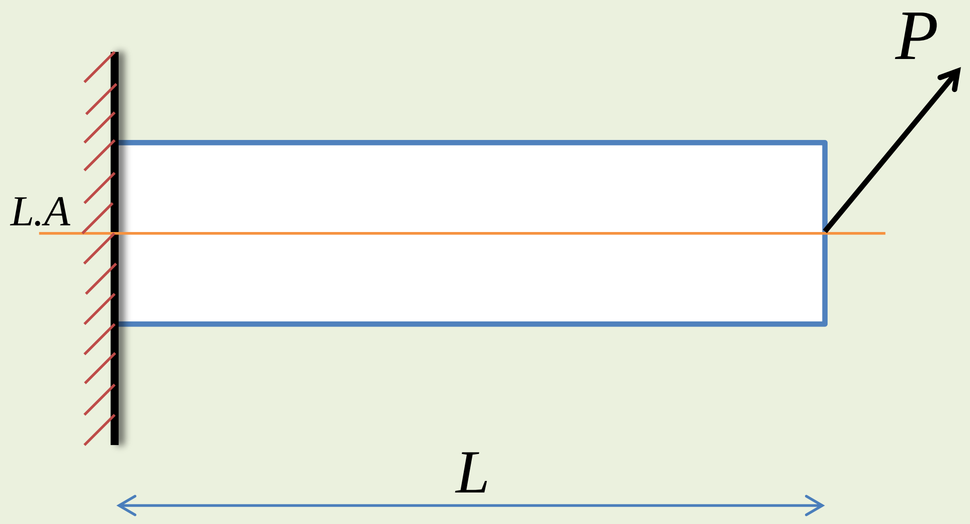
Equivalent Load Under Shear Load

1. Vertical Transverse Shear Load



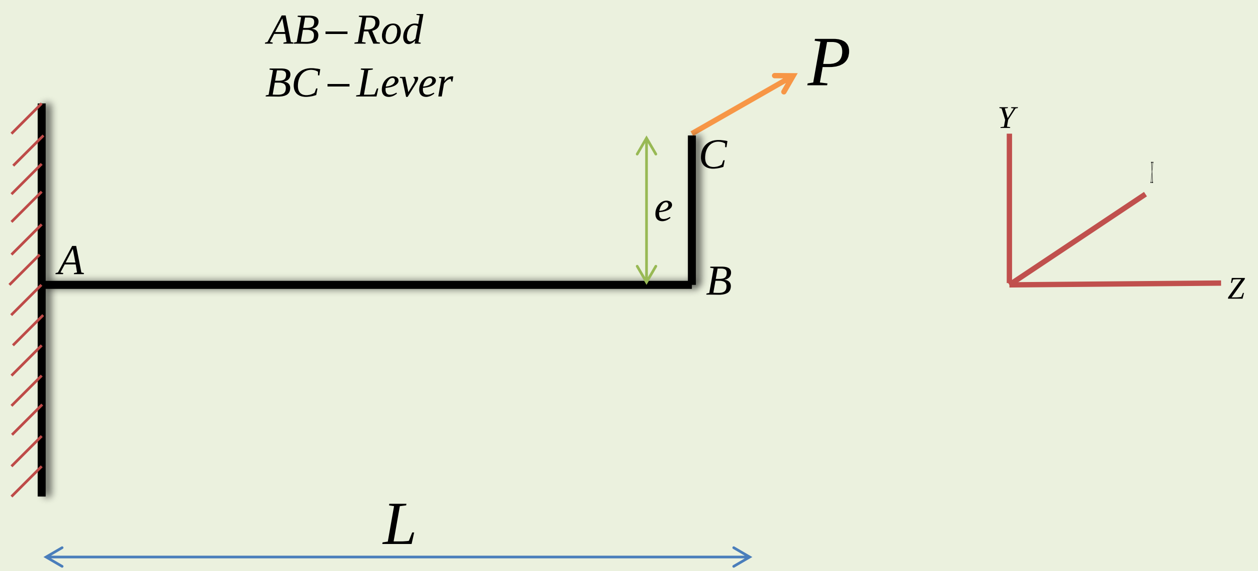
Equivalent Load Under Shear Load

1. Horizontal Transverse Shear Load

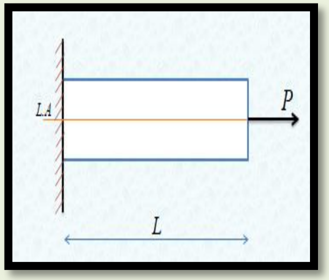
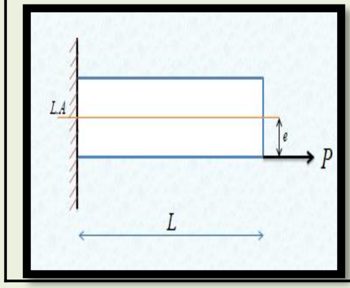


Equivalent Load Under Shear Load

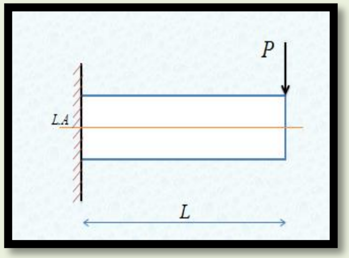
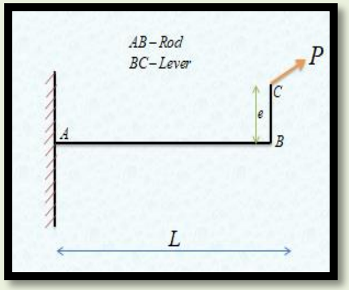
1. Eccentric Transverse Shear Load



Summary of Equivalent Load Diagram

Loading Condition	Axial Load	Shear Load	Bending Moment	Twisting Moment
Pure axial load 	Constant (P)	Zero	Zero	Zero
Eccentric axial loading 	Constant (P)	Zero	Constant (P x e)	Zero

Summary of Equivalent Load Diagram

Loading Condition	Axial Load	Shear Load	Bending Moment	Twisting Moment
T.S.L 	Zero	Constant = P	Variable Free end = 0 Fixed end = PL	Zero
E.T.S.L 	Zero	Constant = P	Variable Free end = 0 Fixed end = PL	Constant = Pe

Practise Question

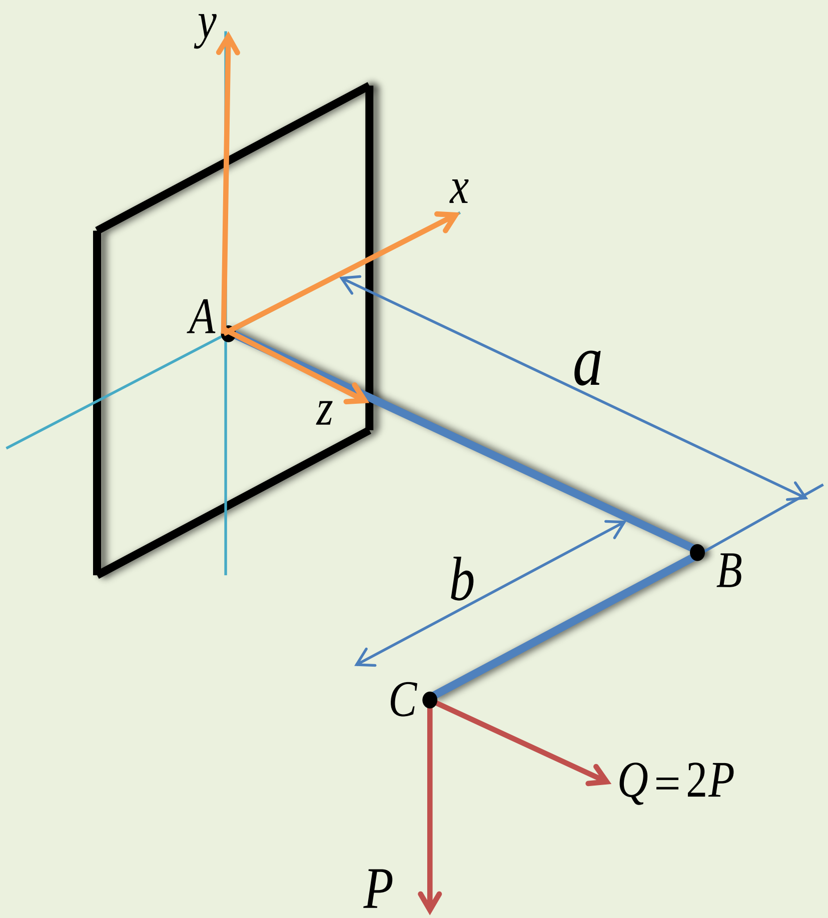
For the rod and lever assembly as shown in the figure determine-

1. **Axial load , shear force, bending moment & torsional moment acting on the rod.**
2. **A.L, S.F, B.M & T.M acting on the x-s/c like A,B,C .**

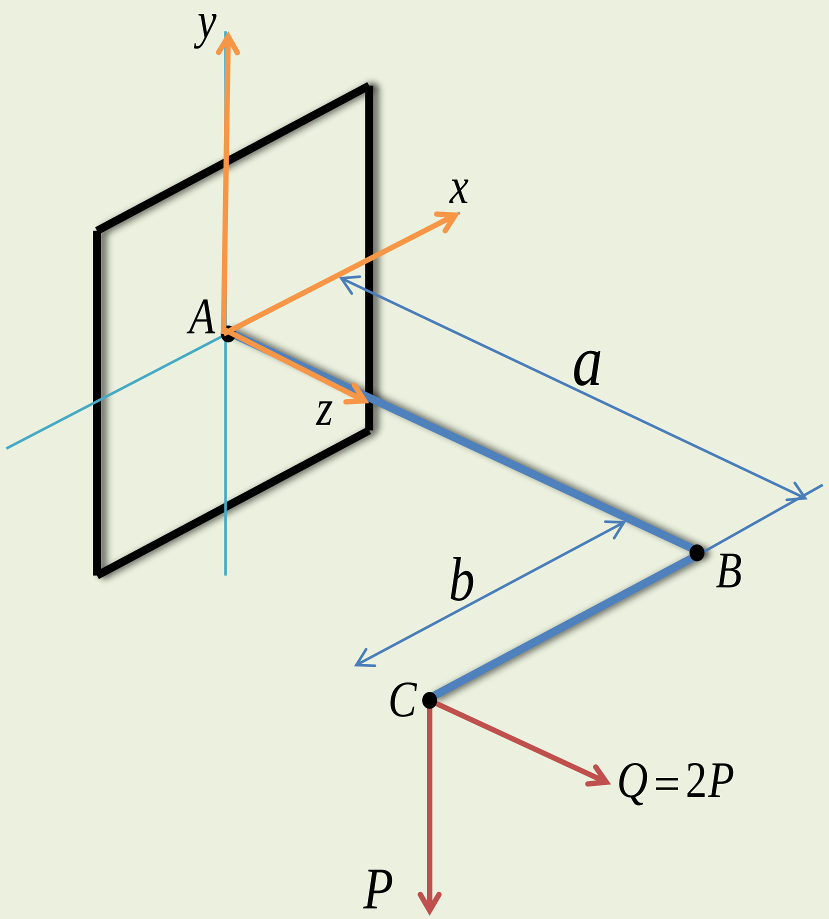
Note:

AB - Rod

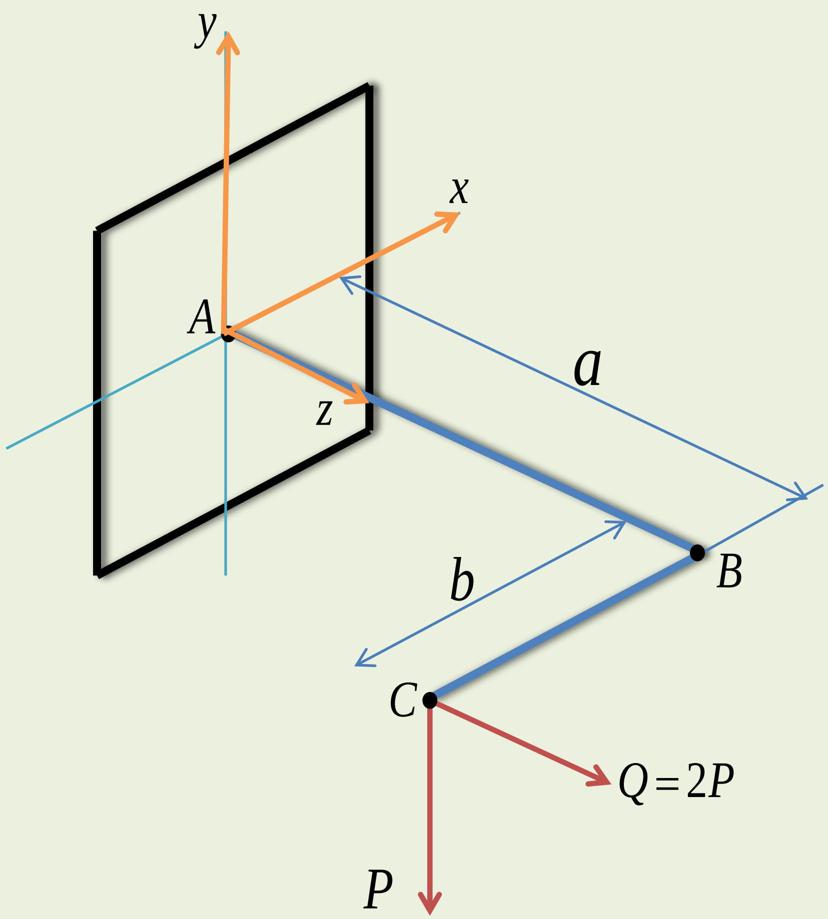
BC - Lever



y
x
A
a
z
B
b
C
Q = 2P
P



y
x
A
a
z
B
b
C
Q = 2P
P



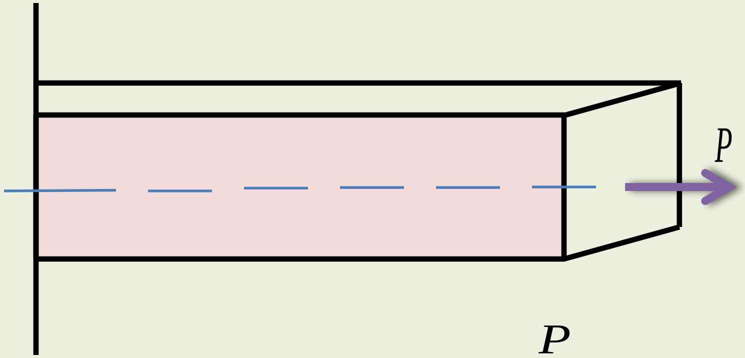
ANKIT SAXENA (saxena01ankit@gmail.com)

STRESS DISTRIBUTION

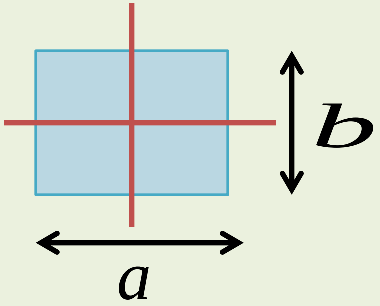
Stress distribution in different loading condition

Axial Loading:

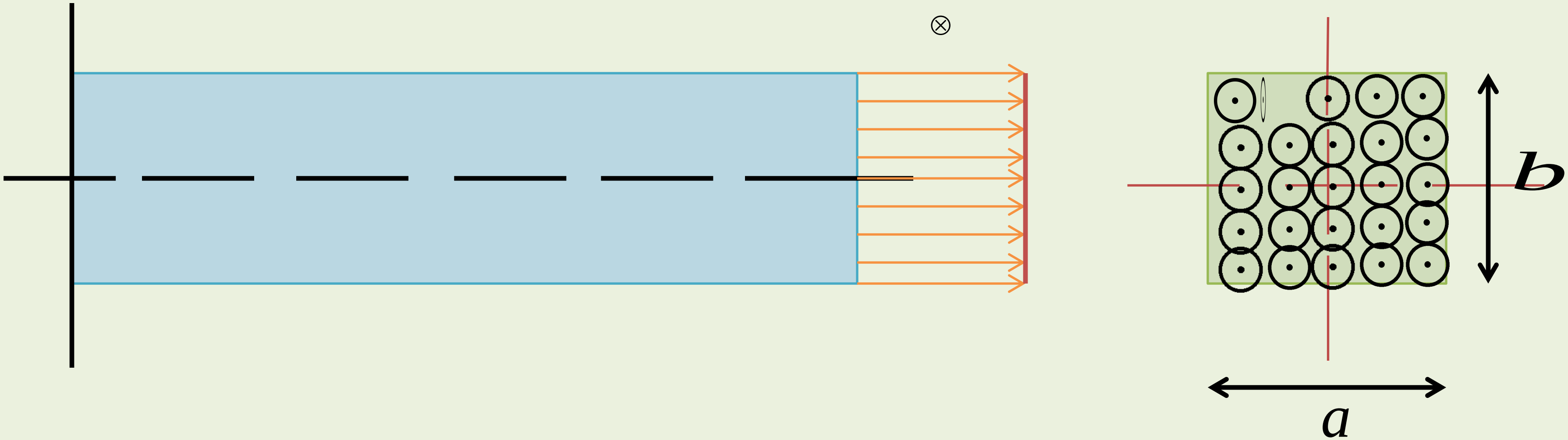
- The intensity of stress is same at each and every point on the cross section.
- Axial stress is denoted by (**σ** **).**
- It is either tensile or compressive in nature.
- Consider a bar having rectangular x-section (A= a x b)



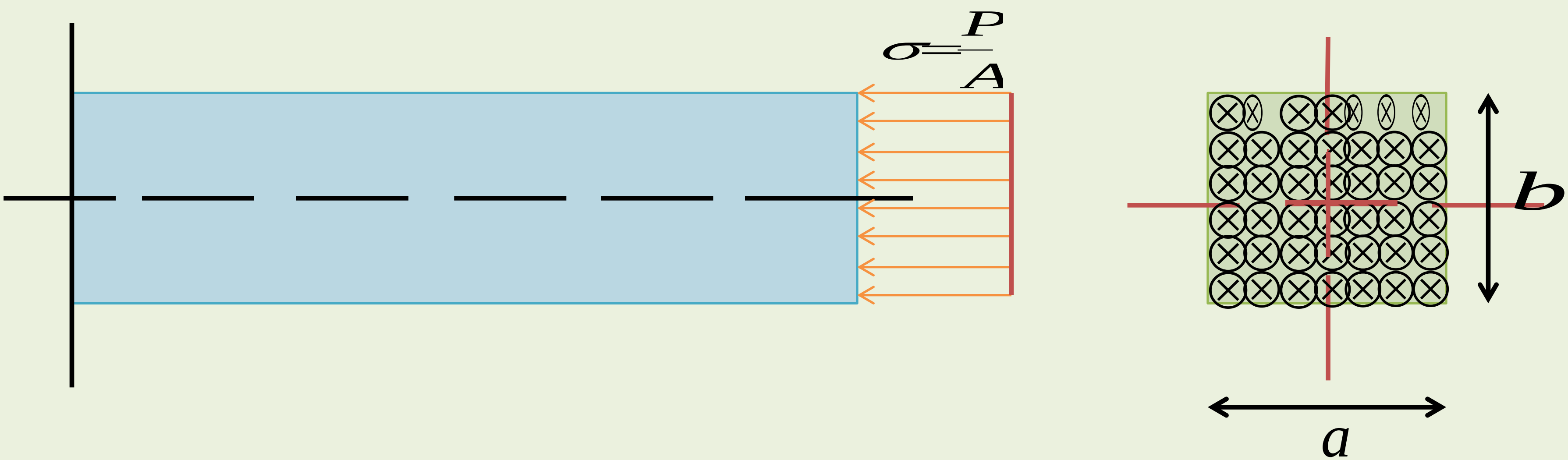
$$\sigma = \frac{P}{A}$$



- If applied external force is tensile in nature.



- If the applied external force is compressive in nature.



Shear Loading:

- Due to shear loading there is shear stress induced in the member.
- Shear stress is denoted by (**τ**).

$$\boldsymbol{\tau = P \frac{A \bar{Y}}{I_{NA} b}}$$

τ = shear stress developed at a fiber on the plane of x-s/c of beam.

A = Area of hatched portion which is above the fiber where τ is to be determined.

$\bar{Y}$ = distance of centroid of hatched portion from the neutral axis (N.A)

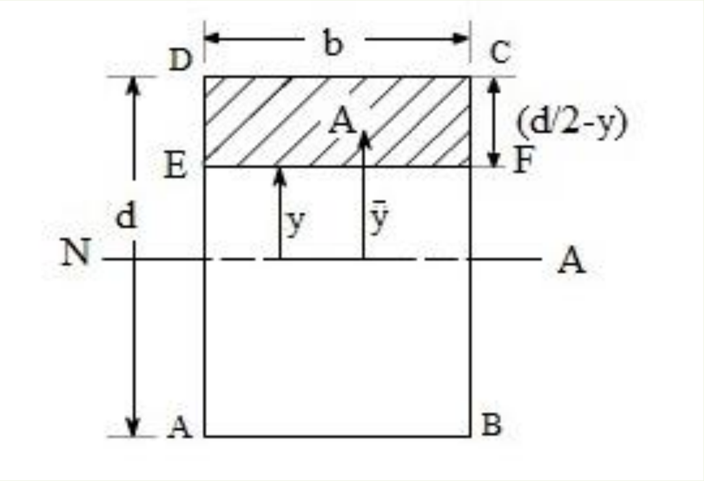
$A\bar{Y}$ = first moment of area of hatched portion about N.A

I_{NA} = moment of inertia of cross section about N.A

b = width of fiber where τ is to be determined.

Rectangular x-section

- Consider a rectangular x-section of dimensions b & d respectively.



- A is the area of hatched x-section cut off by a line parallel to the neutral axis. ȳ is the distance of the centroid of A from the neutral axis.

$$\tau = P\ \frac{A\,\bar{Y}}{I_{NA}b}$$

A = b ($\frac{d}{2}$ - y)

$\bar{y} = [\frac{1}{2}(\frac{d}{2} - y) + y]$

$= \frac{1}{2}(\frac{d}{2} + y)$

$I_{NA} = \frac{bd^3}{12}$

Width of strip = b

Substituting all the values,

$$\tau = \frac{F.b.(\frac{d}{2} - y).\frac{1}{2}.(\frac{d}{2} + y)}{b.\frac{b.d^3}{12}}$$

$$\tau = \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This show there is a parabolic distribution of stress with y.

The maximum value of shear stress at y = 0

So

$$\tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$$

$$= \frac{3.F}{2.b.d}$$

$$\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$$ The value of τ_{max} occurs at the neutral axis

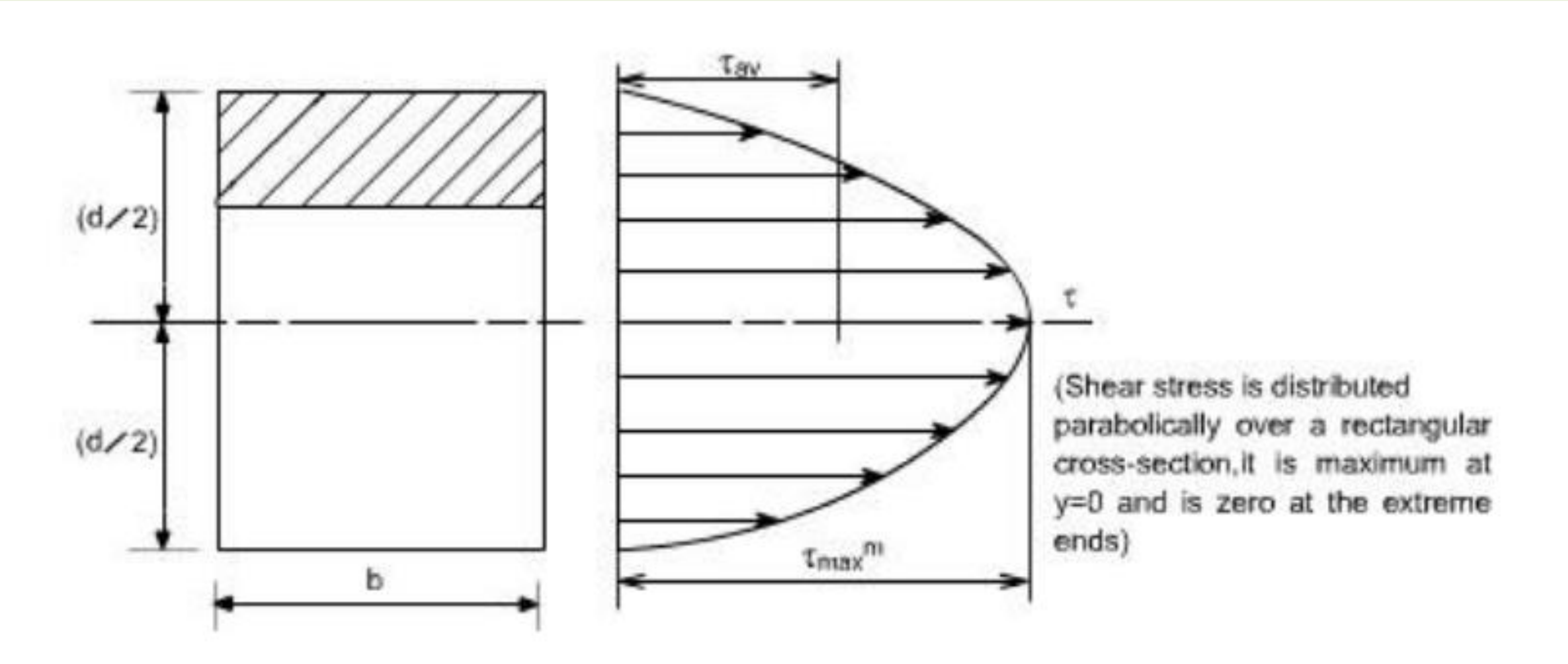
The mean shear stress is defined as -

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

The relation between mean and maximum shear stress is given as -

$$\tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$$

Shear stress distribution in rectangular section is given as -



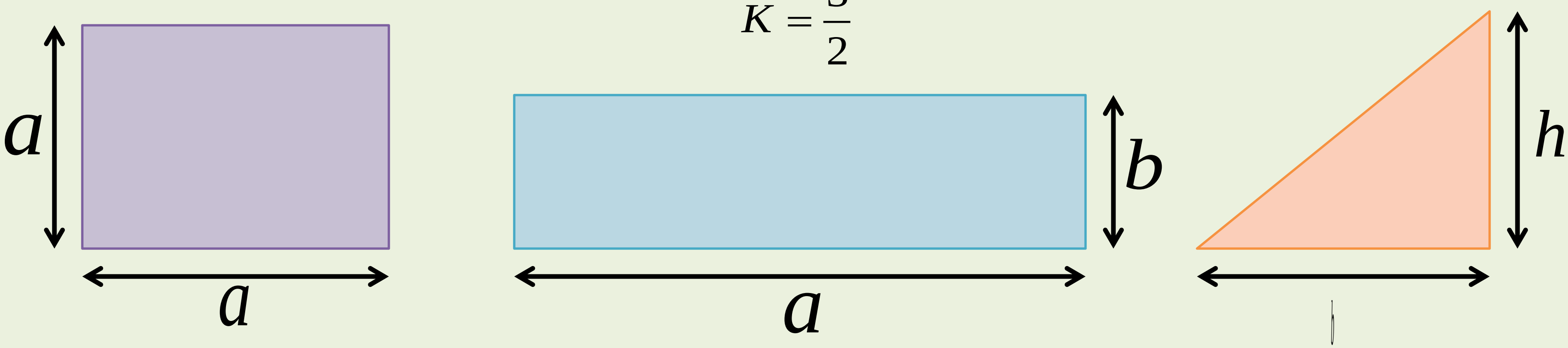
It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y =0 and is zero at the extreme ends.

Important Points

- If P is the applied shear force , then relationship between τ_{Max} & τ_{Avg} for rectangular, square and triangle section is given as -

$$\tau_{Max} = K.\tau_{Avg}$$

$$K = \frac{3}{2}$$



Square section

$a \rightarrow$ *side length*

$$\tau_{Avg} = \frac{P}{a^2}$$

$$\tau_{Max} = (\frac{3}{2}).\tau_{Avg}$$

Rectangular section

$a \rightarrow$ *width*

$b \rightarrow$ *height*

$$\tau_{Avg} = \frac{P}{a.b}$$

$$\tau_{Max} = (\frac{3}{2}).(\frac{P}{a.b})$$

Triangular section

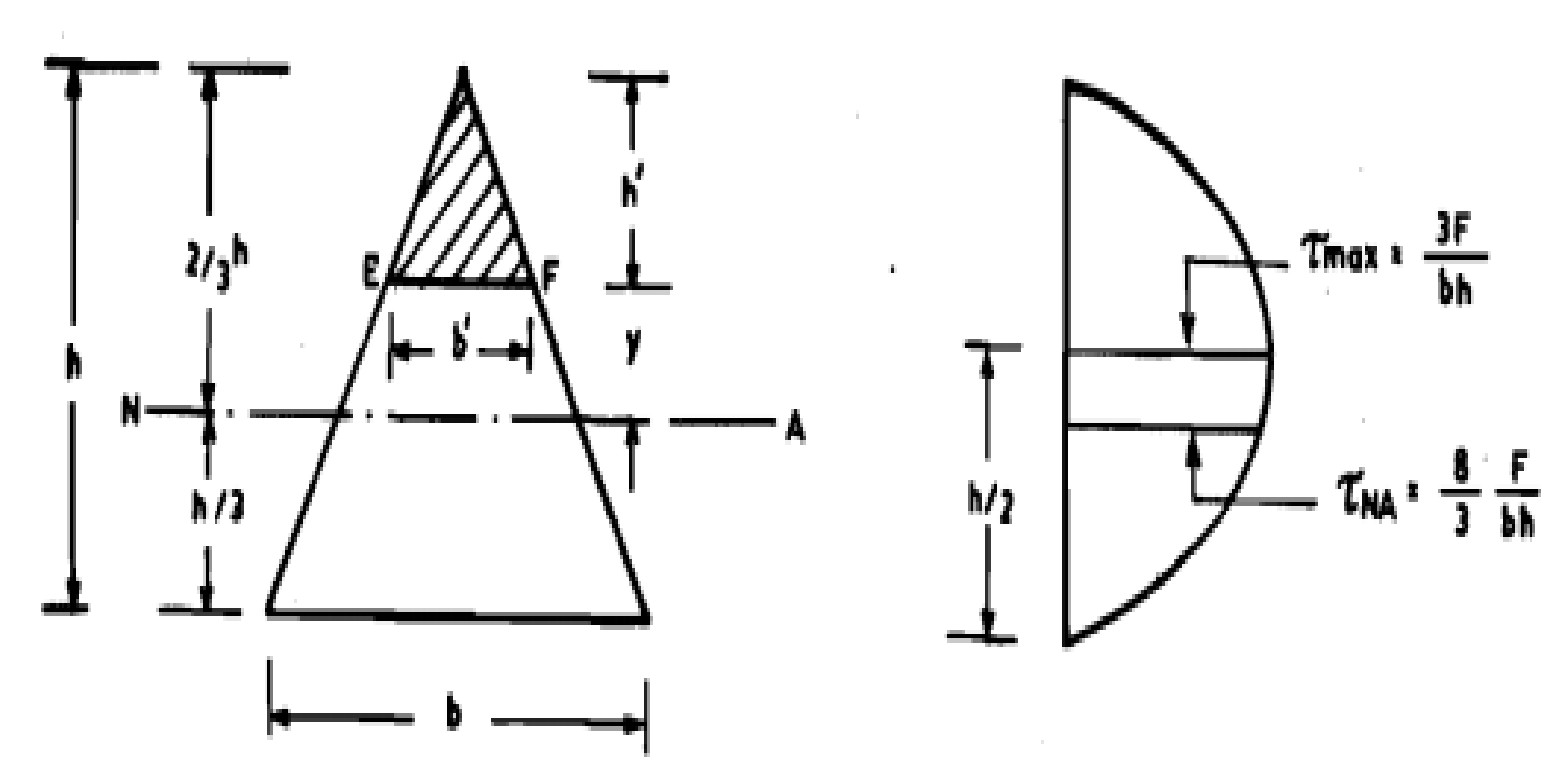
$b \rightarrow$ *base − length*

$h \rightarrow$ *height*

$$\tau_{Avg} = \frac{P}{\frac{1}{2}b.h}$$

$$\tau_{Max} = (\frac{3}{2}).(\frac{P}{\frac{1}{2}a.b})$$

Shear stress distribution in triangular section



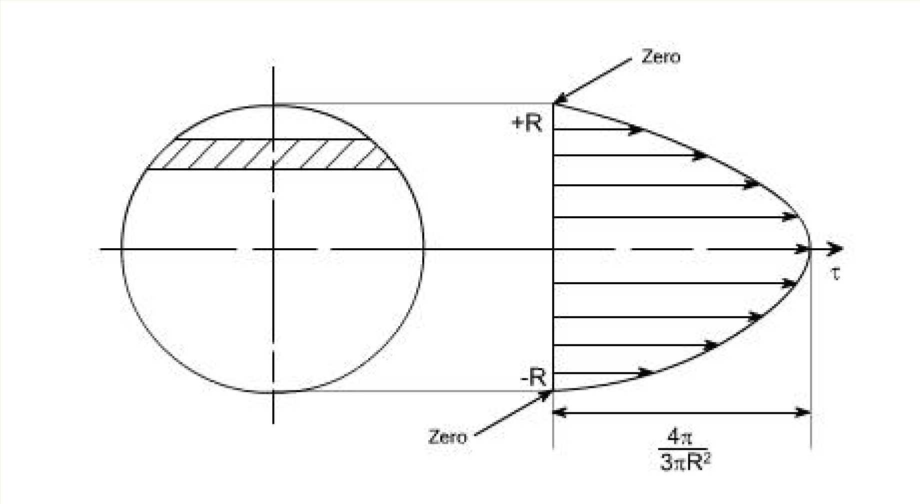
Important Points

- If P is the applied shear force , then relationship between τ_{Max} & τ_{Avg} for circular section is given as -

$K = \frac{4}{3}$

$\tau_{Max} = (\frac{4}{3}) * \tau_{Avg}$

$\tau_{Avg} = \frac{P}{\frac{\pi}{4}d^2}$



Note : R is the radius of the circle.

Important Points

- If P is the applied shear force , then relationship between τ_{Max} & τ_{Avg} for square section having diagonals horizontal & vertical is given as -

$$K = \frac{9}{8}$$

$$\tau_{Max} = \left(\frac{9}{8}\right) * \tau_{Avg}$$

$$\tau_{Avg} = \frac{P}{a^2}$$

I - Section

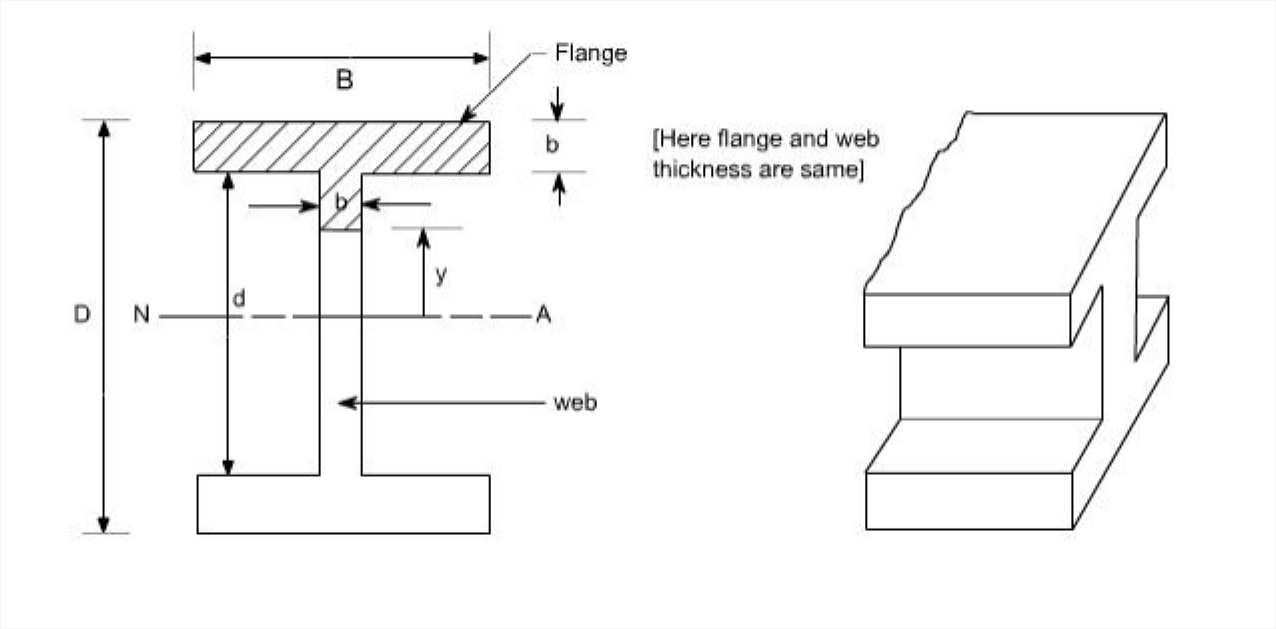
As we know -

$$\tau = P\ \frac{A\,\bar{Y}}{I_{NA}b}$$

If P, A, $\bar{Y}$, I_{NA} are constant. Then

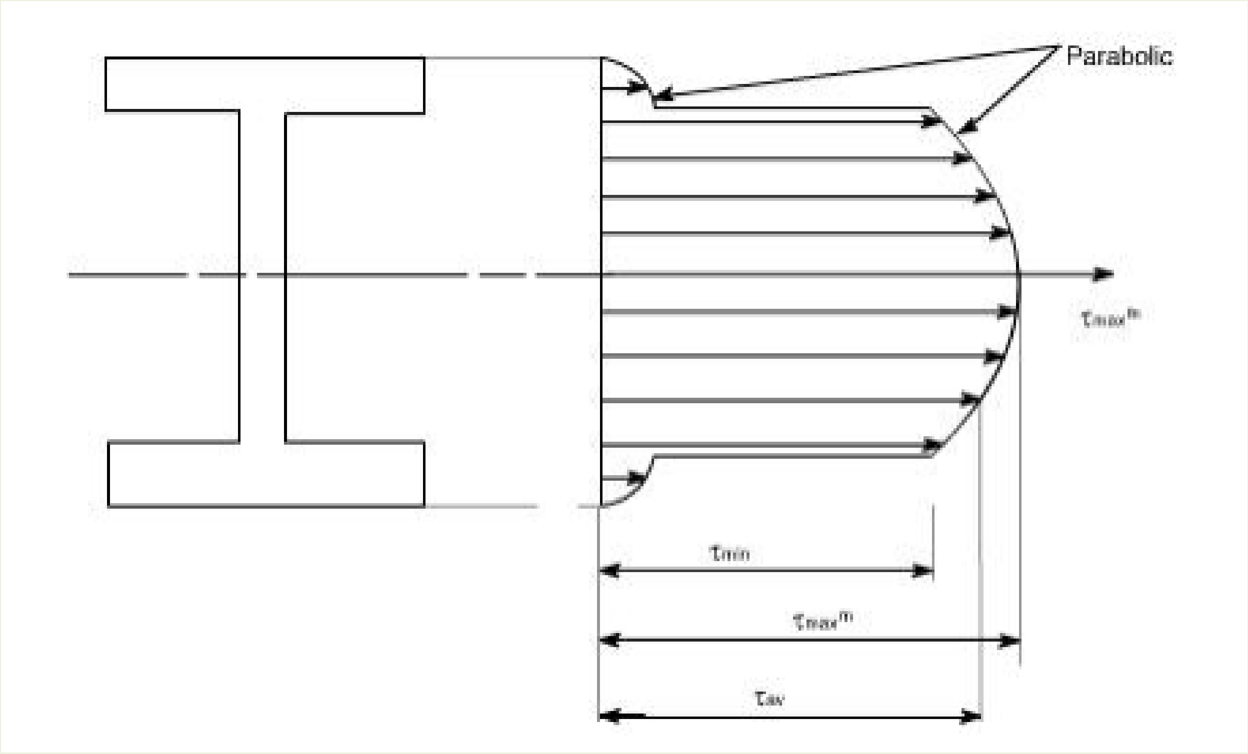
$$\tau \ \alpha\ \frac{1}{b}$$

From we conclude that if section width decrease, the induced shear stress value will go increase and vice versa.



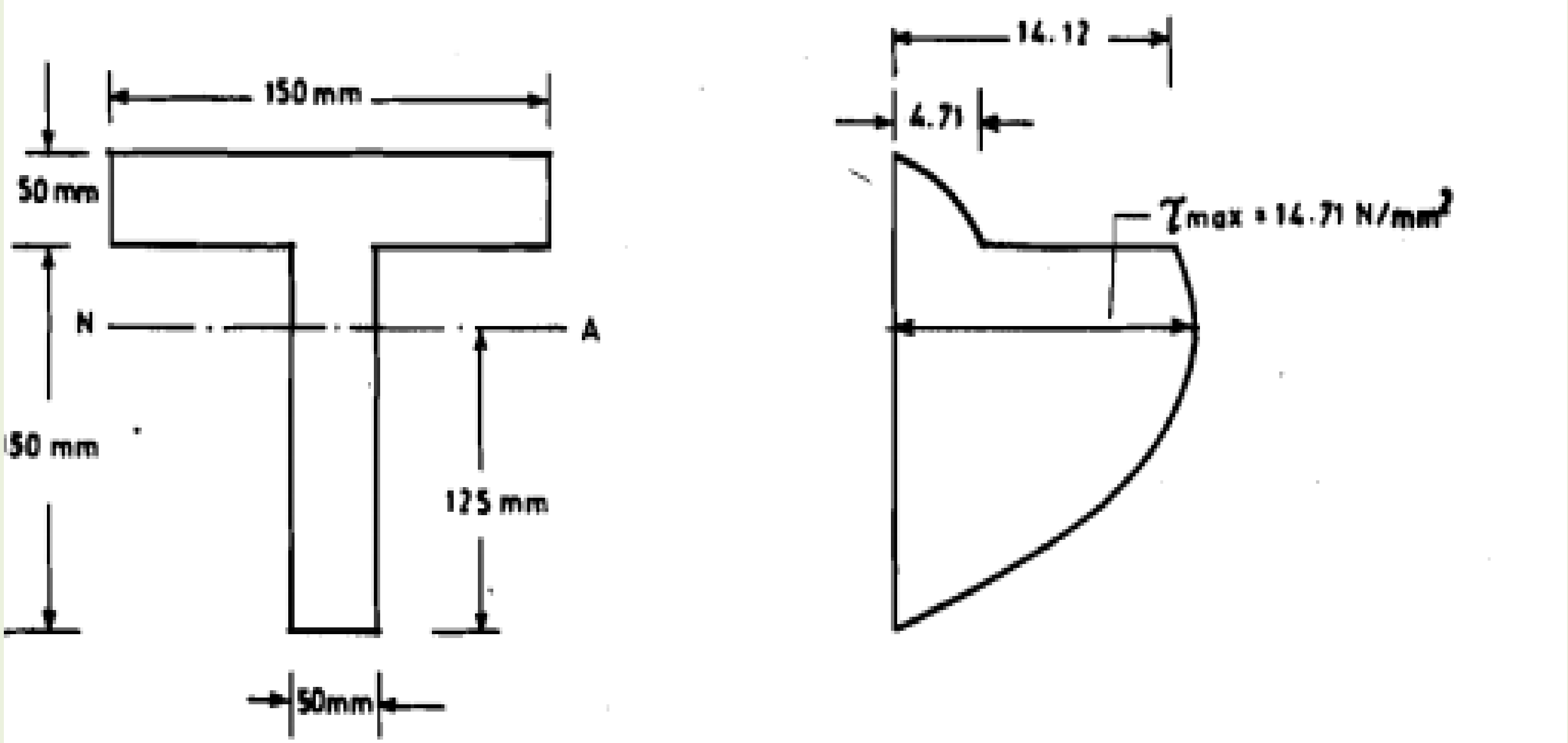
I - Section

- Stress distribution for I-section is given as-



At the junction of web & flange, the section width change suddenly so there is drastically change in shear stress magnitude at web & flange junction.

T - Section



Bending

Bending equation is given as-

$$\frac{M}{I_{NA}} = \frac{\sigma_b}{y} = \frac{E}{R}$$

M = Bending Moment (N-mm)

I_{NA} = Area moment of Inertia (mm^4)

σ_b = Bending stress ($\frac{N}{mm^2}$)

Y = Distance of fiber where we calculate bending stress from the neutral fiber(mm)

E = Young's modulus of elasticity ($\frac{N}{mm^2}$)

R = Radius of Curvature (mm)

Bending Equation Assumptions

- The material is homogeneous and isotropic.
- The value of Young's modulus of elasticity [E] is same in tension and compression.
- The transverse sections which were plane before bending, remain plane after bending also.
- The radius of curvature is large as compared to the dimensions of the cross-section.
- Each fiber of the structure is free to expand or contract.

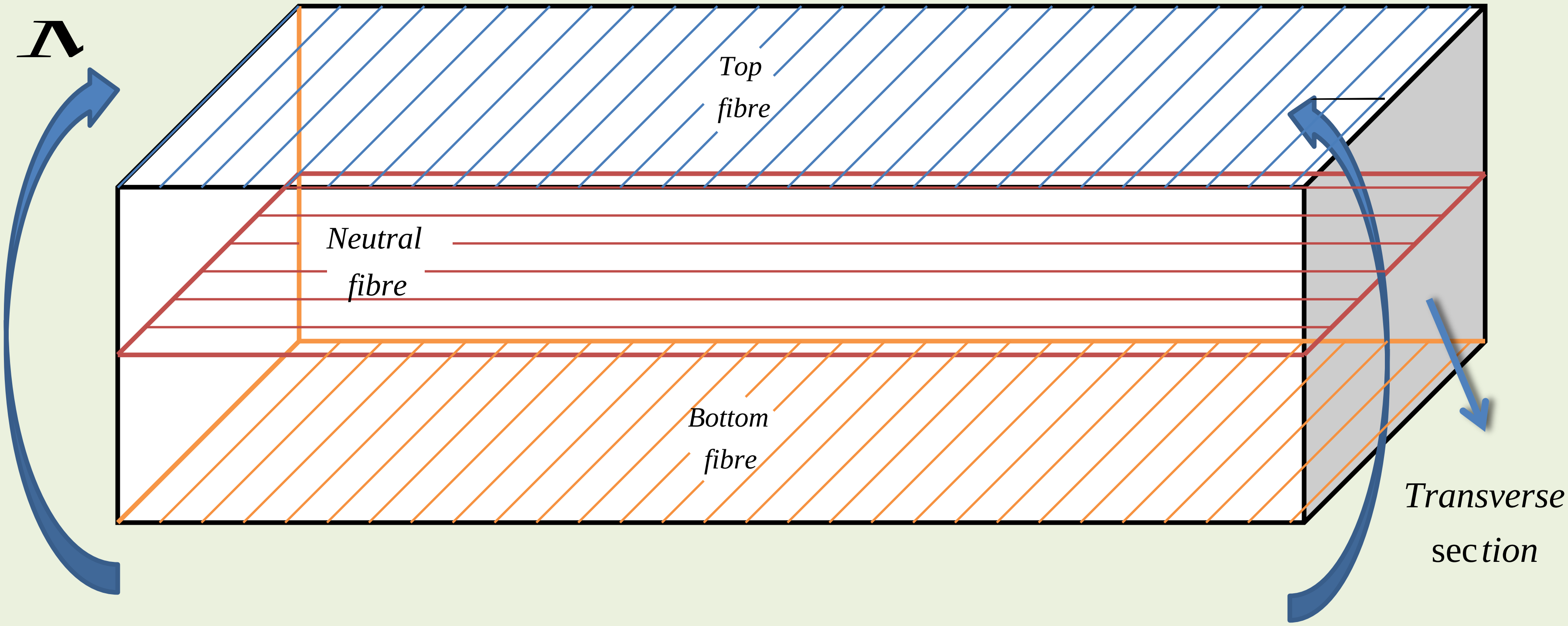
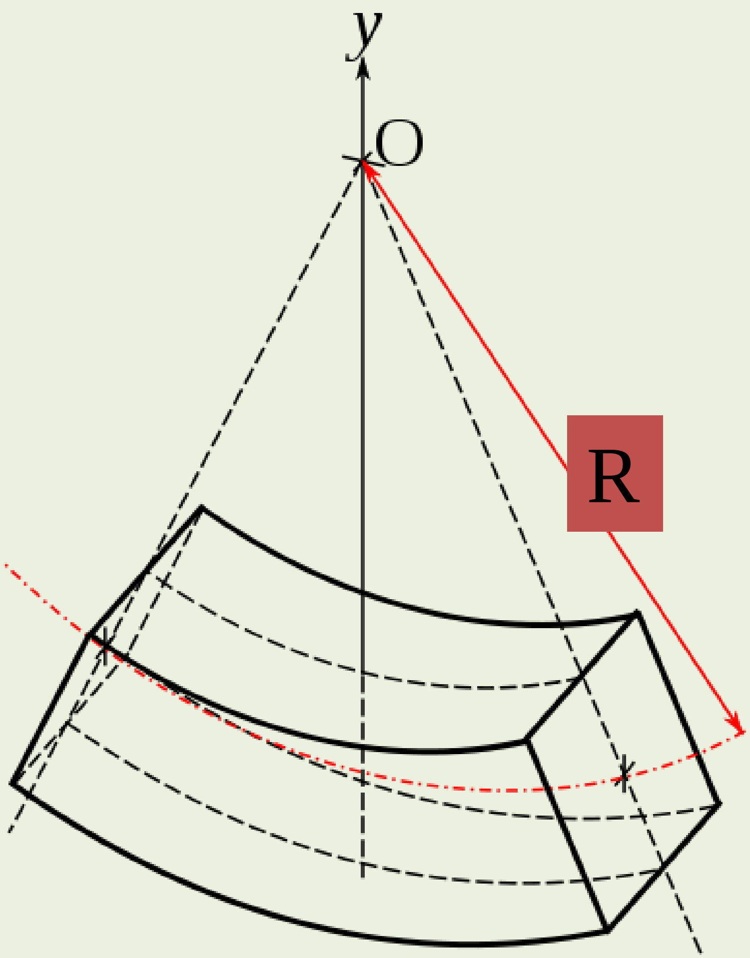
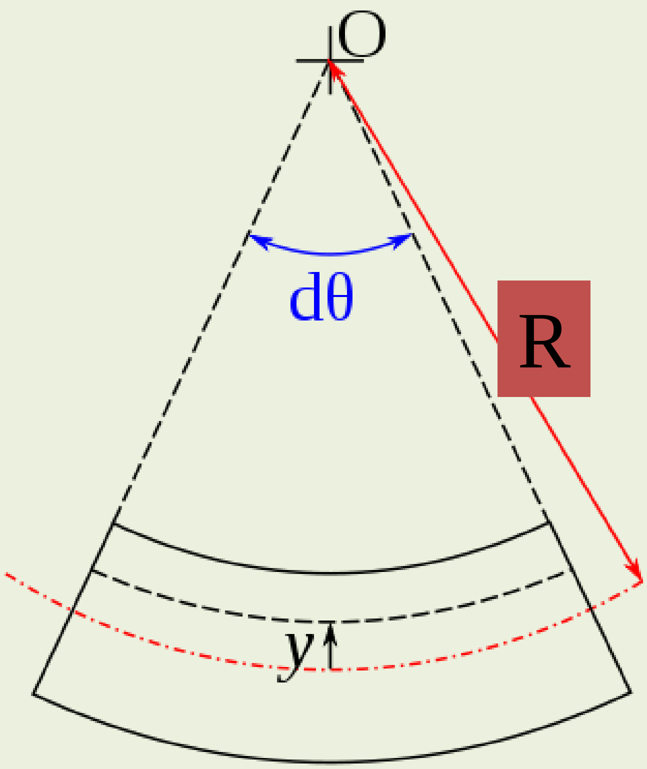


Figure: Type of sagging bending

ANKIT SAXENA (saxena01ankit@gmail.com)

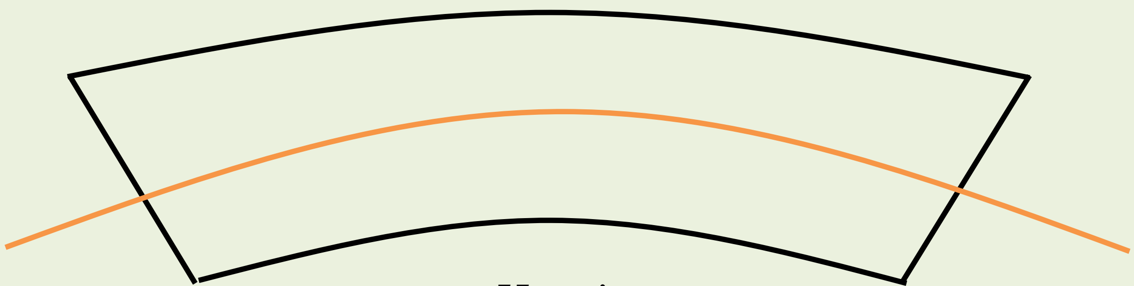
- In type of sagging bending, all fibre above the neutral fibre (top fibre) get compressed & all fibre below the neutral fibre (bottom fibre) get elongated.





Neutral Fibre : All the fibre that lengths get unchanged before and after the bending, is said to be neutral fibre.

- In the same way in type of hogging bending, all fibre above the neutral fibre (top fibre) get elongated & all fibre below the neutral fibre (bottom fibre) get compressed.



Hogging

(or)

$-ve$ *Bending*

Section Modulus: It is the ratio of I_{NA} to the y. It is denoted by Z. Section modulus defined the strength of the section.

$$Z = \frac{I_{NA}}{y}$$

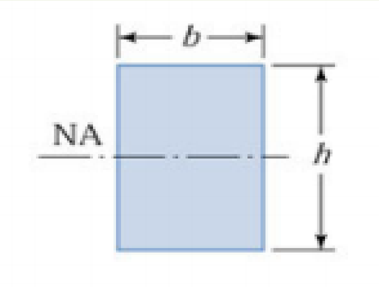
- By increasing y value, bending stress (σ_b) get increased.
- At y = max, σ_b = max

$$\sigma_b \alpha y$$

Section modulus of common x-section About NA :

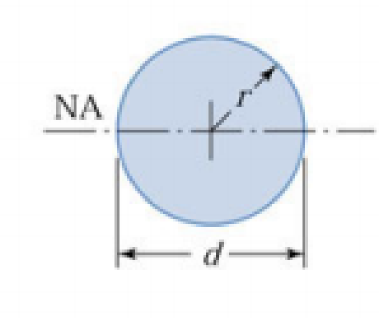
1. Rectangle

$$Z = \frac{bh^2}{6}$$



2. Circle:

$$Z = \frac{\pi d^3}{32}$$

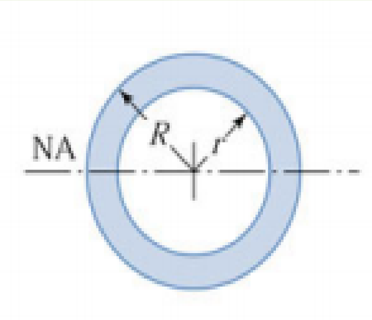


3. Tube:

r = inner radius

R = outer radius

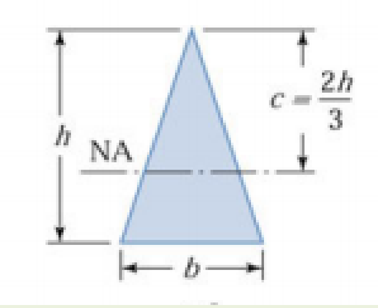
$$Z = \frac{\pi}{4R}(R^4 - r^4)$$



Section modulus of common x-section :

1. **Triangle:**

$$Z = \frac{1}{24}(bh^2)$$



Bending stress distribution

- Consider a cantilever beam having a circular cross section is subjected to sagging type bending.
- The fibre above the neutral fibre will get experience compression so compressive stress induced in upper half of the x-section.
- So maximum compressive stress induced at the top most fibre because

$$y = y_{max} = \frac{d}{2}$$

Bending stress distribution

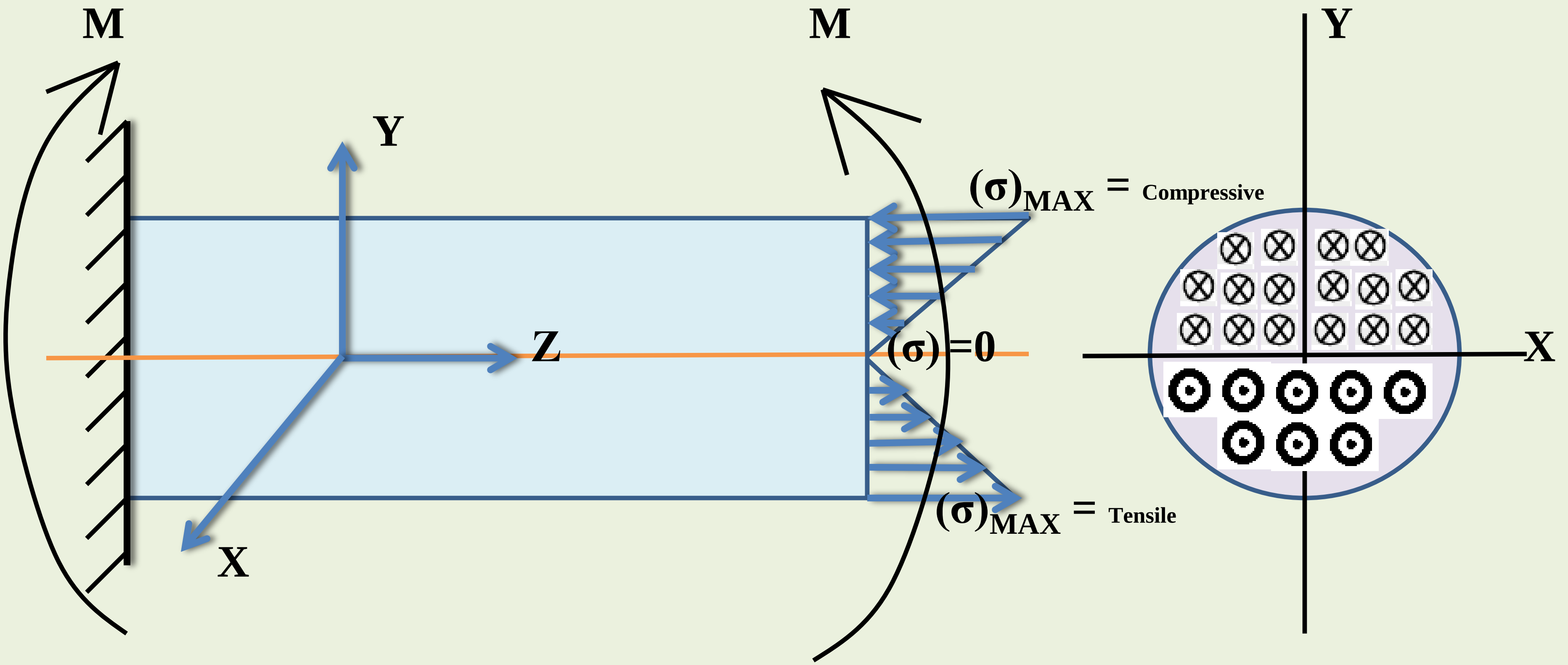
- The fibre below the neutral fibre will get experience tension so tensile stress induced in lower half of the x-section.
- So maximum tensile stress induced at the bottom most fibre because

$$y = y_{max} = \frac{d}{2}$$

- Bending stress (σ_b) is linearly vary with respect to y.

$$\sigma_b \alpha\, y$$

Bending stress distribution



Note:-

1) The nature of bending stress and axial stress is same. That can be add or subtract directly.

2) In the same way direct shear stress and torsional shear stress have same nature, they can also directly add up and subtract.

Example (1)

If the maximum axial stress induced in the member at a point is 10 MPa tensile and maximum bending stress induced 30 MPa compressive. Determine the resultant stress induced at that point.

Answer : 20 MPa (Compressive)

Torsional or Twisting

Torsional equation is given as -

$$\frac{T_R}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

T_R = Twisting couple or Torque (N-mm)

J = Polar moment of inertia (mm^4)

τ_{Max} = Torsional shear stress (Mpa)

r = It is the distance between the rotational axis and the arbitrary point in the section (mm)

G = Shear modulus (MPa)

θ = angle of twist (radian)

L = distance between the fixed end and arbitrary x-section. (mm)

Torsional or Twisting equation assumptions

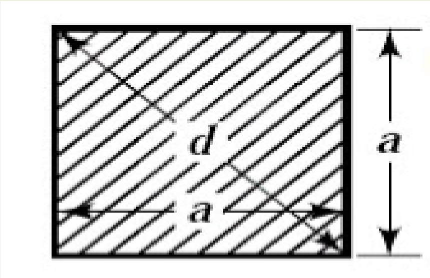
In the development of a torsion formula for a circular shaft, the following assumptions are made:

- Material of the shaft is homogeneous & isotropic throughout the length of the shaft.
- Shaft is straight and of uniform circular cross section over its length.
- Torsion is constant along the length of the shaft.
- Cross section of the shaft which are plane before torsion remain plane after torsion.
- Radial lines remain radial during torsion.
- Stresses induced during torsion are within the elastic limit.

Polar moment of inertia for common shape

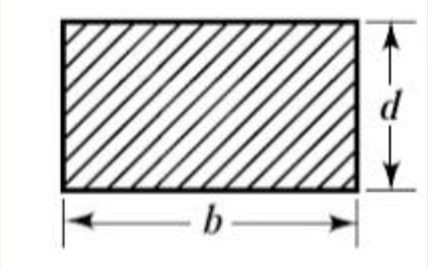
1. Square section:

$$J = \frac{a^4}{6}$$



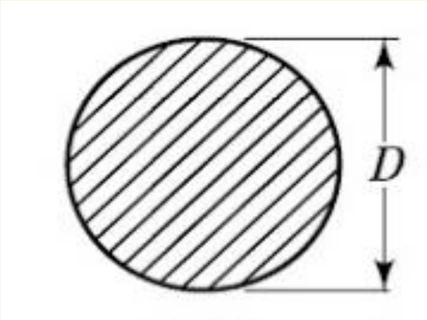
2. Rectangular section

$$J = \frac{bd(b^2 + d^2)}{12}$$



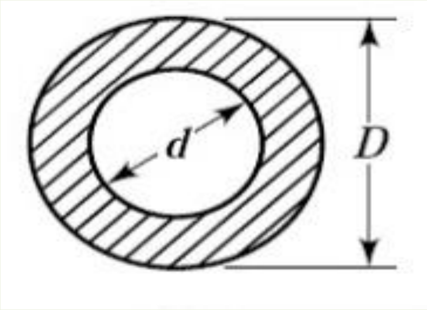
Polar moment of inertia for common shape

3. Solid circular section:



$$J = \frac{\pi D^4}{32}$$

4. Hollow circular section (Tube section)



$$J = \frac{\pi}{32}(D^4 - d^4)$$

Torsional shear stress distribution

From the above equation -

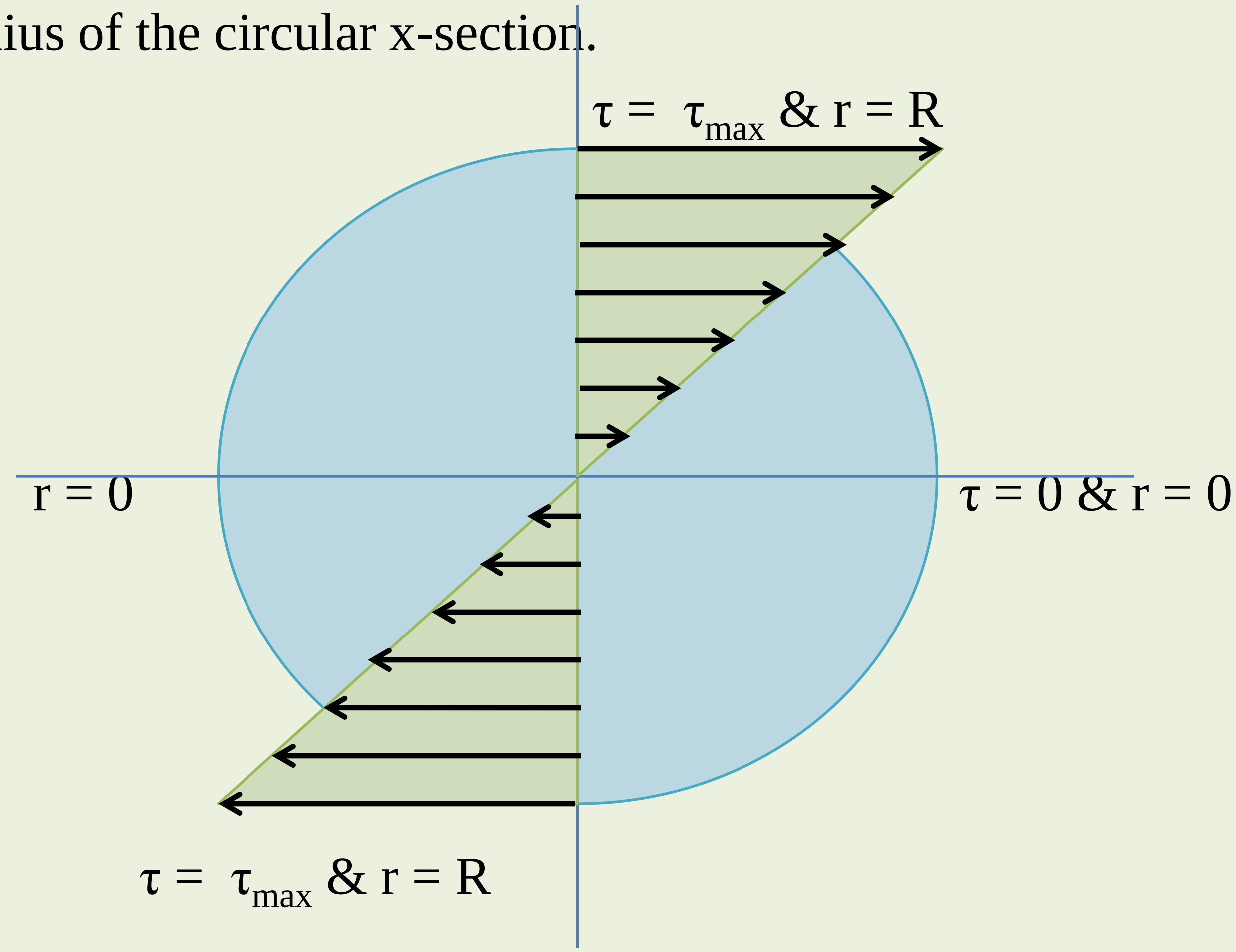
$$\frac{T_R}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

$$\tau \propto r$$

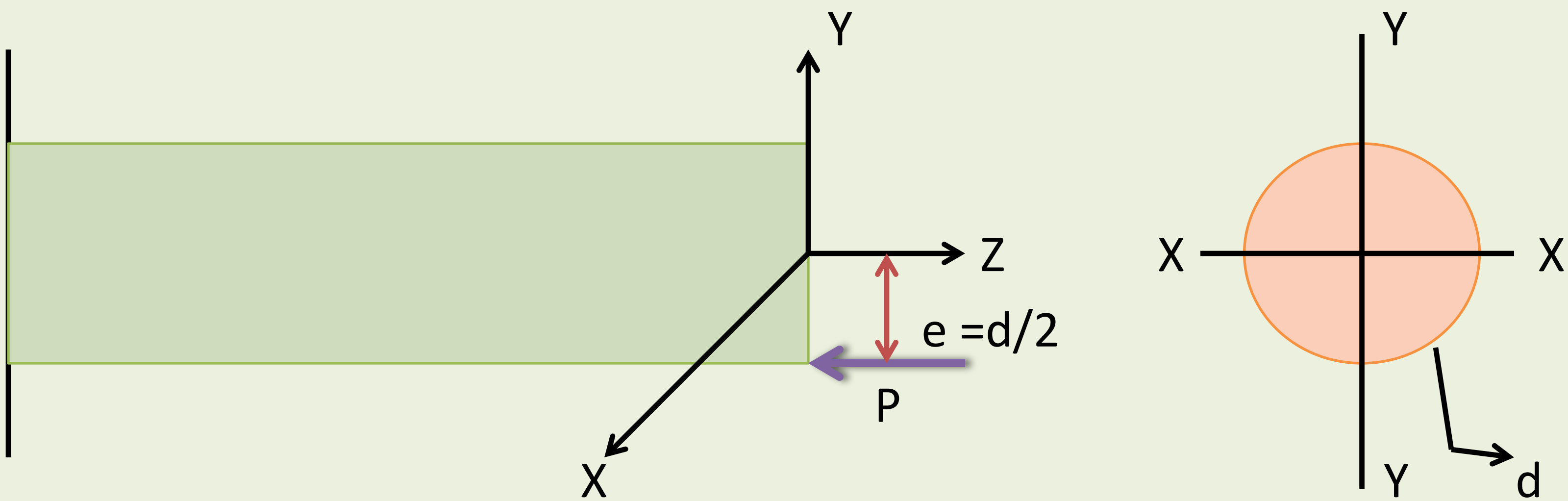
- There is a linear relationship between τ & r.
- If we increase r value τ value will be increase and vice versa.
- For $\tau = \tau_{Max}$, $r = r_{max}$ = outer most point

Torsional shear stress distribution

if R is the radius of the circular x-section.



Q. For the member as shown in the figure determine the maximum stress developed on the x-section of the member.



Options:

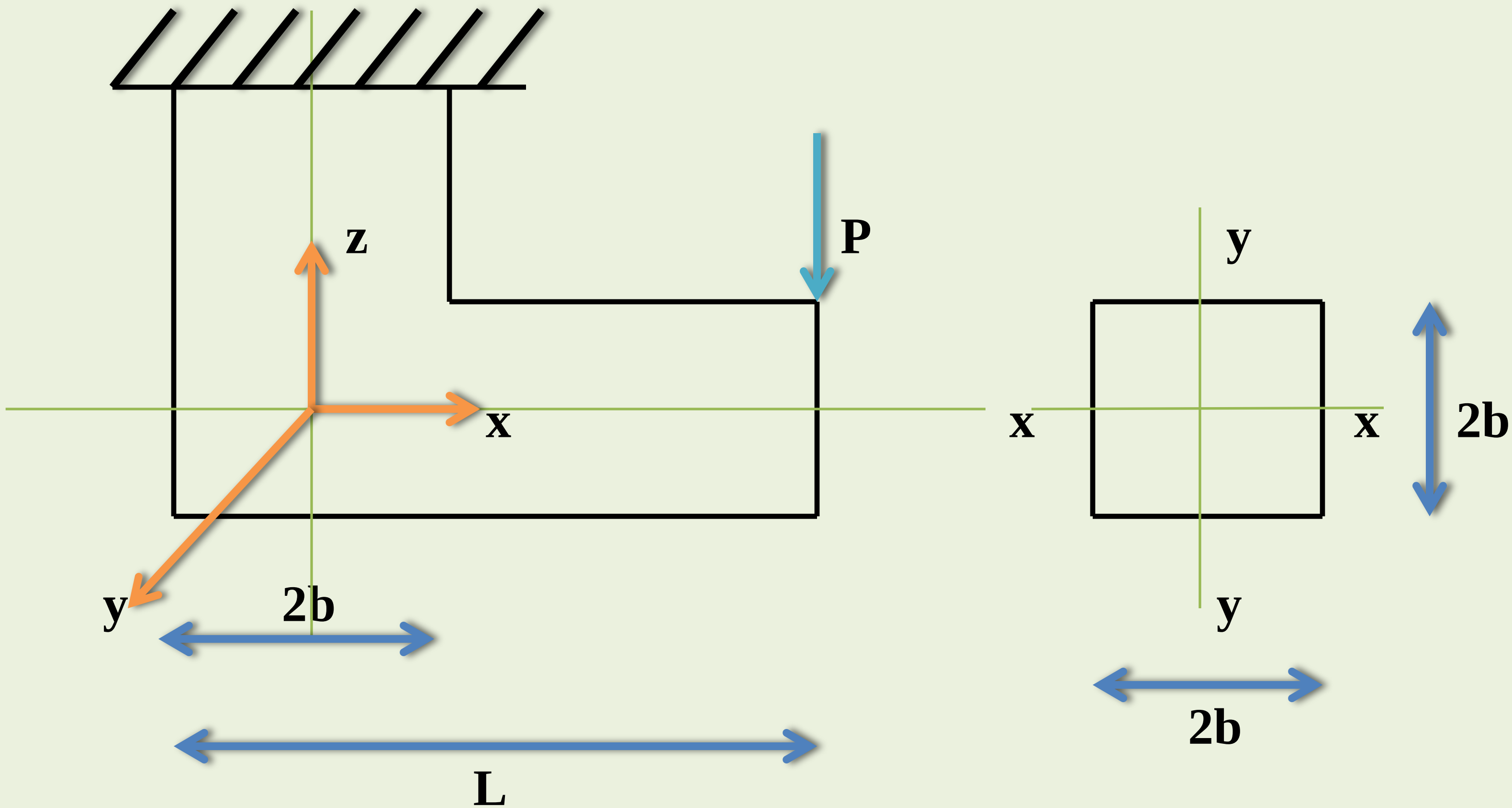
$$20\left(\frac{P}{\pi d^2}\right)$$

$$16\left(\frac{P}{\pi d^2}\right)$$

$$24\left(\frac{P}{\pi d^2}\right)$$

$none$

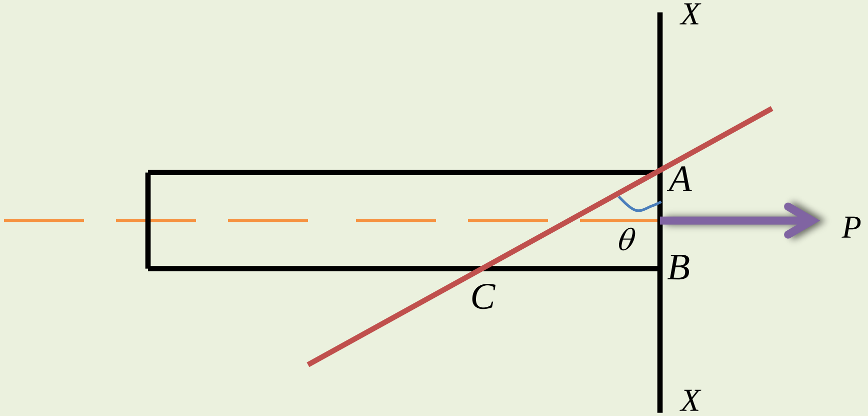
Q. For the member as shown in the figure determine the maximum stress developed on the vertical x-section of the member.



State of Stress

Basic Terminology

Oblique plane: Plane cut at an angle of θ with respect to cross sectional plane X-X.



State of stress: It is representation of all the normal stresses (σ) & shear stresses (τ) acting at a point in a material in all the three mutual perpendiculars directions.

Uni-axial state of stress :

- In uniaxial state of stress or 1-D state of stress, normal stress is acting only in one dimensional.
- All the stress components (normal & shear) acting in other 2 direction (y & z) are zero.
- The state of stress is given as-

$$[\sigma]_{1-D} = [\sigma_{xx}]$$

Bi-axial state of stress (2-D state of stress):

- When a component is subjected to different loading in such a way that the stresses produced act on two planes perpendicular to each other.
- All the stress components (normal & shear) acting in 3 direction (i.e. z direction) are zero.

- The state of stress is given as-

$$[\sigma]_{2-D} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{pmatrix}$$

<u>Tri-axial state of stress (3-D state of stress):</u>

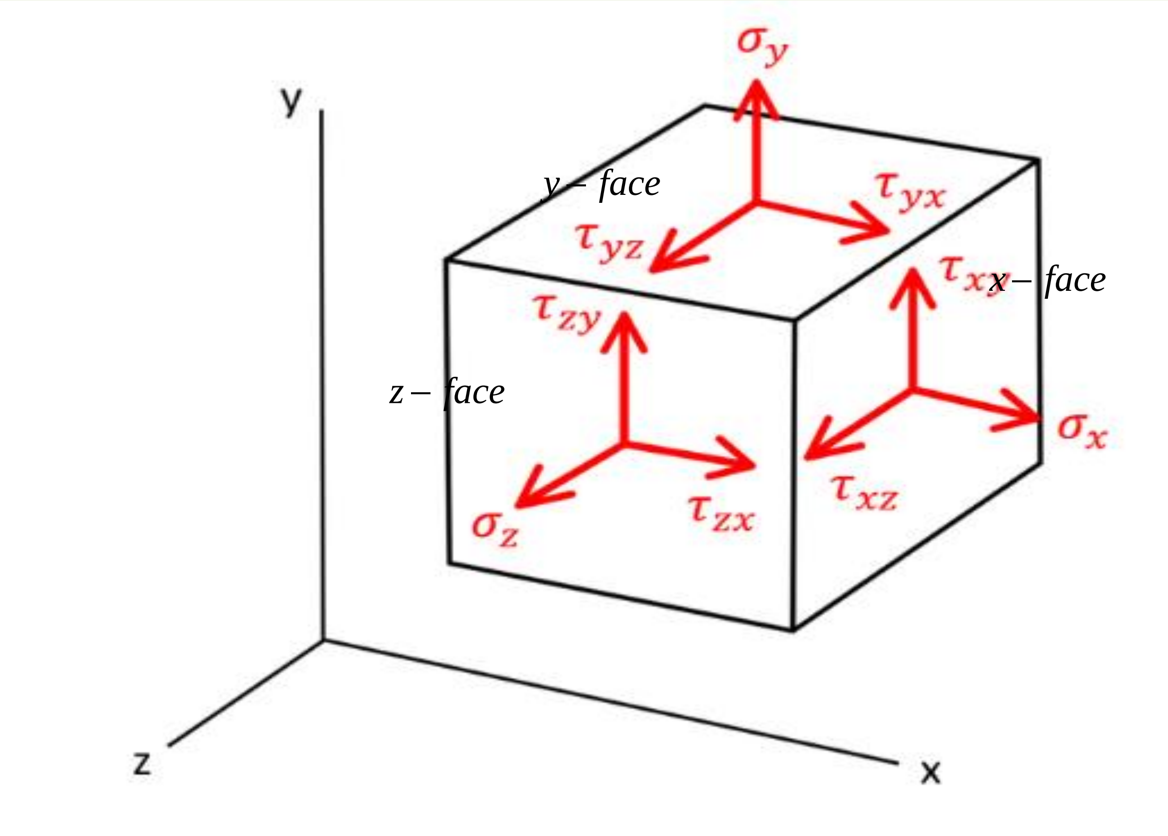
- when a component is subjected to different loading in such a way that the stresses produced act on three mutual perpendicular planes.
- There are total 9 stress components (3 normal stress & 6 shear stress).
- The state of stress is given as -

$$[\sigma]_{3-D} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

Normal stress representation

Normal stress (σ_A)

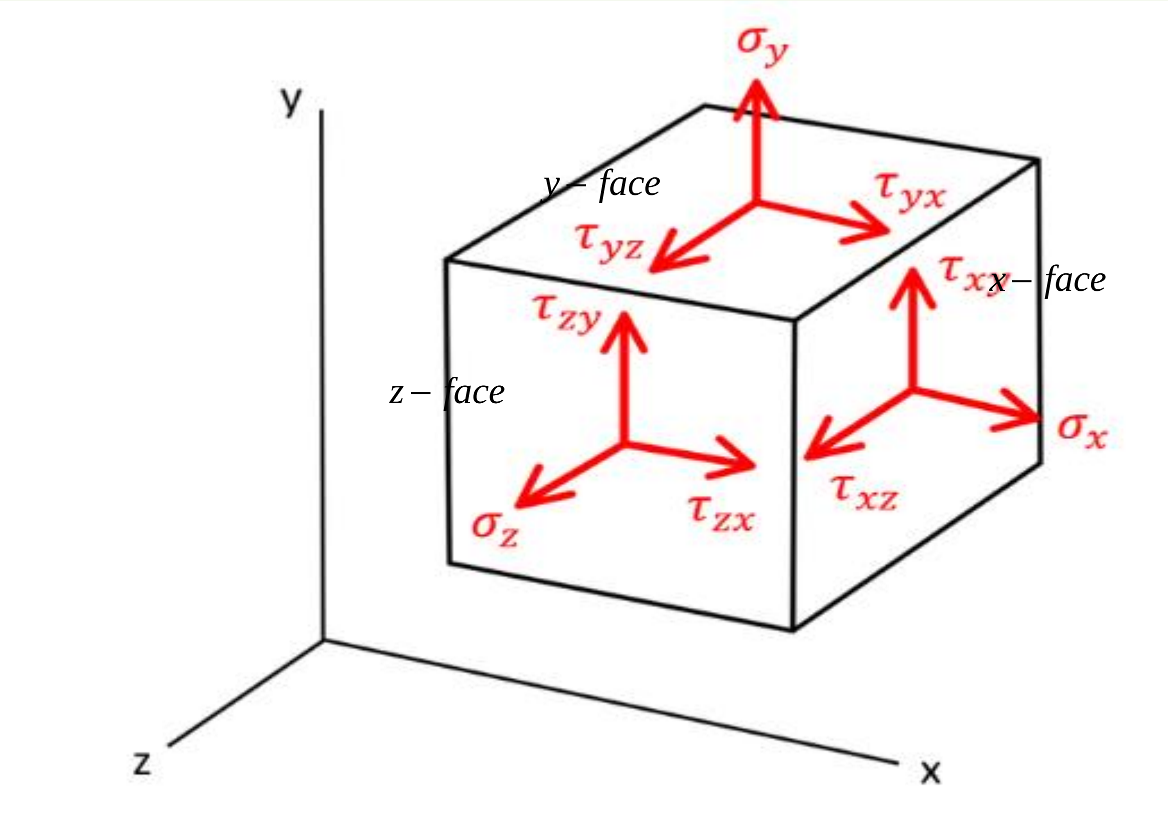
- A = It is a subscript, which represents the face on which normal stress is acting.
- (σ_x) = Normal stress is acting on x-face.
- (σ_y) = Normal stress is acting on y-face.
- (σ_z) = Normal stress is acting on z-face.



Shear stress representation

Shear stress (τ_{AB})

- A = It is a subscript, which represents the face on which shear stress is acting.
- B = It is a subscript, which represents the direction on which shear stress is acting.
- (τ_{xy}) = Shear stress is acting on x-face & y-direction..
- (τ_{yz}) = Shear stress is acting on y-face & z-direction.

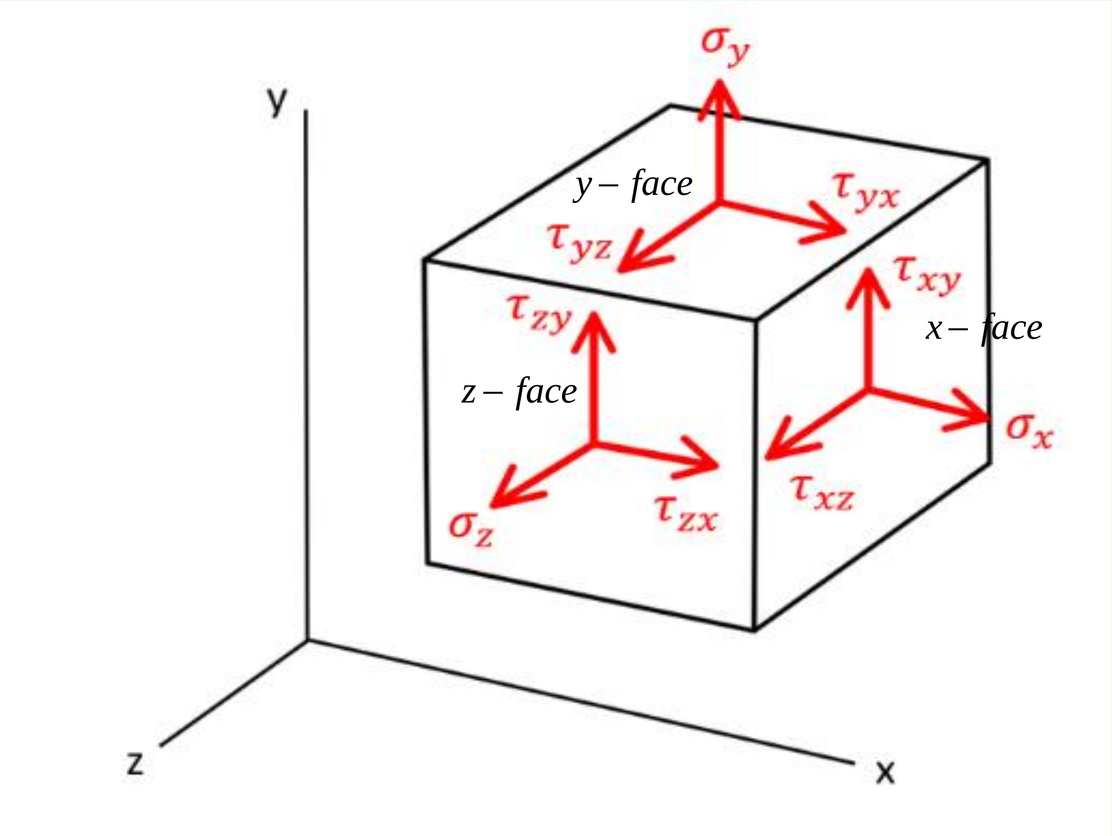


For static equilibrium

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$



τ_{xy} & τ_{yx} are also called complimentary shear stress.

UNIAXIAL STATE OF STRESS

Stress components on y-face & z-face are zero.

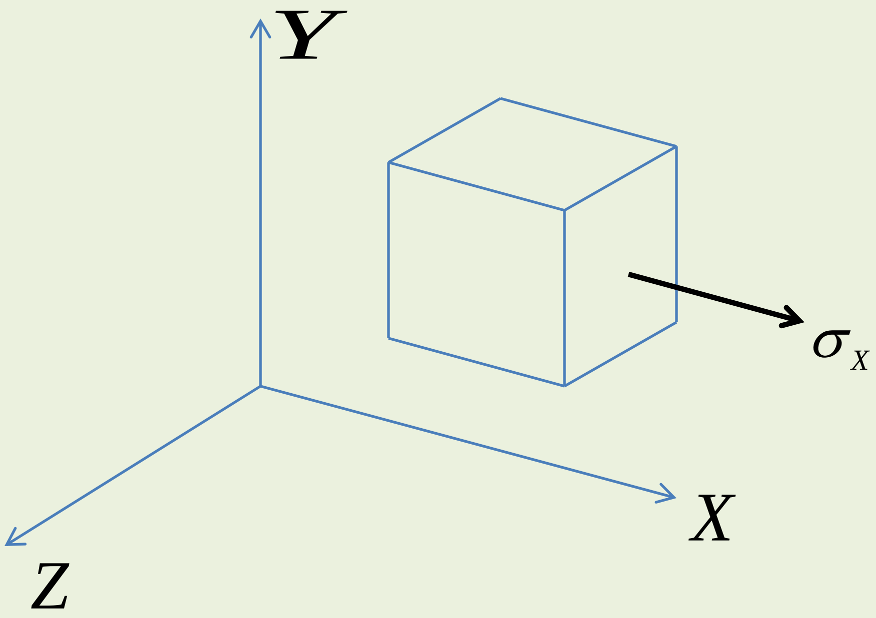
$$\sigma_y = 0$$

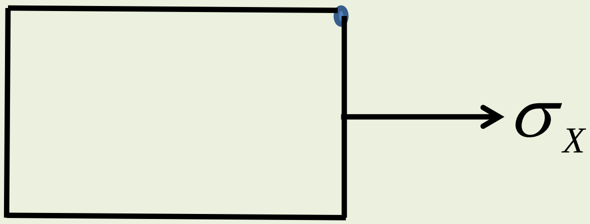
$$\sigma_z = 0$$

$$\tau_{zy} = \tau_{yz} = 0$$

$$\tau_{xy} = \tau_{yx} = 0$$

$$\tau_{zx} = \tau_{xz} = 0$$





BI-AXIAL STATE OF STRESS

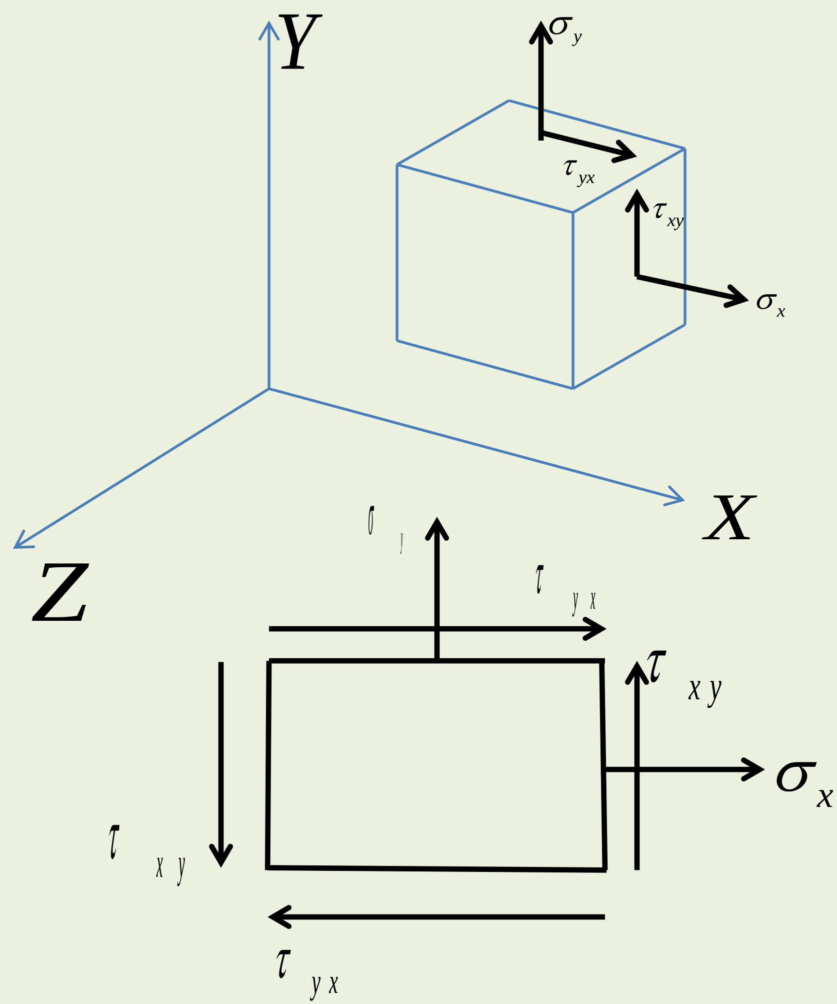
Stress components on z-face are zero.

$$\sigma_z = 0$$

$$\tau_{zy} = \tau_{yz} = 0$$

$$\tau_{zx} = \tau_{xz} = 0$$

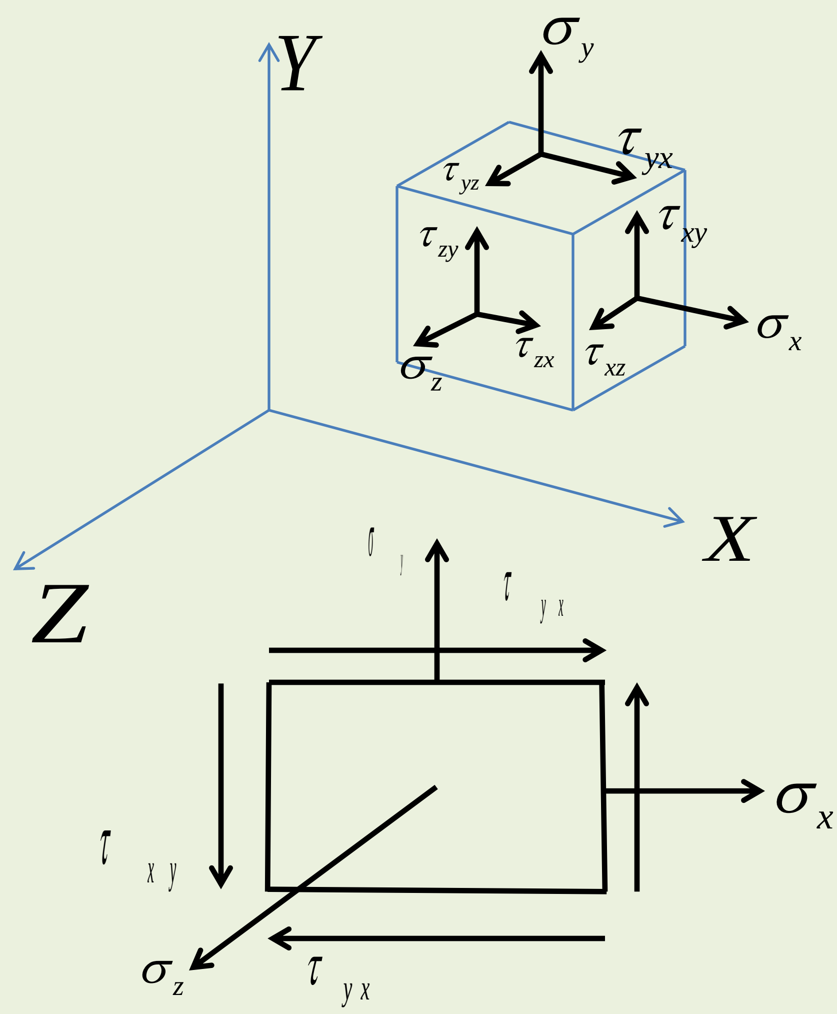
$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



TRI-AXIAL STATE OF STRESS

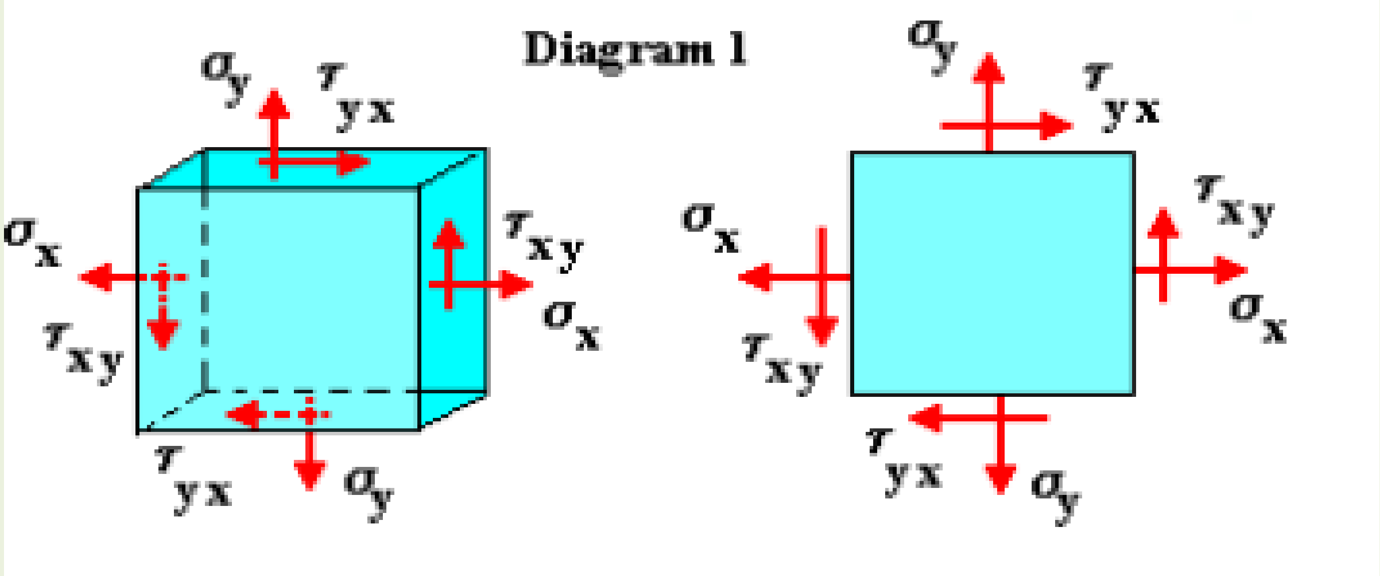
- All stress components (3 Normal stress & 6 shear stress) will acting at a point.
- Stress tensor is given as-

$$[\sigma]_{3-D} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

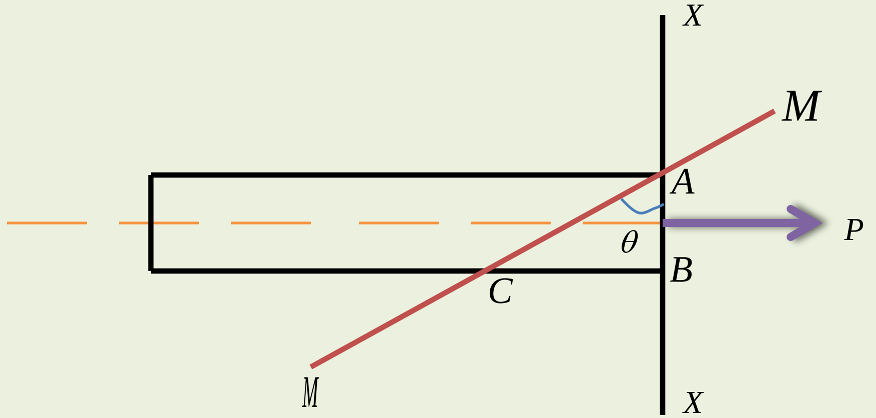


<u>Sign convention for shear stress</u>

- The Shear Stress will be considered positive when a pair of shear stress acting on opposite sides of the element produce a clockwise (CW) torque (couple).



Expression for normal stress and shear stress on oblique plane under uni-axial state of stress



Consider a oblique plane M-M, cut at an angle θ in clockwise direction from the x-sectional plane X-X.

P - Axial tensile load (N)

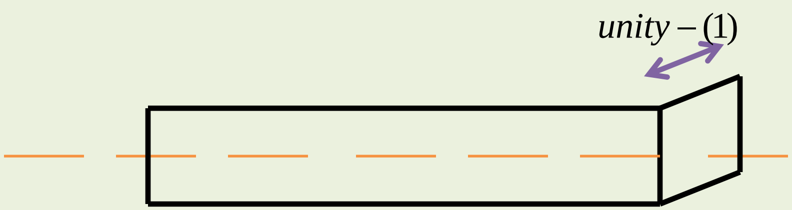
A - X-sectional area on X-X plane.

Tensile axial stress on X-X section is given as-

$$\sigma = \frac{P}{A}$$

θ = Angle made by oblique plane w.r.t x-sectional plane X-X.

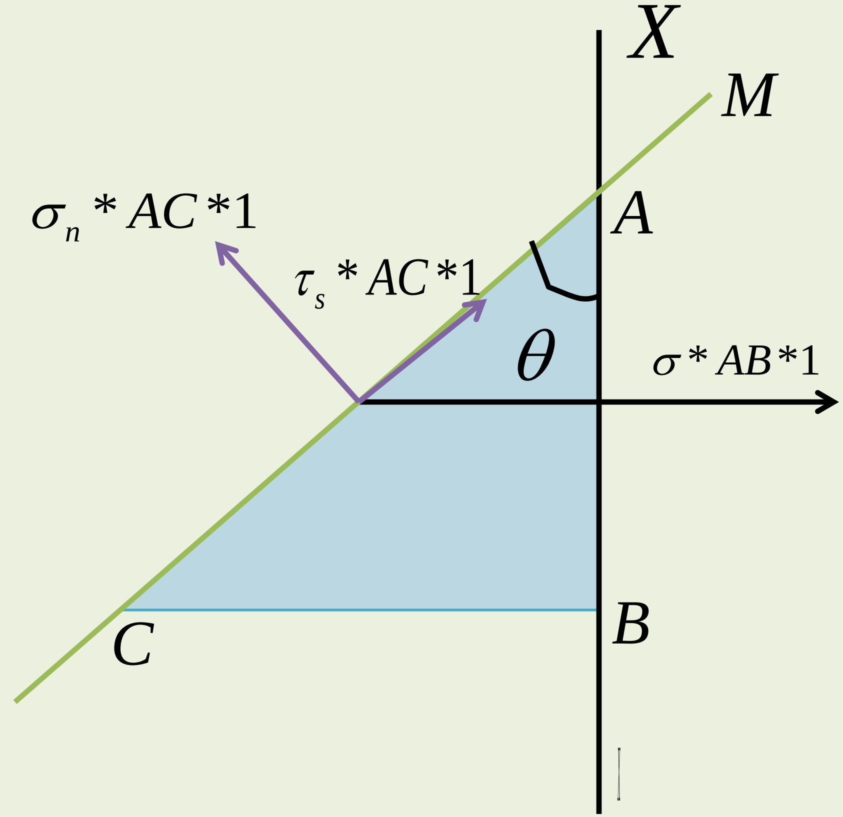
Let assume the depth of the x-section is unity.



σ_n = Normal stress on oblique plane

($\perp$ to the plane M-M plane)

τ_s = shear stress on oblique plane (II to the M-M plane)



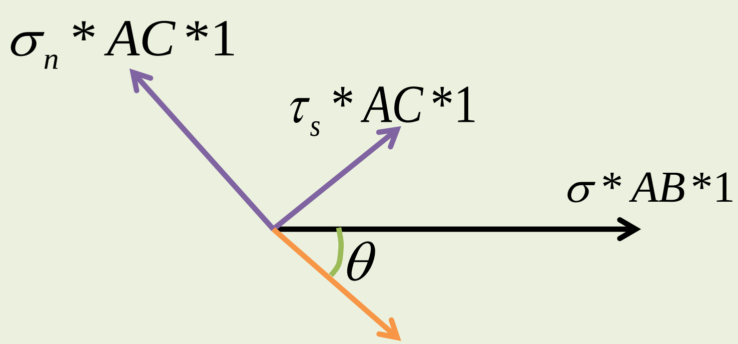
Taking all the force component $\perp$ to the oblique plane.

$$(\sigma_n * AC) - (\sigma * AB * \cos\theta) = 0$$

$$(\sigma_n * AC) = (\sigma * AB * \cos\theta)$$

$$\sigma_n = \sigma * \left(\frac{AB}{AC}\right) * \cos\theta$$

$$\sigma_n = \sigma * \cos^2\theta$$



Taking all the force components parallel to the oblique plane :

$$(\tau_s * AC) + (\sigma * AB * \text{Sin}\,\theta) = 0$$

$$(\tau_s * AC) = -(\sigma * AB * \text{Sin}\,\theta)$$

$$\tau_s = -(\sigma * \frac{AB}{AC} * \text{Sin}\,\theta)$$

$$\tau_s = -\sigma * \text{Cos}\,\theta * \text{Sin}\,\theta$$

$$\tau_s = -(\frac{\sigma}{2} * \text{Sin}\,2\theta)$$

1. When θ = 0°, oblique plane become x-sectional plane. Axial & shear stress on oblique plane is given as-

$$\sigma_n = \sigma \cos^2 \theta = \sigma$$

$$\tau_s = -\frac{\sigma}{2} \sin 2\theta = 0$$

Important point

1. Axial tensile stress is always taken positive.
2. θ is always measured in CW (clockwise) direction from the x-sectional plane (X-X) & taken as positive.
3. If θ is measured ACW (anti-clockwise) direction from the x-sectional plane (X-X) taken as negative ($-\theta$).

Principal plane (Plane of zero shear):

A plane on which normal stress is either maximum or minimum & shear stress is zero, it said to be principal plane.

On principal plane-

$$\sigma_n = \text{Maximum or minimum}$$

$$\tau_s = \text{zero}$$

Principal stress:

The normal stress value of the principal plane is said to be principal stress.

- In **uniaxial state of stress** only 1 principal stress present. (σ_1)
- In biaxial state of stress condition, there are 2 type of principal stress-

1. Maximum principal stress (σ_1)
2. Minimum principal stress (σ_2)

- In tri-axial state of stress condition, there are 3 type of principal stress induced. ($\sigma_{1,}\ \sigma_{2.}\ \sigma_3$).

Maximum shear stress plane:

1. A plane on which shear stress is maximum.
2. On maximum shear stress plane the normal stress is non zero.
3. The non zero value of normal stress is the average of maximum and minimum principal stress.

Pure shear plane

A plane on which normal stress is zero, it said to be plane of pure shear.

On pure shear plane-

$$\sigma_n = 0$$

- When principal stresses are equal in magnitude but unlike in nature.

$$\sigma_1 = - \sigma_2$$

Conclusion

Under the uniaxial loading, the normal & shear stress on a oblique plane is given as-

$$\sigma_n = \sigma * \cos^2\theta \quad \& \quad \tau_s = -\frac{\sigma}{2}\sin 2\theta$$

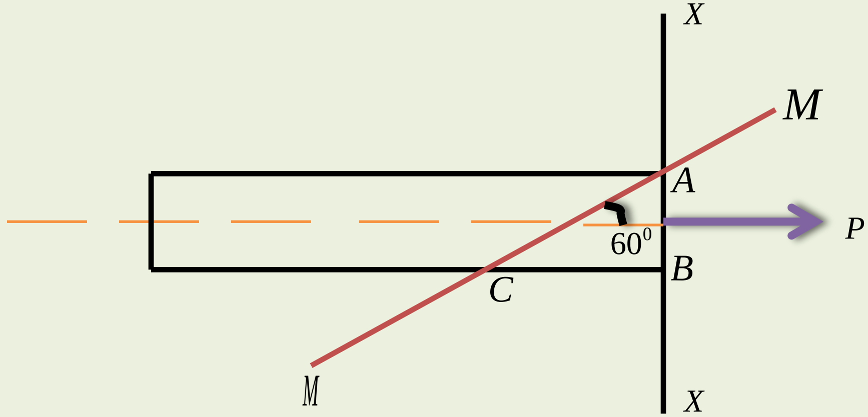
Plane of location	Normal stress (σ_n)	Shear stress (τ_s)	Remark
$\theta = 0°$	σ	0	1. This is a major principal plane. 2. Normal stress = Maximum 3. Shear stress = 0
$\theta = 45°$	$\sigma/2$	$-\sigma/2$	
$\theta = 90°$	0	0	1. This is a minor principal plane. 2. Normal stress = Minimum 3. Shear stress = 0
$\theta = 135°$	$\sigma/2$	$\sigma/2$	

Q 1. Determine the following when a prismatic bar is subjected to a axial tensile load of 20 KN. Assume cross sectional area of the bar is 200 mm^2.

a) Normal and shear stress developed on an oblique plane making an angle 60° from longitudinal axis.

b) Resultant stress on the maximum shear stress plane.

c) Maximum tensile and compressive stress & maximum shear stress developed on a prismatic bar.



Important

[Stress is a II^{nd} order tensor quantity]

- Zero order tensor example = All Scalar Quantity (speed, distance etc.)
- First order tensor example = Vector quantity (Magnitude & Direction)
- 2nd order tensor example = stress (Magnitude, direction & face)

<u>**Important**</u>

While solving any numerical problem in mechanics of solids-

- Always put load value in Newton (N)
- Always put all the dimensions in mm.
- Stress unit - MPa ($\frac{N}{mm^2}$)

Expression for normal stress and shear stress on oblique plane under bi-axial state of stress

BI-AXIAL STATE OF STRESS

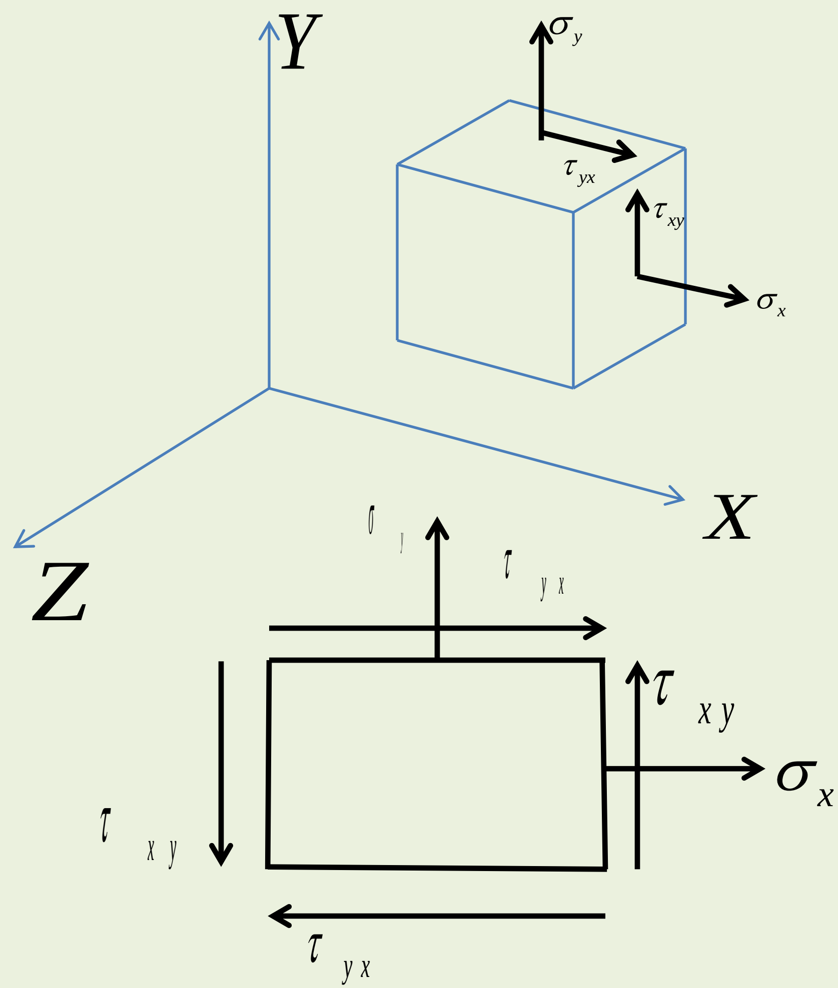
Stress components on z-face are zero.

$$\sigma_z = 0$$

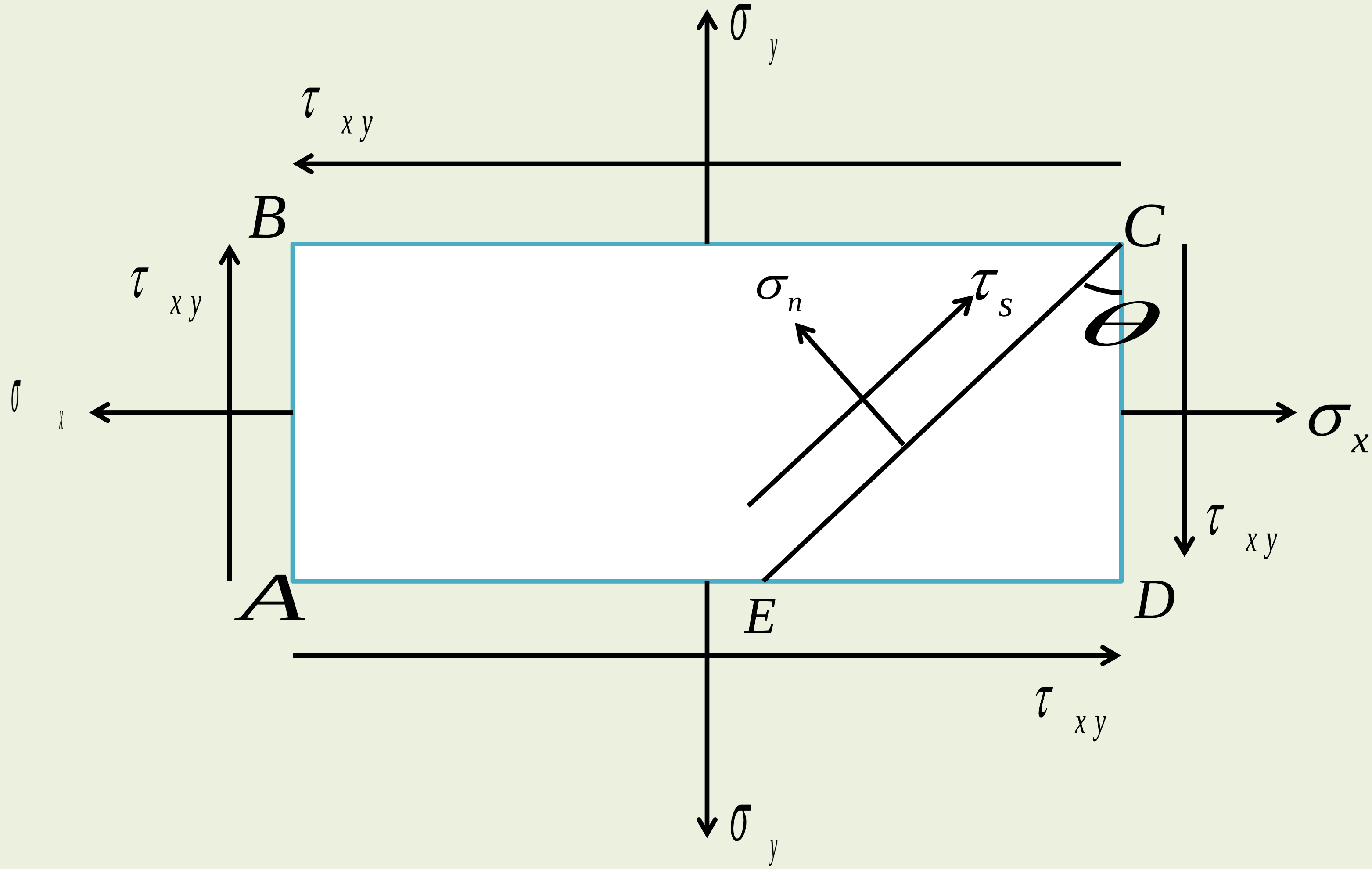
$$\tau_{zy} = \tau_{yz} = 0$$

$$\tau_{zx} = \tau_{xz} = 0$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

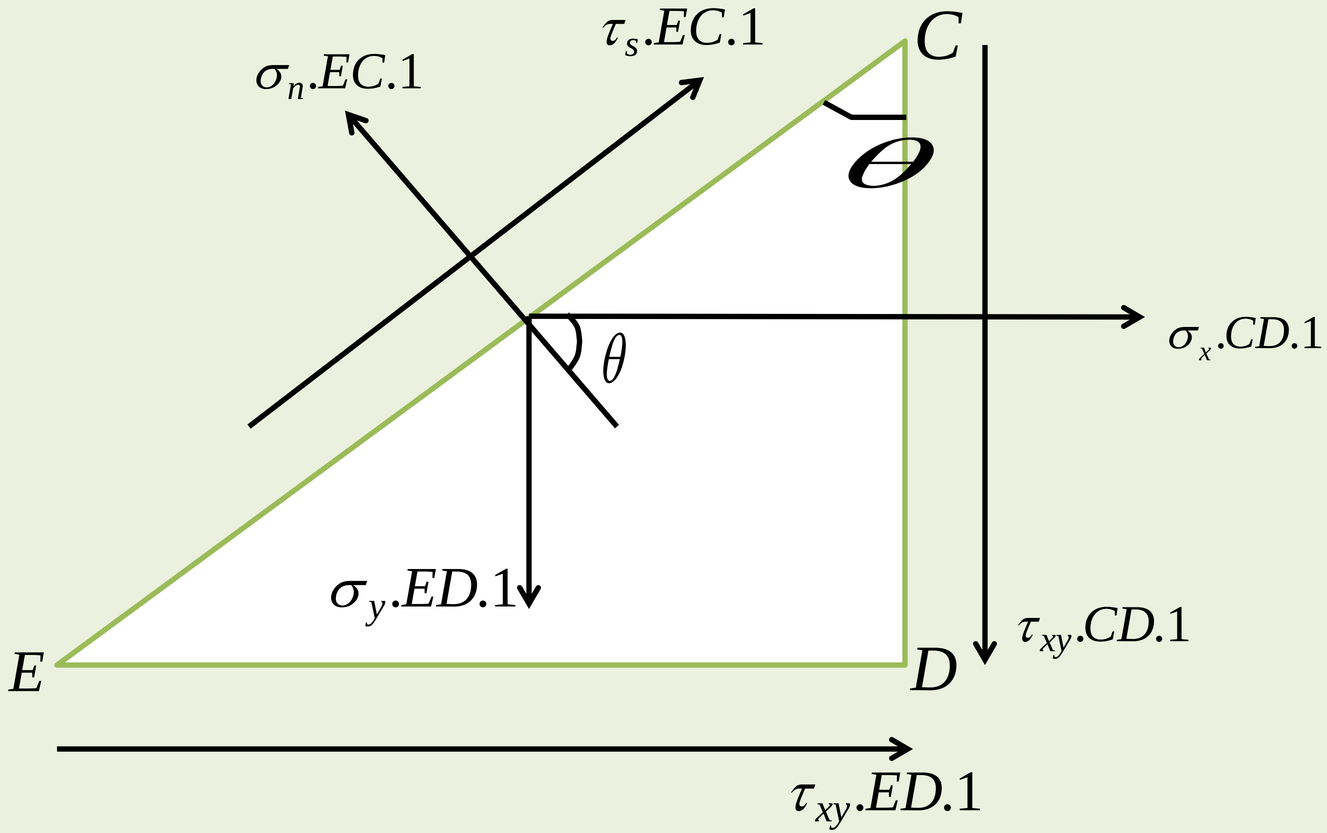


ANKIT SAXENA (saxena01ankit@gmail.com)



Let assume that the member has a unity depth.

Force balance on oblique plane



Resolving the force ⊥ to the oblique plane :

Normal stress on oblique plane is given as-

$$\sigma_n.CE-\sigma_x.CD.\cos\theta-\sigma_y.ED.\sin\theta-\tau_{xy}.CD.\sin\theta-\tau_{xy}.ED.\cos\theta=0$$

$$\sigma_n=\sigma_x(\frac{CD}{CE})\cos\theta+\sigma_y(\frac{ED}{CE})\sin\theta+\tau_{xy}(\frac{CD}{CE})\sin\theta+\tau_{xy}(\frac{ED}{CE})\cos\theta$$

$$\sigma_n=\sigma_x\cos^2\theta+2\tau_{xy}\cos\theta\sin\theta+\sigma_y\sin^2\theta$$

$$\sigma_n=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+\tau_{xy}\sin 2\theta$$

$$\sigma_n=\sigma_x(\frac{1+\cos 2\theta}{2})+\sigma_y(\frac{1-\cos 2\theta}{2})+\tau_{xy}\sin 2\theta$$

$$\sigma_n=\frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta+\tau_{xy}\sin 2\theta$$

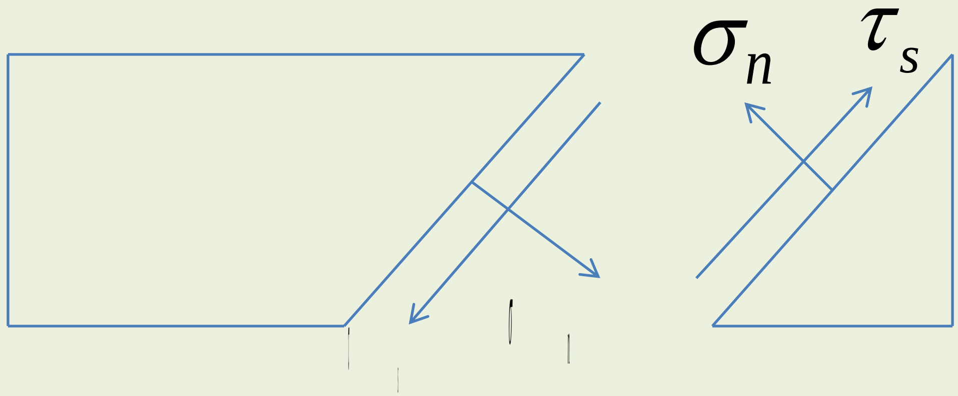
Resolving the force II to the oblique plane :

Shear stress on oblique plane is given as-

$$\tau_s = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

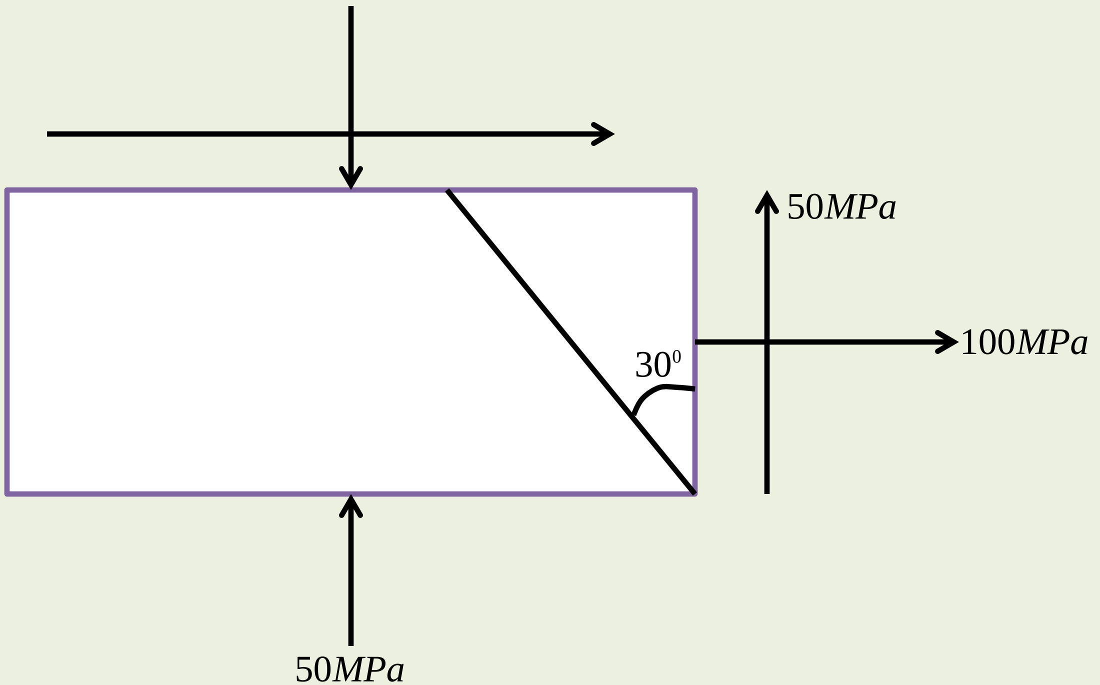
Note:-

1. Axial tensile stress is taken as positive.
2. The angle of inclination (θ) of oblique plane w.r.t x-sessional plane is taken as positive in clockwise direction (CW).
3. The shear stress τ_{xy} on x-face is taken as positive when it is in CW (clockwise direction).

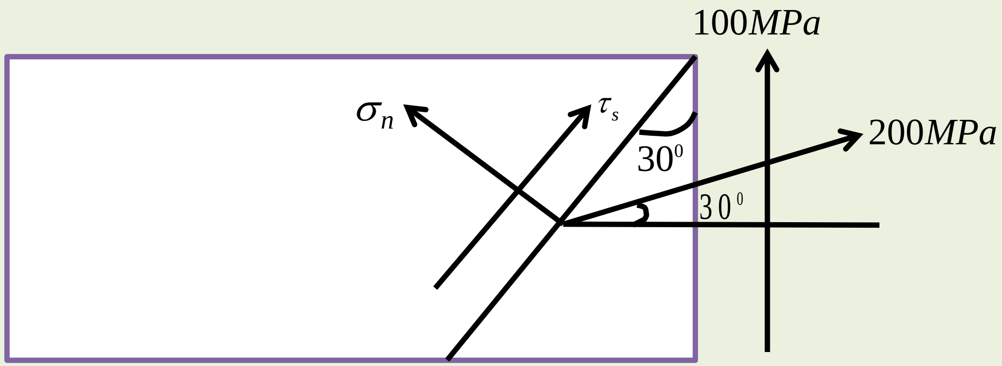


Shear stress on oblique plane is taken as positive when it forms a couple in anti-clockwise direction (ACW).

Q1. Determine the normal stress and shear stress on an oblique plane for a given state of stress.



Q2. Determine the normal stress and shear stress on an oblique plane for a given state of stress.



Q3. A point is subjected to equal & unlike normal stress on 2 mutually ⊥ plane passing through a point. Determine normal stress & shear stress on an oblique plane inclined 45° to the x-section plane.

Expression for principal plane location & Principal stress

On principal plane, normal stress is maximum and shear stress is zero.

Method 1:

$$\frac{d}{d\theta}(\sigma_n) = 0$$

$$\frac{d}{d\theta}[\frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta+\tau_{xy}\sin 2\theta] = 0$$

$$\frac{1}{2}(\sigma_x-\sigma_y)(-2\sin 2\theta)+\tau_{xy}(2\cos 2\theta)=0$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

Method 2:

$$\tau_s = 0$$

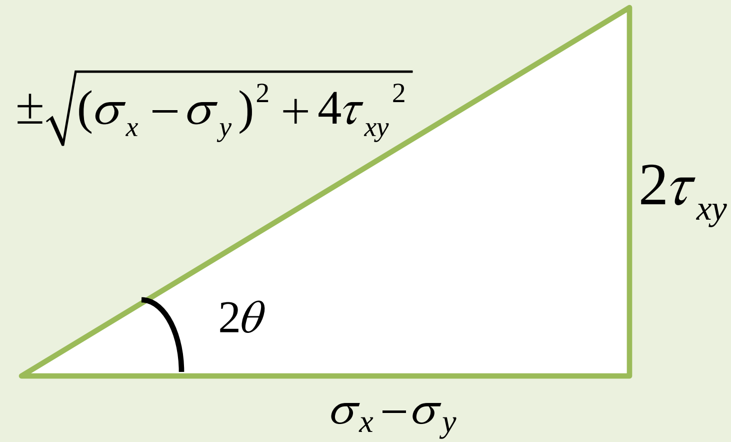
$$-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

From both the methods we get the same result. Apply trigonometry rule, we get

$$\sin 2\theta = \frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$



Substitute the value of sin2θ & cos2θ in normal stress (σ_n) on oblique plane equation and we get the principal stresses σ_1 & σ_2 value in biaxial state of stress condition.

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{1/2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y).\frac{(\sigma_x - \sigma_y)}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} + \tau_{xy}.\frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sigma_{1/2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}[\frac{(\sigma_x - \sigma_y)^2}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} + \frac{4\tau_{xy}^2}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}]$$

$$\sigma_{1/2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2})$$

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2})$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2})$$

Determination of maximum shear stress & orientation of maximum shear stress plane

For τ_s to be Maximum -

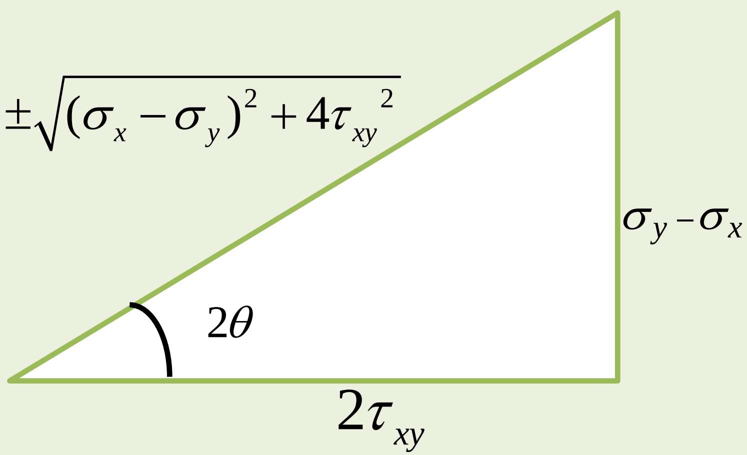
$$\frac{d}{d\theta}(\tau_s) = 0$$

$$\frac{d}{d\theta}[-\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

$$-\frac{1}{2}(\sigma_x - \sigma_y).(2\cos 2\theta) + \tau_{xy}.(-2\sin 2\theta) = 0$$

$$\tan 2\theta = \frac{-\frac{1}{2}(\sigma_x - \sigma_y)}{\tau_{xy}}$$

$$\tan 2\theta = \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}$$



Apply trigonometry rule, we get

$$\sin 2\theta = \frac{\sigma_y - \sigma_x}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2}}$$

$$\cos 2\theta = \frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2}}$$

Substitute the value of sin2θ & cos2θ in τ_s equation, we get τ_{Max}

$$\tau_{Max} = \pm\frac{1}{2}\left(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2}\right)$$

$$\tau_{Max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

Shear Stress (τ_{Max})

In plane τ_{Max}	Absolute τ_{Max}

In plane τ_{Max}

- Calculate τ_{Max} on a plane
- Plane is a 2D
- Always consider bi-axial state of stress.
- It consists of only 2 principal stress.

Absolute τ_{Max}

- Calculate τ_{Max} at a point
- Point is a 3D
- Always consider tri-axial state od stress
- It consists of 3 principal stress.
- If 3rd principal stress is not given, its value taken as zero.

Tri-axial state of stress

In tri-axial state of stress condition Absolute (Abs.) τ_{Max} is given as -

$$abs.\tau_{Max} = Max^m \left| \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2} \right|$$

Biaxial state of stress

In biaxial state of stress condition Absolute (Abs.) τ_{Max} is given as -

Tri-axial $\Rightarrow$ Biaxial

$\sigma_3 \Rightarrow 0$

$$abs.\tau_{Max} = Max^m \left| \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2}{2}, \frac{\sigma_1}{2} \right|$$

Biaxial state of stress

In biaxial state of stress condition In plane τ_{Max} is given as -

$$In\ plane\ \tau_{Max} = \left|\frac{\sigma_1 - \sigma_2}{2}\right|$$

Important point

The summation of normal stress on complimentary plane (90° to each other) is constant at given state of stress or at given point.

Biaxial state of stress $\Rightarrow$ $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$

Tri-axial state of stress $\Rightarrow$ $\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z$

Plane of pure shear (τ*)

σ_n =0

θ = ?

$$\tau^* = \sqrt{-\sigma_1\sigma_2} \quad \text{Mpa}$$

If both σ_1 & are σ_2 +ve

Then $\tau^* = \sqrt{-ve}$

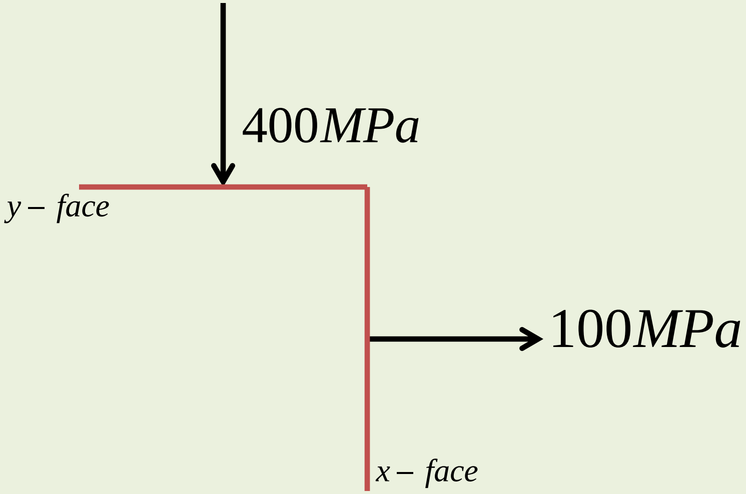
⇒ No plane of pure shear

$$\tan\beta = \sqrt{\frac{\sigma_1}{\sigma_2}}$$

β = Angle made by plane of pure shear with respect to the principal plane.

Q. for a biaxial state of stress at a point as shown in figure, determine the following -

1. Principal plane & principal stress
2. Shear stress on a plane of pure shear
3. Maximum tensile and compressive stress
4. Maximum shear stress developed at a point.



Q. for the given stress tensor at a point , determine the following -

1. Principal stress at that point
2. Shear stress at that point

$$[\sigma]=\begin{pmatrix} 40 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

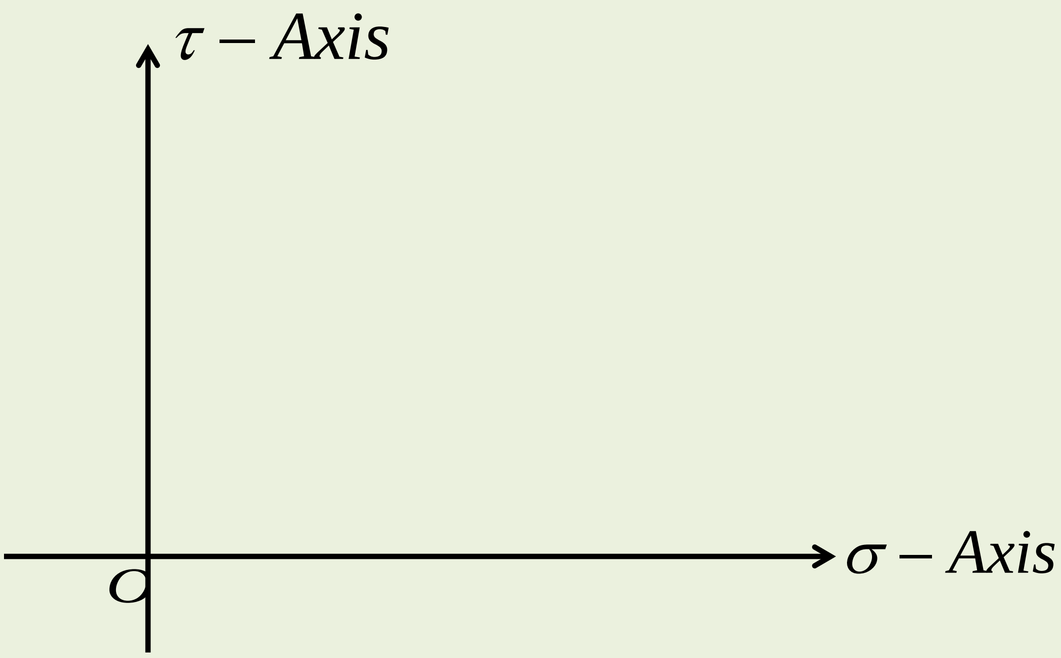
Mohr's Circle

Mohr's Circle

- It is a graphical representation of Biaxial (2D) state of stress.
- It is a stress circle which is used to determine the normal stress & shear stress on any oblique plane.
- Mohr's circle is also used to determine the principal stresses and principal planes location.

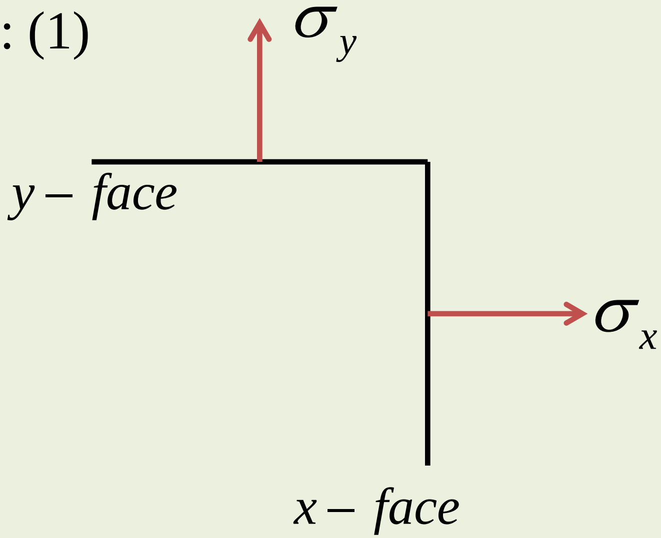
Sign convention used in Mohr's circle

1. On X-axis, we represents normal stress & on Y-axis we represents shear stress.

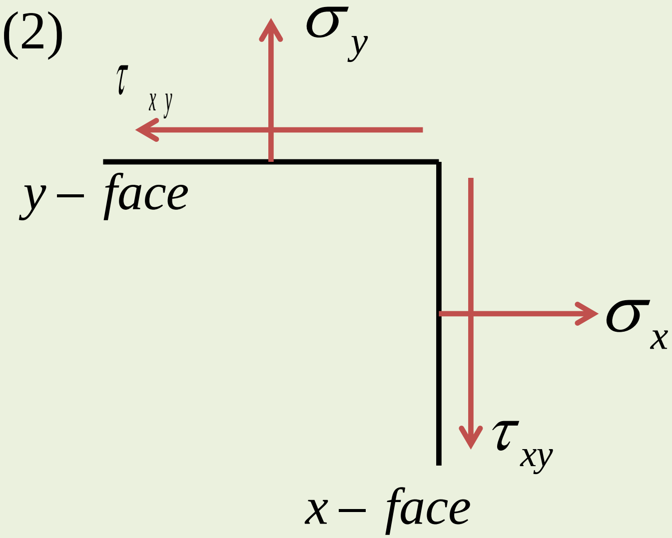


2. Draw state of stress of a given point.

Example: (1)



(2)

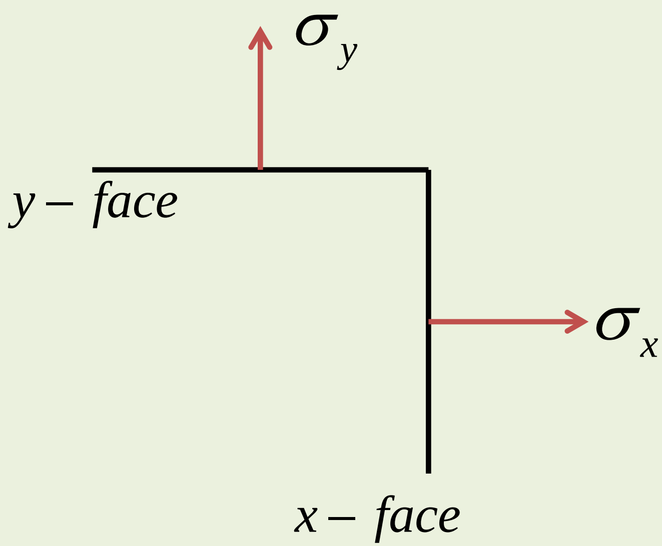


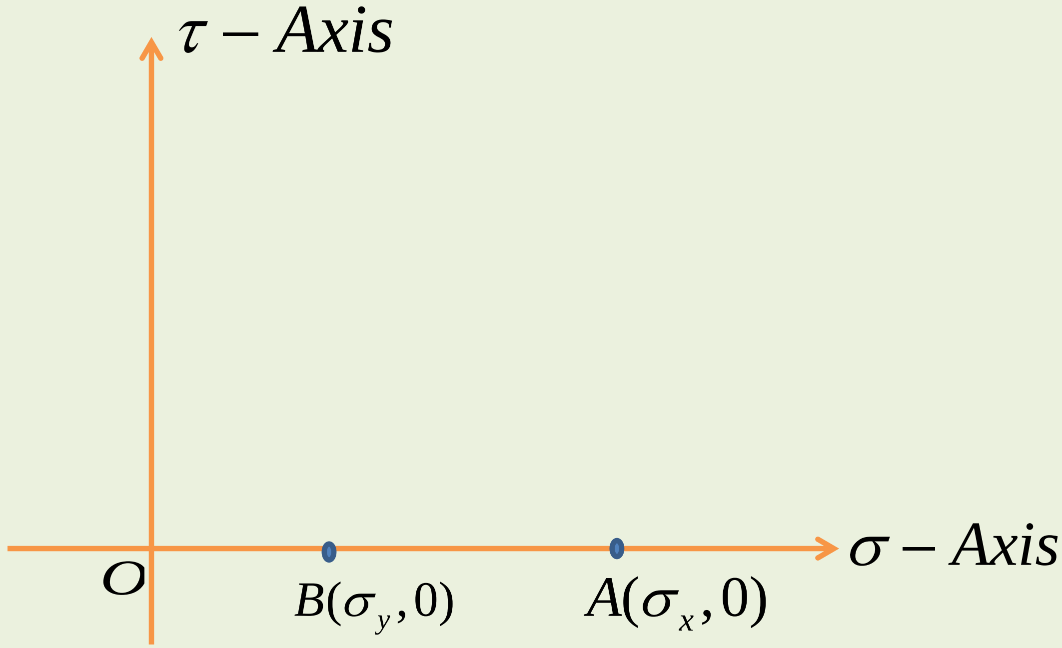
3. Locate a point A corresponding to the state of stress on x-face.

4. Locate a point B corresponding to the state of stress on y-face.

Example 1

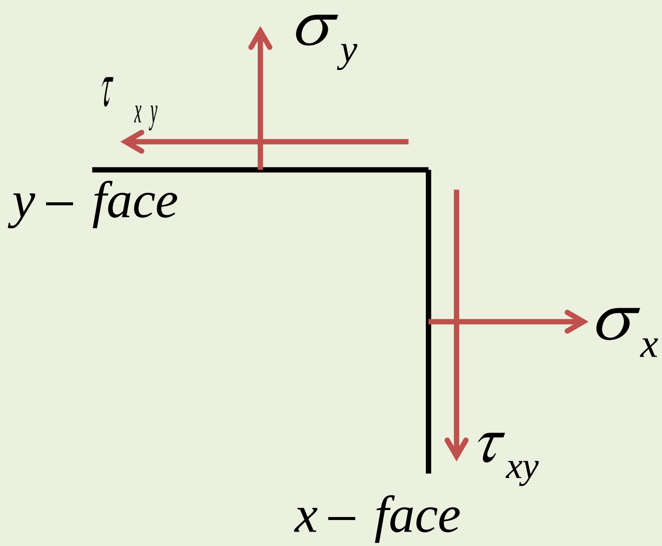
(Let assume $\sigma_x > \sigma_y$)

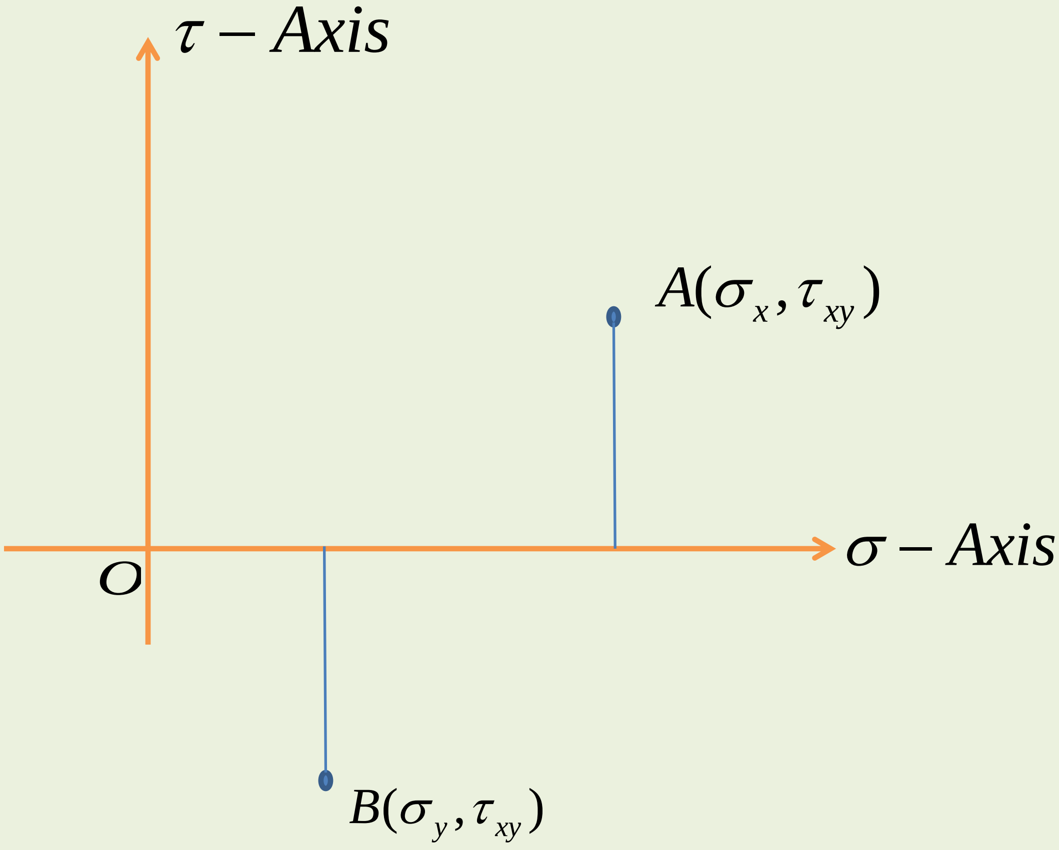




Example 2

(Let assume $\sigma_x > \sigma_y$)



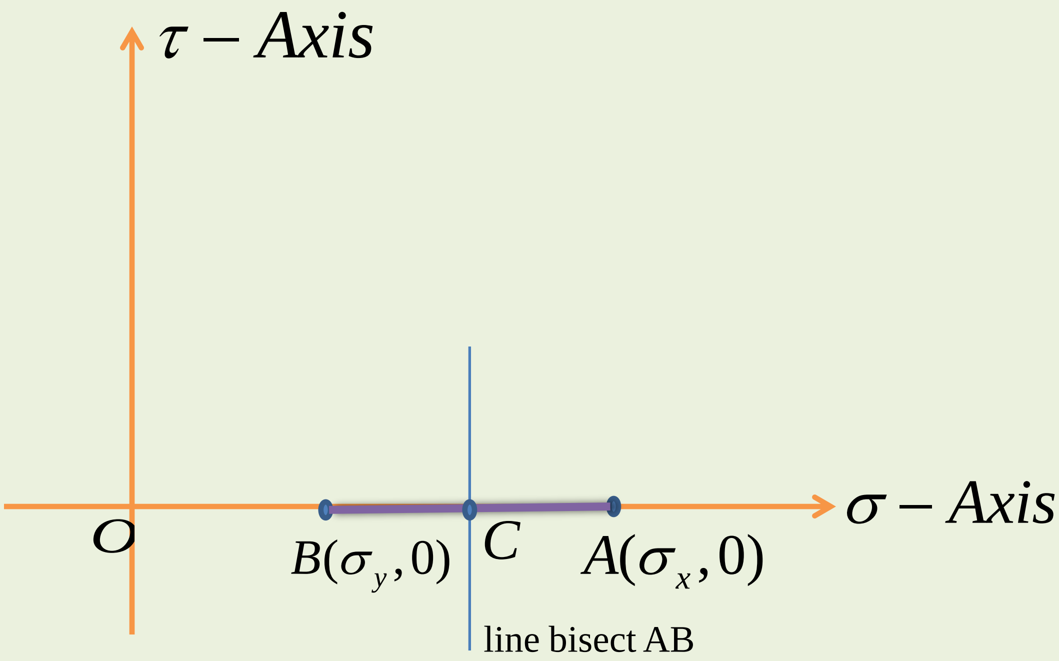


5. Join A & B point

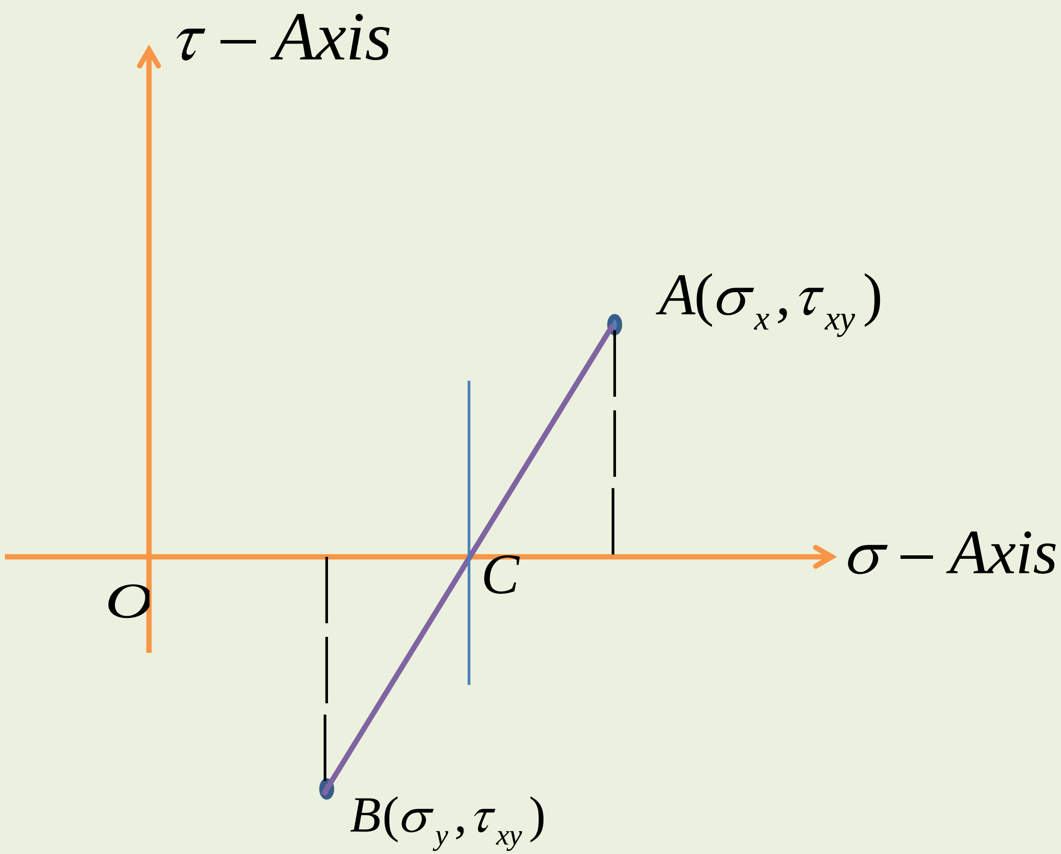
6. Bisect the line joining A & B, cut the σ axis at point C.

7. Point C is the centre of Mohr's circle. (Always remember that the centre of Mohr's circle will lie on σ-axis.) .

Example 1



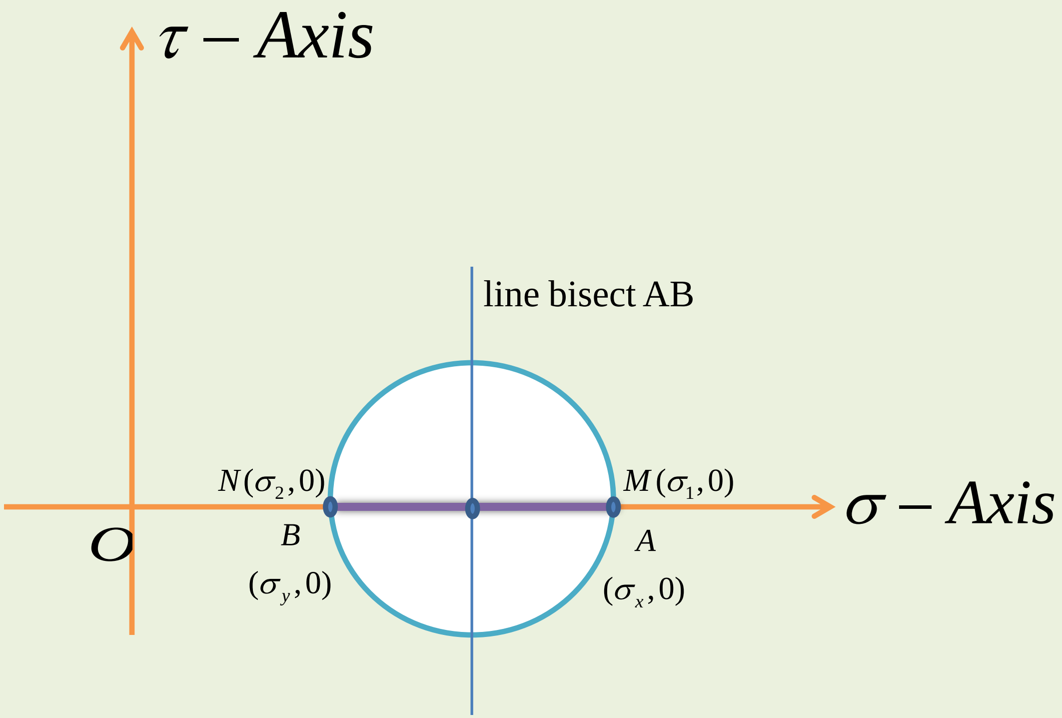
Example 2



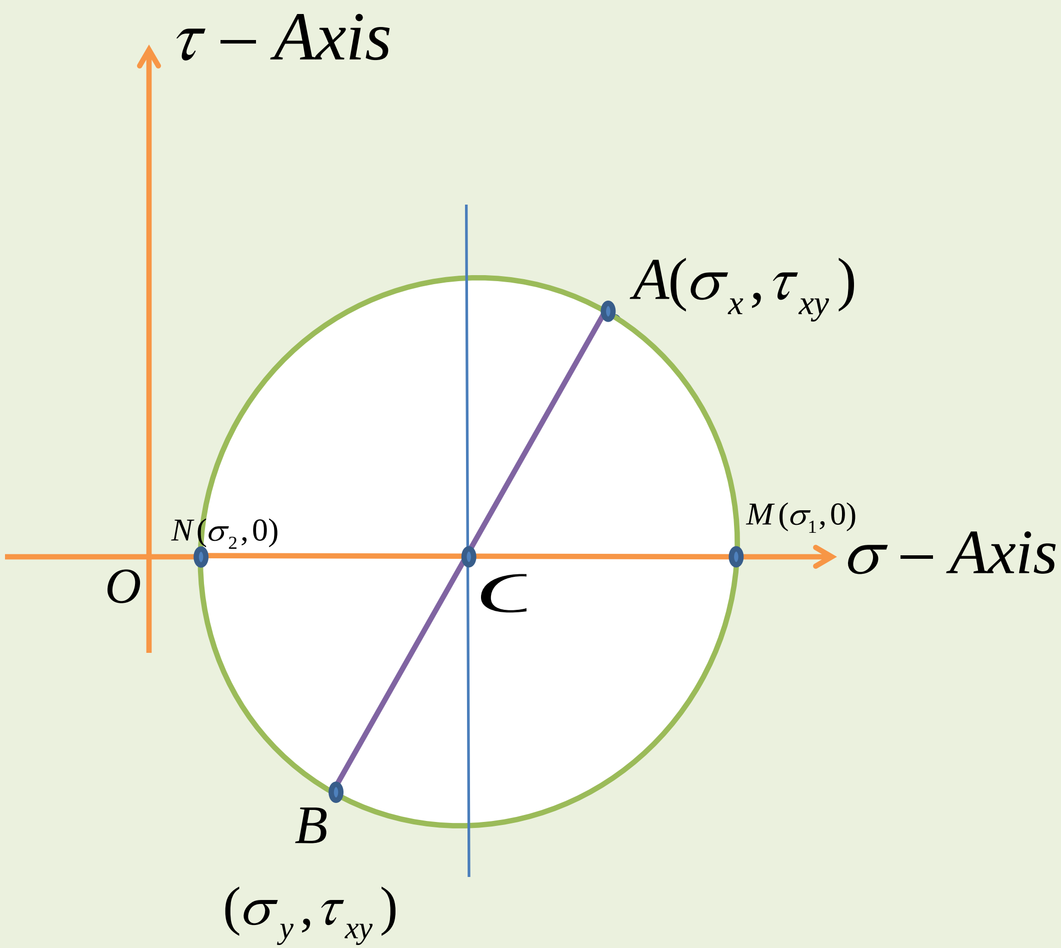
8. By Taking AC & BC as radius and C as centre draw a circle.

9. The points (M& N) at which Mohr's circle cut the σ-axis represents principal stress and radial lines CM &CN represents the principal planes.

Example 1



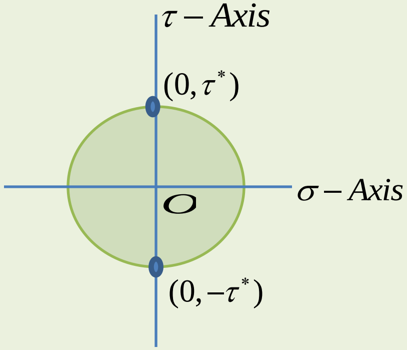
Example 2



$\tau - Axis$
$A(\sigma_x, \tau_{xy})$
$N(\sigma_2, 0)$
$M(\sigma_1, 0)$
$\sigma - Axis$
O
C
B
(σ_y, τ_{xy})

Observations

1. Every radial line of circle is representing a plane.
2. Coordinates of any point on Mohr's circle circumference represents the normal stress & shear stress on that radial plane (radial line by joining coordinate point with the centre of circle).
3. Every plane in Mohr's circle is representing by the double angle from the actual represents in state of stress.
4. The centre of Mohr's circle will always lie on σ-axis.
5. On the plane of pure shear the value of normal stress is zero. The centre of Mohr's circle will be at origin (O).



Observations

6. Centre co-ordinate of Mohr's circle is given as-

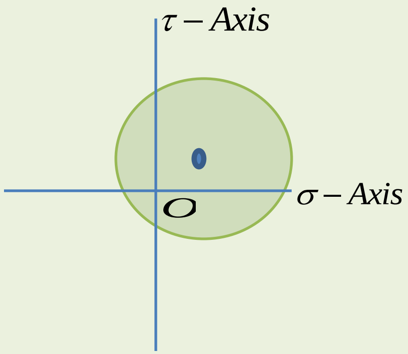
$$[C] = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right)$$

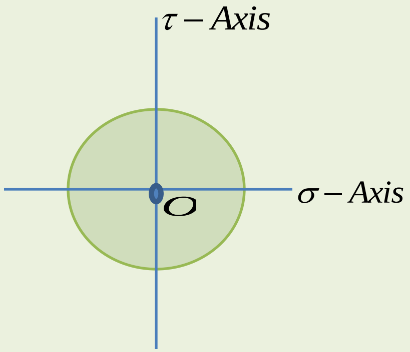
7. Radius of Mohr's circle is given as -

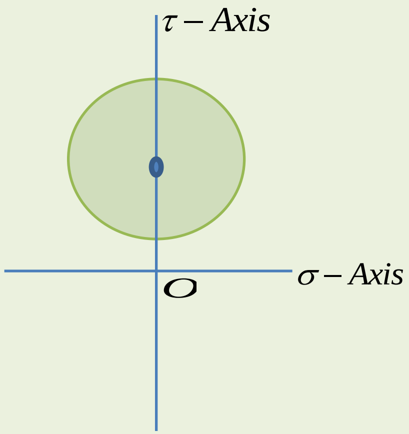
$$r = \frac{1}{2}(\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2})$$

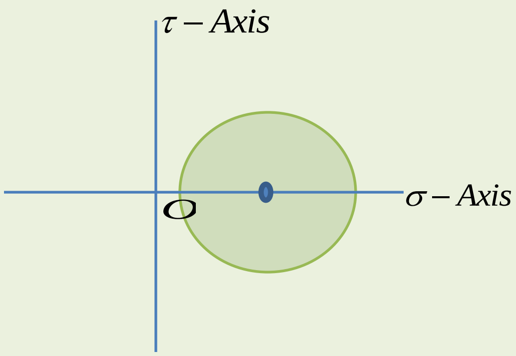
8. A radial line II to the τ-axis represents the plane of maximum shear stress.

Q. Which of the following Mohr's circle is possible?



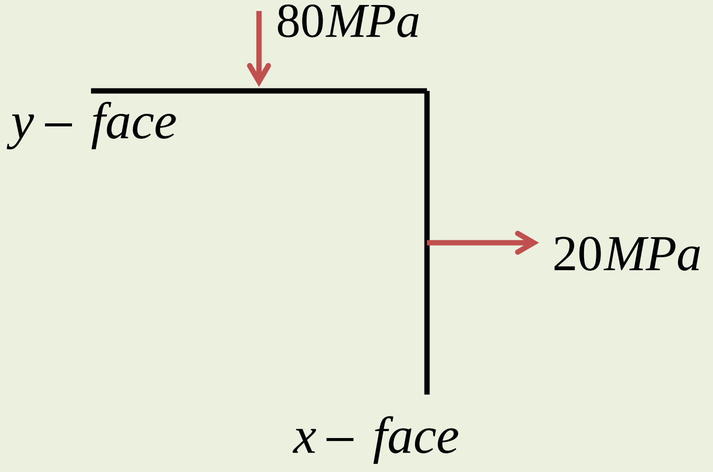






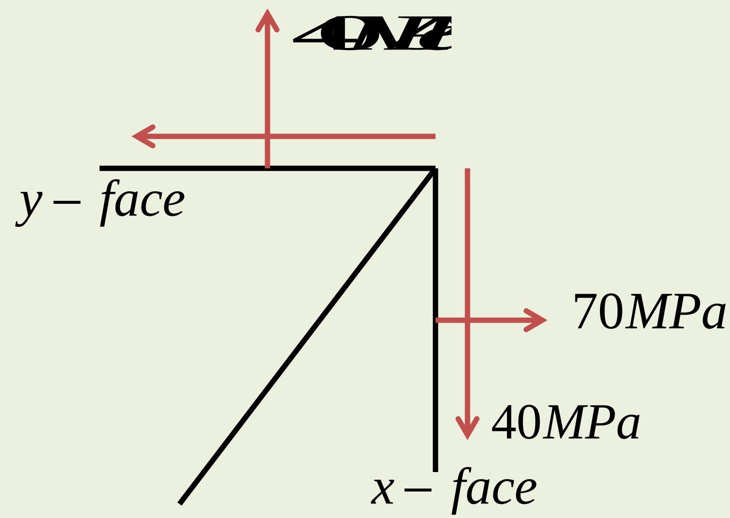
Q. Draw the Mohr's circle for a given state of stress of a point & determine the followings:

1. Co-ordinates of Mohr's circle centre.
2. Principal stress
3. Normal stress on τ_{max} plane.
4. Resultant stress on τ_{max} plane.
5. Maximum tensile & compressive stress

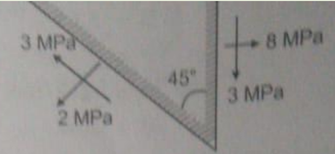


Q. Draw the Mohr's circle for a given state of stress of a point & determine the followings:

1. Co-ordinates of Mohr's circle centre.
2. Principal stresses
3. Normal stress & shear stress on a given oblique plane (O.P) which is at 70° w.r.t X-face.
4. Resultant stress on τ_{max} plane.



Q. At a point in a material subjected to 2 direct stresses on a plane at right angle. The resultant stress on a plane A is 80 MPa and inclined at 30° to the normal stress and the resultant stress on a plane B is 40 MPa which is inclined to 45° to the normal stress. Find the principal stresses & show the position of two planes A & B relative to the principal plane.



(a) 8.242, 0.658 (b) 9.242, 0.758
(c) 9.242, 0.758 (d) 8.242, 0.758

[1993 : 2 Marks]

Common Data Questions Q.3.2 and Q.3.3

The state of stress at a point *P* in a two dimensional loading is such that the Mohr's circle is a point located at 175 MPa on the positive normal stress axis.

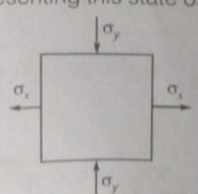
3.2 The maximum and minimum principal stresses respectively from the Mohr's circle are
(a) +175 MPa, –175 MPa
(b) +175 MPa, +175 MPa
(c) 0, –175 MPa
(d) 0, 0

[2003 : 2 Marks]

3.3 The directions of maximum and minimum principal stresses at the point *P* from the Mohr's circle are
(a) 0, 90° (b) 90°, 0
(c) 45°, 135° (d) All directions

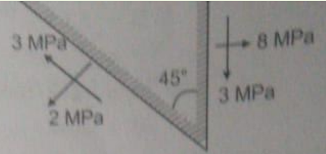
[2003 : 2 Marks]

3.4 The figure shows the state of stress at a certain point in a stressed body. The magnitudes of normal stresses in the x and y directions are 100 MPa and 20 MPa respectively. The radius of Mohr's stress circle representing this state of stress is



(a) 120 MPa (b) 80 MPa
(c) 60 MPa (d) 40 MPa

[2004 : 1 Mark]



(a) 8.242, 0.658 (b) 9.242, 0.758
(c) 9.242, 0.758 (d) 8.242, 0.758

[1993 : 2 Marks]

Common Data Questions Q.3.2 and Q.3.3

The state of stress at a point P in a two dimensional loading is such that the Mohr's circle is a point located at 175 MPa on the positive normal stress axis.

3.2 The maximum and minimum principal stresses respectively from the Mohr's circle are

(a) +175 MPa, –175 MPa
(b) +175 MPa, +175 MPa
(c) 0, –175 MPa
(d) 0, 0

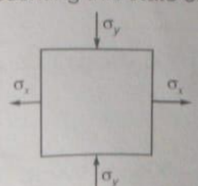
[2003 : 2 Marks]

3.3 The directions of maximum and minimum principal stresses at the point P from the Mohr's circle are

(a) 0, 90° (b) 90°, 0
(c) 45°, 135° (d) All directions

[2003 : 2 Marks]

3.4 The figure shows the state of stress at a certain point in a stressed body. The magnitudes of normal stresses in the x and y directions are 100 MPa and 20 MPa respectively. The radius of Mohr's stress circle representing this state of stress is



(a) 120 MPa (b) 80 MPa
(c) 60 MPa (d) 40 MPa

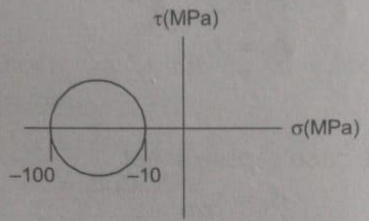
[2004 : 1 Mark]

3.
3.
3.8
3.9

ANKIT SAXENA (saxena01ankit@gmail.com)

Mohr's Circle

3.5 The Mohr's circle of plane stress for a point in a body is shown. The design is to be done on the basis of the maximum shear stress theory for yielding. Then, yielding will just begin if the designer chooses a ductile material whose yield strength is



(a) 45 MPa (b) 50 MPa
(c) 90 MPa (d) 100 MPa

[2005 : 2 Marks]

3.6 A two dimensional fluid element rotates like a rigid body. At a point within the element, the pressure is 1 unit . Radius of the Mohr's circle, characterizing the state at that point, is

(a) 0.5 unit (b) 0 unit
(c) 1 unit (d) 2 units

[2008 : 2 Marks]

3.7 The state of stress at a point under plane stress condition is

$\sigma_{xx} = 40$ MPa, $\sigma_{yy} = 100$ MPa and

$\tau_{xy} = 40$ MPa

The radius of Mohr's circle representing the given

(a) 40 (b) 50

(c) 60 (d) 100 [2012 : 2 Marks]

3.8 In a plane stress condition, the components of stress at a point are $\sigma_x = 20$ MPa, $\sigma_y = 80$ MPa and $\tau_{xy} = 40$ MPa. The maximum shear stress (in MPa) at the point is

(a) 20 (b) 25

(c) 50 (d) 100

[2015 : 2 Marks, Set-2]

3.9 The state of stress at a point is $\sigma_x = \sigma_y = \sigma_z = \tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0$ and $\tau_{xy} = \tau_{yx} = 50$ MPa. The maximum normal stress (in MPa) at that point is ______. [2017 : 1 Mark, Set-2]

Mohr's circle

minimum stres

3.4 (c)

Rac

Given,

∴ Ra

Strain & Elastic Constants

Types of Strain

- *Normal strain* (ε)
  - *Longitudinal strain*
  - *Lateral strain*
  - *Volumetric strain*
- *Shear strain* (γ)

Normal strain (ε)

It is defined as the ratio of change in dimension to the original dimension. It is denoted by the symbol ε. It is given as-

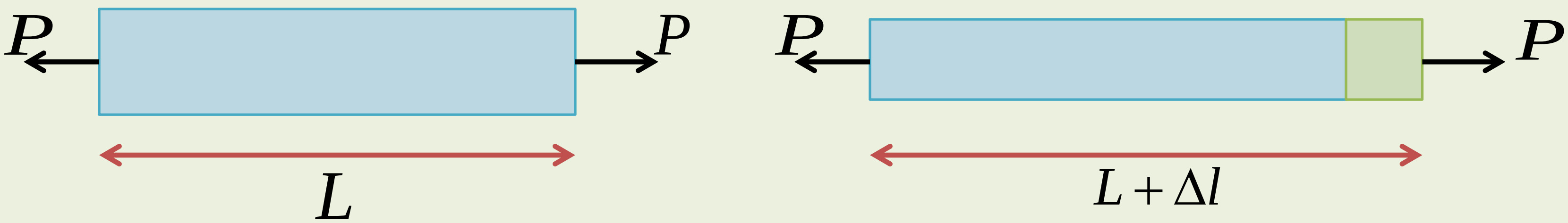
Δl = Change in dimension

L = Original dimension

$$\varepsilon = \frac{\Delta l}{L}$$

Longitudinal strain -

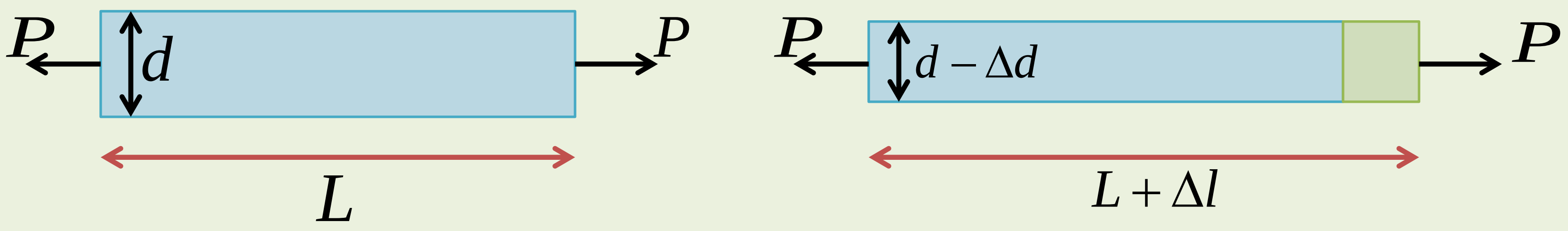
It is defined as the ratio of change in dimension along the direction of line of action of action of load to the original dimension.



Lateral strain

It is defined as the ratio of change in dimension perpendicular (⊥) to the direction of line of action of action of load to the original dimension.

Consider a bar having a circular x-section with diameter (d). After apply the tensile axial load, the length (L) will increase and diameter (d) will get decrease.



Important points

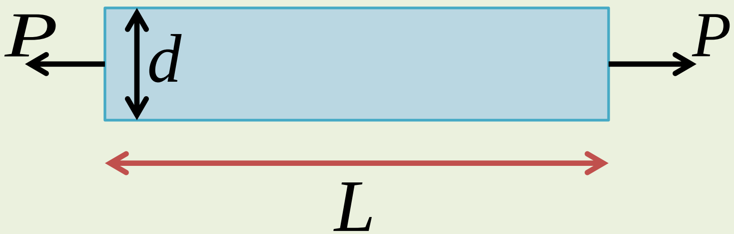
1. Every longitudinal strain is associated with 2 lateral strain.
2. Longitudinal & lateral strain are unlike in nature.
3. Under tri-axial state of stress, in strain tensor there are total nine strains (3 normal strain + 6 shear strain) components developed.
4. Under tri-axial loads, total normal strain in any direction is equal to the algebraic sum of one longitudinal strain & 2 lateral strain in that direction.

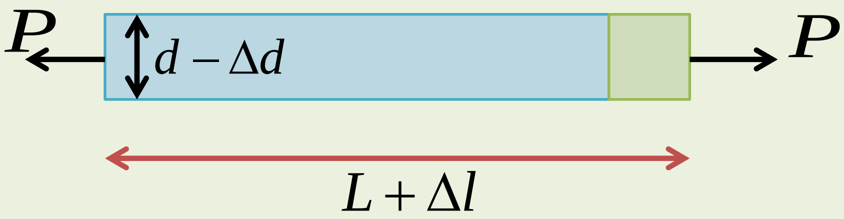
Important points

Longitudinal & lateral strain is given as-

$$\varepsilon_{long.} = \frac{\Delta l}{L}$$

$$\varepsilon_{lateral} = \frac{\Delta d}{d}$$





Volumetric strain

It is defined as the ratio of change in volume to the original volume. It is denoted by ε_v.

$$\varepsilon_v = \frac{\Delta v}{V}$$

Poisson's ratio (μ)

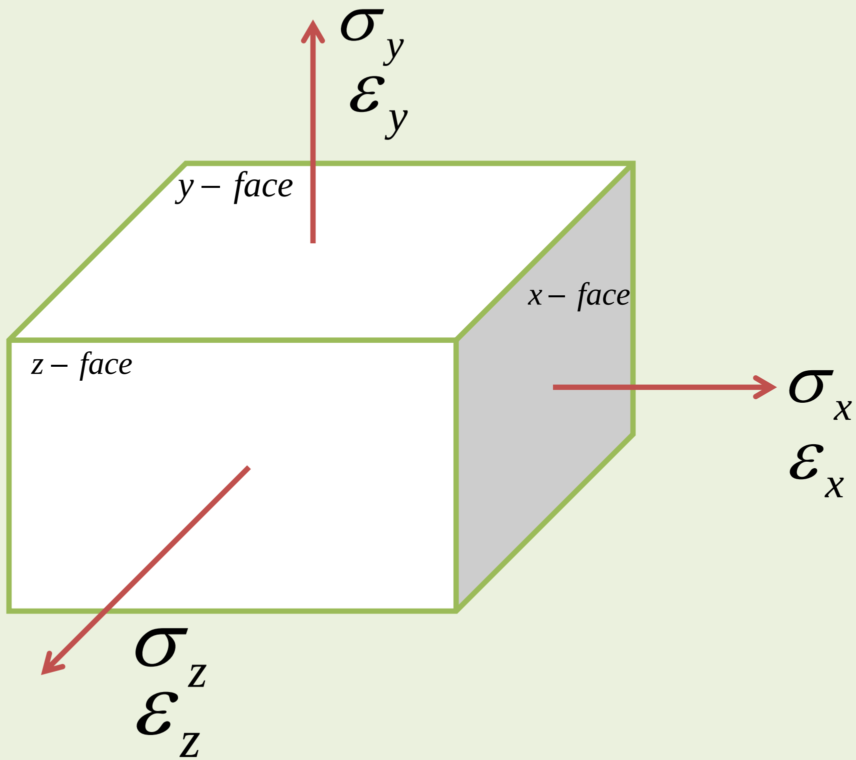
It is ratio of lateral strain to the longitudinal strain. It is denoted by μ -

$$\mu = -\left(\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}\right)$$

(-ve) sign indicate lateral strain and longitudinal strain are unlike in nature.

Relationship between normal stress and normal strain

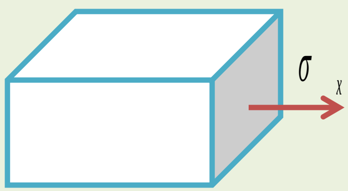
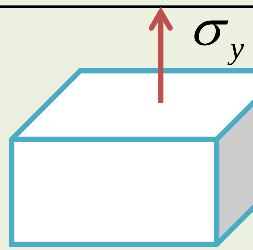
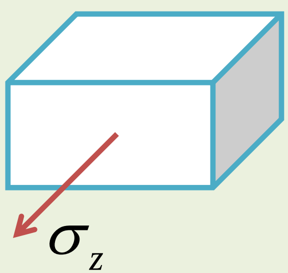
Consider a cuboid, subjected to tri-axial loading -



As per Hook's law,

Stress ∝ corresponding strain

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Strain → / **Loading Condition** ↓	**x-direction** ε_x	**y-direction** ε_y	**z-direction** ε_z
	$\frac{\sigma_x}{E}$	$-\frac{\mu\sigma_x}{E}$	$-\frac{\mu\sigma_x}{E}$
	$-\frac{\mu\sigma_y}{E}$	$\frac{\sigma_y}{E}$	$-\frac{\mu\sigma_y}{E}$
	$-\frac{\mu\sigma_z}{E}$	$-\frac{\mu\sigma_z}{E}$	$\frac{\sigma_z}{E}$

Total normal strain in x-direction is given as-

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)]$$

In the same way ε_y & ε_z is given as-

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_y + \sigma_x)]$$

Volumetric strain is given as-

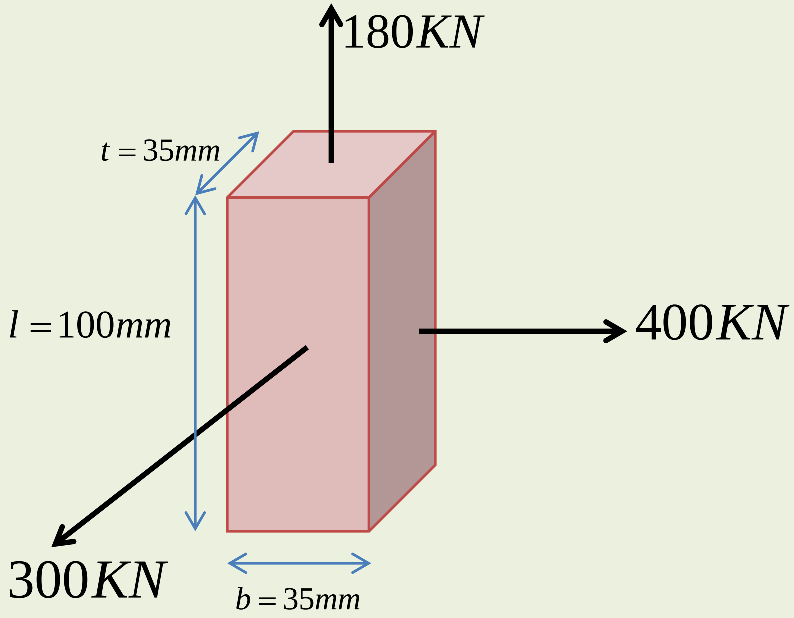
$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\varepsilon_v = \frac{1}{E}(1-2\mu)(\sigma_x + \sigma_y + \sigma_z)$$

Q. A steel wire having 35*35 mm^2 x-section and length 100 mm is acting upon by a tensile load of 180 KN along its longitudinal axis and 400 KN & 300 KN along the axis of lateral surfaces. Determine-

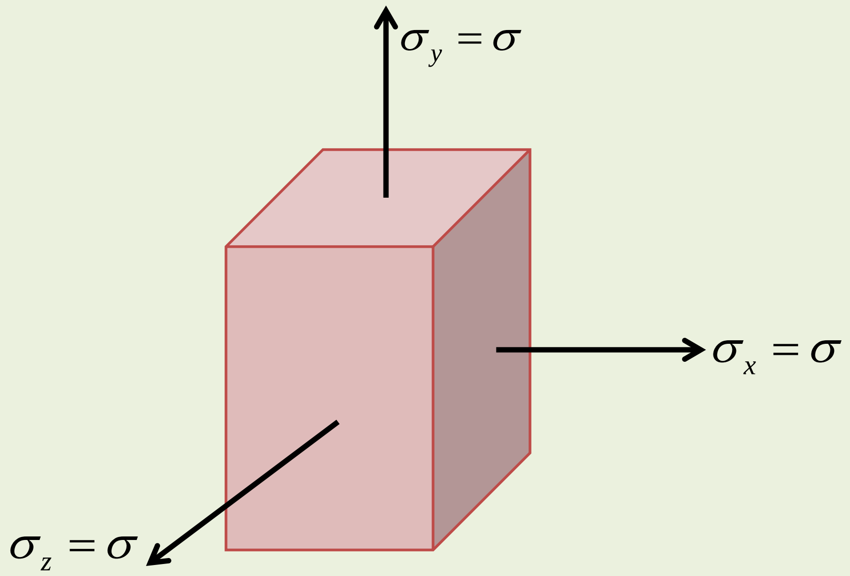
1. Change in the dimensions of the bar
2. Change in volume
3. Longitudinal axial load acting alone to produce the same longitudinal strain as in 1.

Given E = 205 MPa & $\mu = 0.3$



Hydrostatic state of stress

- Under the hydrostatic state of stress, every plane is principal plane.
- Under the hydrostatic state of stress, shear stress is zero on each & every oblique plane.

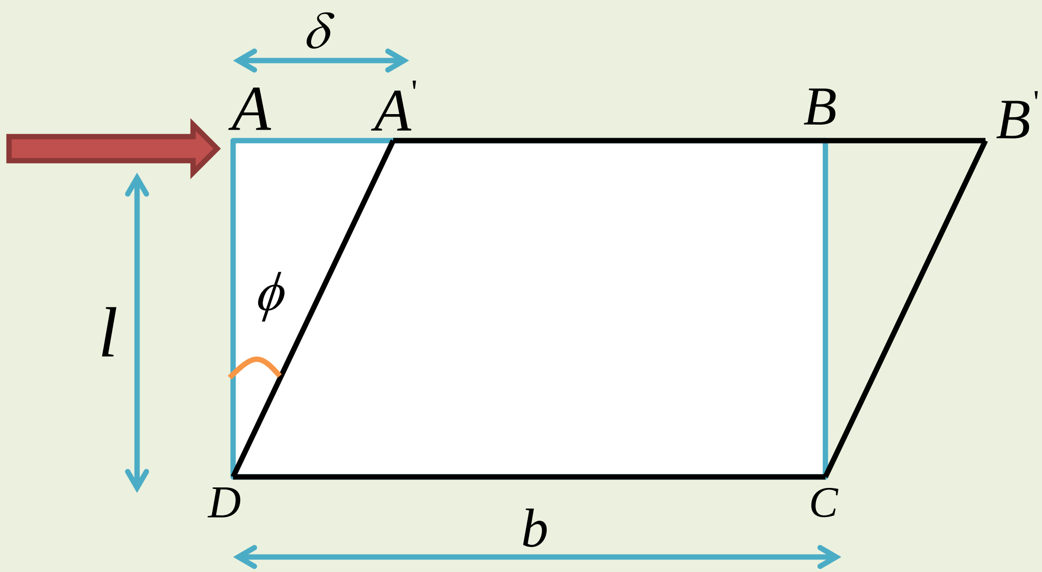


- In hydrostatic state of stress - $\sigma_x = \sigma_y = \sigma_z = \sigma$
- Mohr's circle is a point under hydrostatic state of stress condition. **(M.Imp)**

ANKIT SAXENA (saxena01ankit@gmail.com)

Shear strain

- It is defined as the change in initial right angle between two line elements which are parallel to X & Y axis.



- It is denoted by (γ)

$$In - \Delta ADA^{'}$$

$$\tan\phi = \frac{\delta}{l}$$

$$if\,(\phi) - small\_angle$$

$$\tan\phi \approx \phi$$

$$\phi = \frac{\delta}{l}$$

$$\gamma = \phi = \frac{\delta}{l}$$

Strain tensor

- For Tri-axial state of stress (3D state of stress), Strain tensor is given as-

$$[\varepsilon]_{3D} = \begin{pmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_z \end{pmatrix}$$

Normal strain & shear strain on oblique plane

- Normal strain (ε_n) is given as -

$$\varepsilon_n = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

- Shear strain (γ_s) is given as -

$$\frac{\gamma_s}{2} = -\frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Principal strains under bi-axial state of stress

$$\varepsilon_{1/2} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + 4(\frac{\gamma_{xy}}{2})^2}$$

Elastic Constant

- Elastic constant are used to determine strain theoretically.
- These are used to obtain relationship between stress & strain.
- For a homogeneous and isotropic material there are 4 elastic constants.

$$E, G, K, \mu$$

1. [E] Young's modulus & Modulus of elasticity
2. [G] shear modulus & modulus of rigidity
3. [K] bulk modulus
4. [μ] Poisson's ratio

- For a homogeneous and isotropic material the number of independent elastic constants are 2. i.e. [E & μ].

Important table

Material	Number of Independent Elastic constant
Homogeneous & isotropic	2
Orthotropic	9
Anisotropic	21

Relationship between elastic constant

$$E = 2G(1 + \mu)$$

$$E = 3K(1 - 2\mu)$$

$$E = \frac{9KG}{3K + G}$$

Young's Modulus:

Under uniaxial state of stress condition,

$$E = \frac{\textit{Normal stress}}{\textit{Longitudinal strain}}$$

$$E = \frac{\sigma}{\varepsilon}$$

$$\varepsilon_{long} \alpha \frac{1}{E}$$

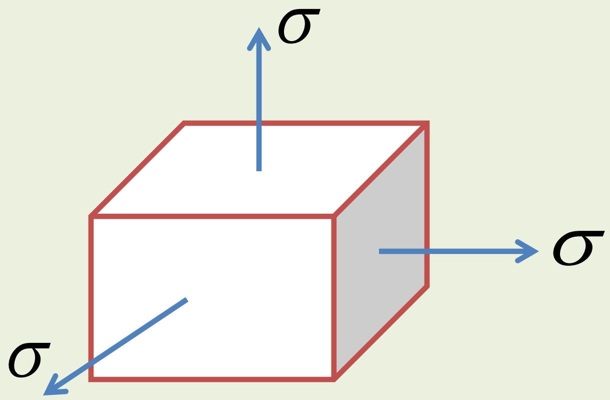
It is also defined as the slope of engineering stress & engineering strain curve up to proportional limit.

Shear Modulus [G] :

$$G = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\gamma}$$

Bulk Modulus [K] :

Under hydrostatic state of stress condition-



$$K = \frac{\text{Hydrostatic stress}}{\text{Volumetric stress}}$$

$$K = \frac{\sigma}{\varepsilon_v}$$

$$K = \frac{\sigma}{\frac{(1-2\mu)}{E}(\sigma_x + \sigma_y + \sigma_z)}$$

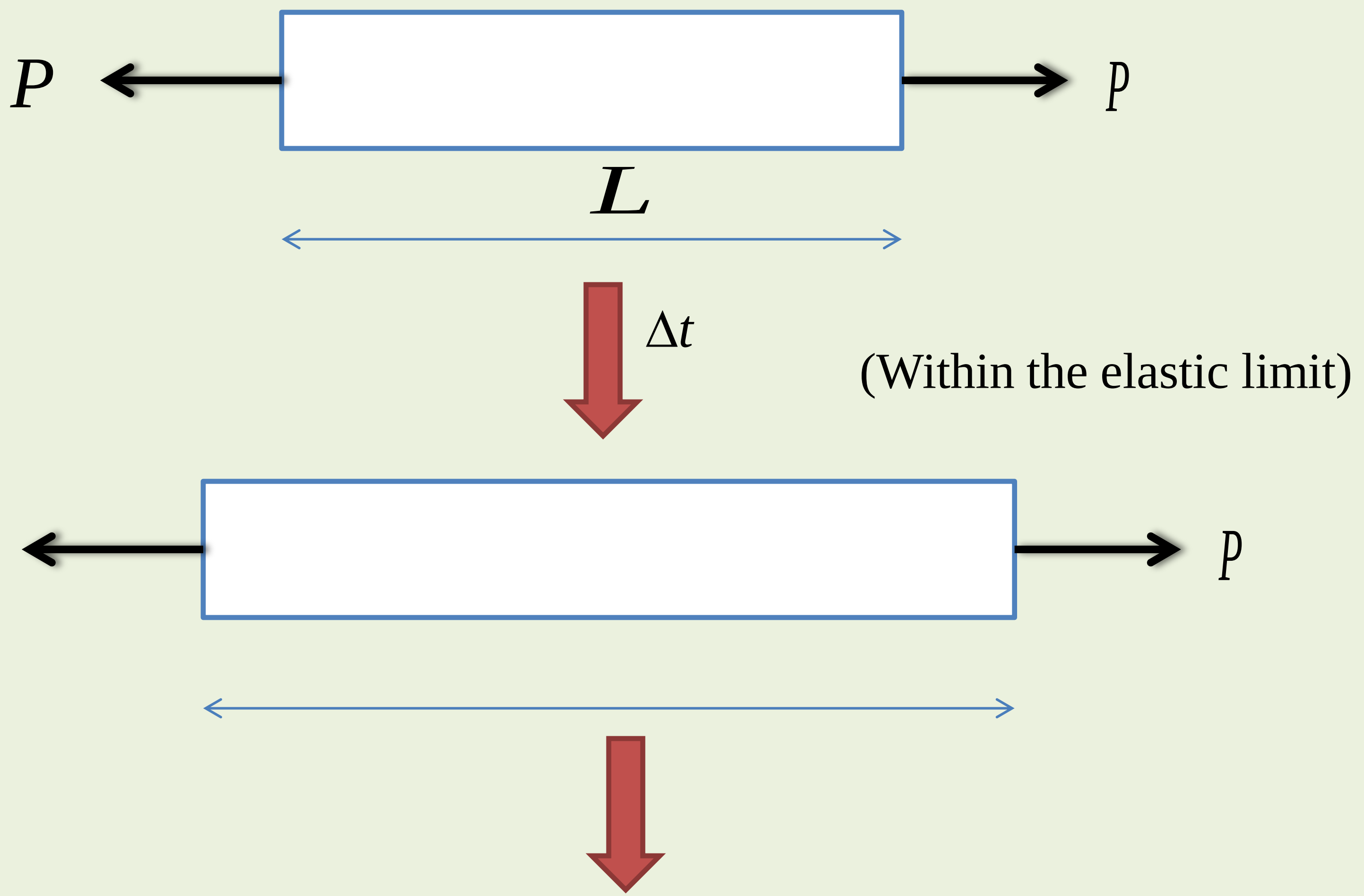
$$K = \frac{\sigma}{\frac{(1-2\mu)}{E}(\sigma + \sigma + \sigma)}$$

$$K = \frac{\sigma}{\frac{(1-2\mu)}{E}.3\sigma}$$

$$K = \frac{E}{3(1-2\mu)}$$

$$E = 3K(1-2\mu)$$

Strain energy due to principal stresses

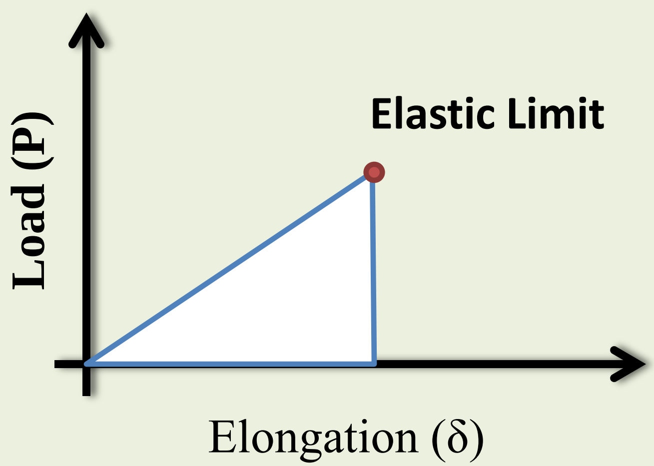


System execute a work

This work is stored within the body in form of internal energy

This energy is known as a strain energy.

- It is denoted by 'U'
- It is also defined as the area under the load-elongation curve (P-δ) within elastic limit.



U = Area under (P-δ) curve

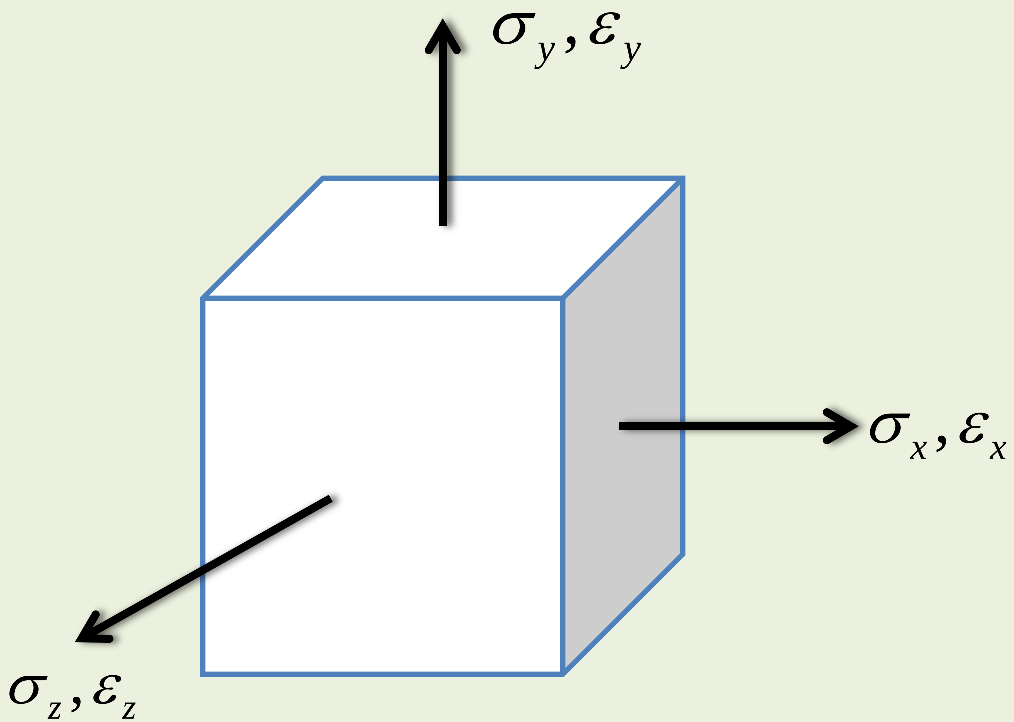
$$U = \frac{1}{2}[P.\delta]$$

$$U = \frac{1}{2}.\sigma.A.\varepsilon.l$$

$$U = \frac{1}{2}(\sigma \times \varepsilon)A.L$$

$$U = \frac{1}{2}(\sigma \times \varepsilon) \times (Volume)$$

$$\frac{Strain - Energy}{Volume} = \frac{1}{2}(\sigma \times \varepsilon)$$



$$\frac{S.E}{Volume} = \frac{1}{2}\left[\left(\sigma_x.\varepsilon_x\right) + \left(\sigma_y.\varepsilon_y\right) + \left(\sigma_z.\varepsilon_z\right)\right]$$

Resilience

It is defined as the energy stored within the elastic limit.

Proof Resilience

It is defined as the energy stored at the elastic limit.

Modulus of Resilience

It is defined as the energy stored per unit volume within the elastic limit.

$$\sigma \alpha \varepsilon$$

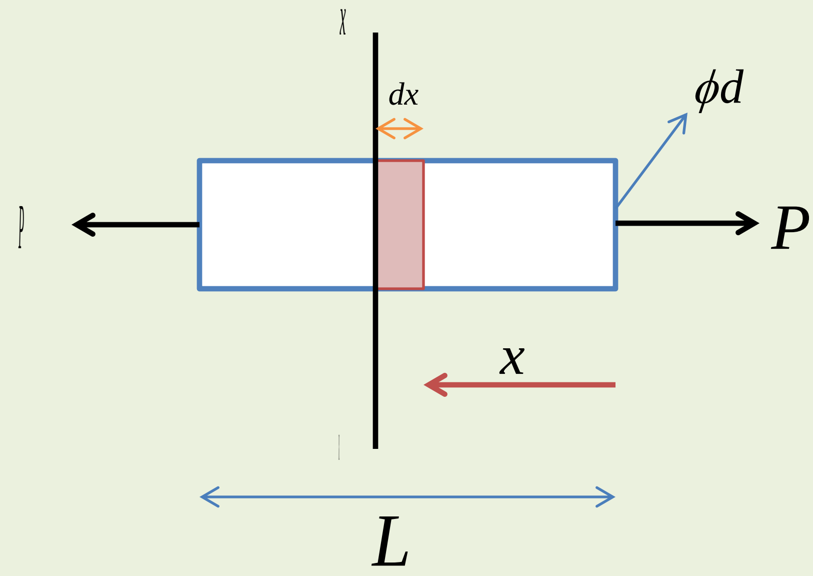
$$\sigma = E\varepsilon$$

$$\frac{P}{A} = E.\frac{\delta}{L}$$

$$\delta = \frac{PL}{AE}$$

$$S.E = \frac{1}{2}(P.\delta)$$

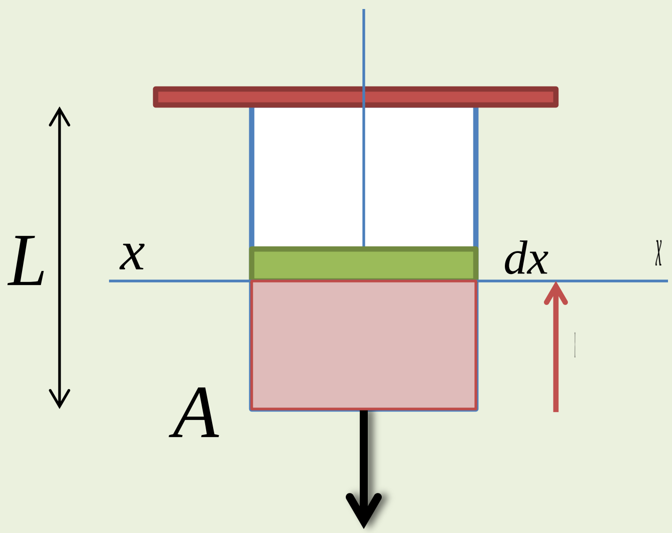
$$S.E = \frac{1}{2}\left(\frac{P^2 L}{AE}\right)$$



General equation for strain energy when bar is subjected to pure axial load

$$S.E = \int_0^L \frac{P^2{}_{x-x}\,dx}{2A_{x-x}E}$$

Strain energy of the bar due to its self weight



Let γ = Weight density (weight per unit volume)

Weight of the member = γAL

$$S.E = \int_0^L \frac{P^2_{x-x}dx}{2A_{x-x}E}$$

$$= \int_0^L \frac{\gamma^2 A^2 L^2 dx}{2AE}$$

$$= \gamma^2 \frac{A}{E} \int_0^L x^2 dx$$

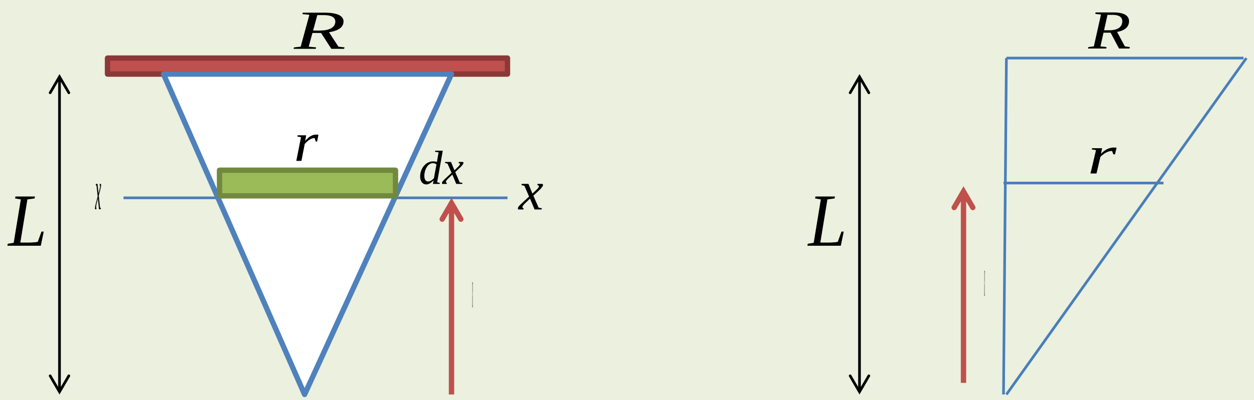
$$= \frac{1}{2}\gamma^2 \frac{A}{E} \cdot \frac{L^3}{3}$$

$$= \frac{\gamma^2 AL^3}{6E}$$

$$= \frac{(\gamma AL)^2 .L}{3 \times 2AE}$$

$$SE = \frac{1}{3}\left(\frac{W^2 L}{2AE}\right)$$

Strain energy of conical bar due to its self weight



Let γ = Weight density (weight per unit volume)

Weight of the member

$$W = \gamma . \frac{1}{3} \pi R^2 . L$$

$$P_{x-x} = \gamma \times vol^{m}$$

$$P_{x-x} = \gamma.\frac{1}{3}\pi r^{2}.x$$

$$\frac{R}{r} = \frac{L}{x}$$

$$r = \frac{Rx}{L}$$

$$A_{x-x} = \pi.r^{2.}$$

$$S.E = \int_{0}^{L} \frac{P_{x-x}{}^{2} dx}{2A_{x-x}E}$$

$$S.E = \int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$= \int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi r^2.x^2.dx}{2E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{Rx}{L}\right)^2 .x^2.dx}{2E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{R}{L}\right)^2 .x^4.dx}{2E}$$

$$S.E = \frac{\gamma^2 \pi R^2}{18EL^2}\left(\frac{x^5}{5}\right)_0^L$$

$$= \frac{\gamma^2 \pi R^2 L^3}{90E}$$

$$= \frac{\gamma^2 (A.L) L^2}{90E}$$

$$= \frac{\gamma^2 \left(A^2.L^2\right) L}{90AE}$$

$$= \frac{1}{10} \frac{\left(\frac{1}{3}\gamma AL\right)^2 .L}{AE}$$

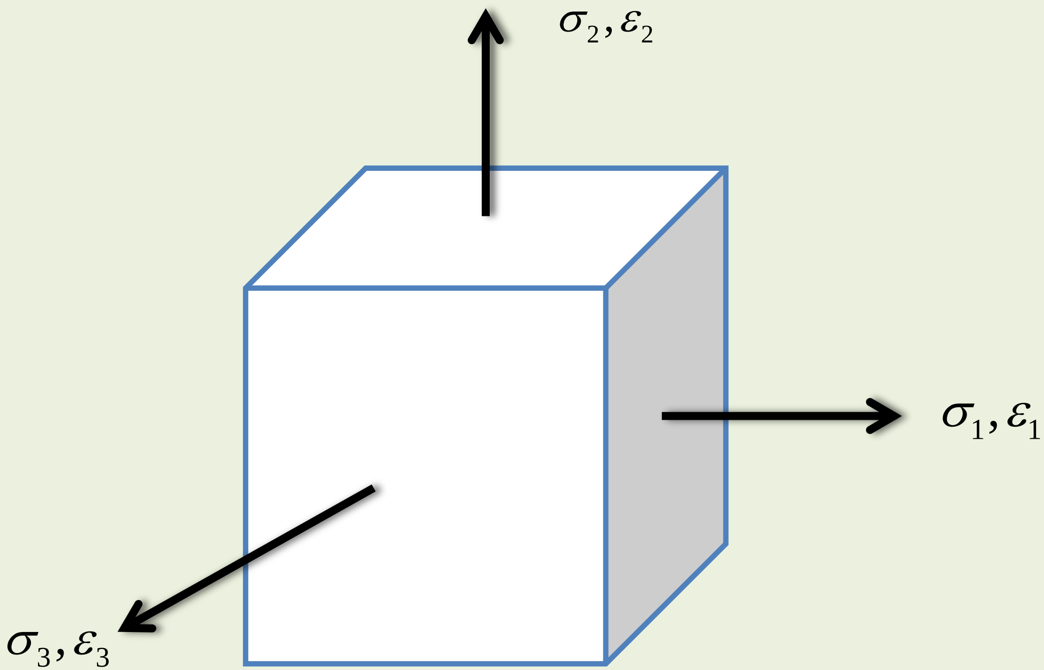
$$= \frac{1}{10} \frac{W^2.L}{AE}$$

Strain energy of bar under axial load

$$U = \frac{1}{K}\left[\frac{P^2 L}{AE}\right]$$

- K = 2, Prismatic bar under pure axial load. [P axial load]
- K = 6, Prismatic bar under self weight. [P is the weight of prismatic bar]
- K = 10, Conical bar under self weight. [P is weight of conical bar]

Strain energy due to principal stress



$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\left(\sigma_1 + \sigma_3\right)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right]$$

$$\frac{SE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

Strain energy per unit volume is also denoted the total strain energy (U).

So,

$$U = \frac{1}{2E}\left[\left(\sigma_1^{\ 2} + \sigma_2^{\ 2} + \sigma_3^{\ 2}\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

This total strain energy [U] has two components-

1. Volumetric Energy
2. Distortion Energy

Total energy = [Volumetric Energy + Distortion Energy]

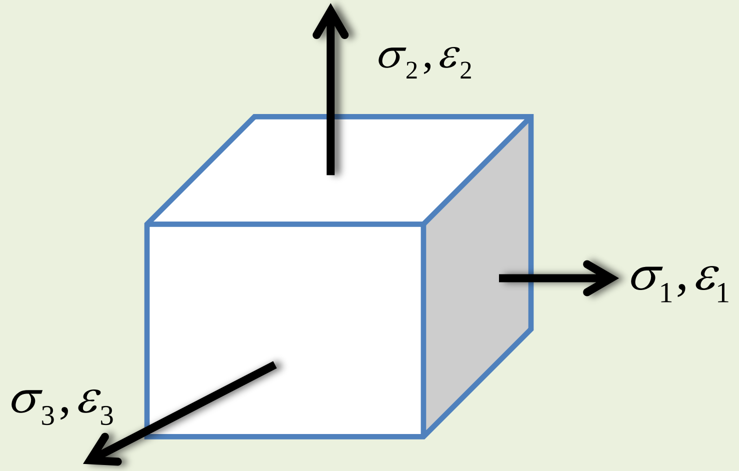
Volumetric Energy per unit volume

$$\frac{Volumetric - Energy}{Volume} = \frac{1}{2}.\sigma_{avg}.\varepsilon_v$$

$$\sigma_{avg} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

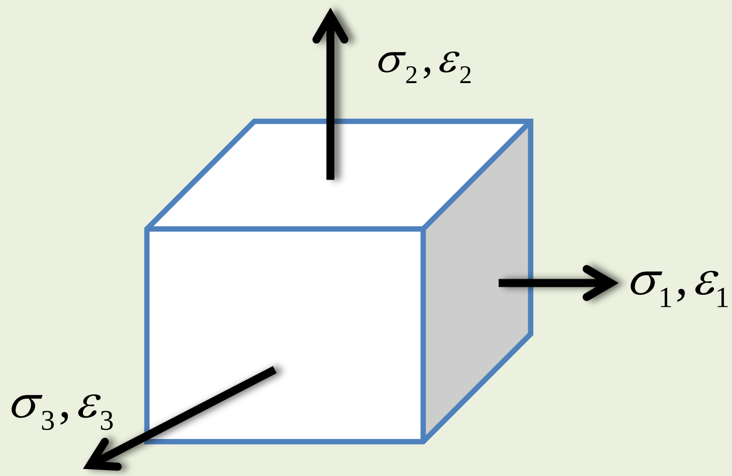
$$\varepsilon_v = \frac{(1-2\mu)}{E}.(\sigma_1 + \sigma_2 + \sigma_3)$$

$$\frac{Volumetric - Energy}{Volume} = \frac{(1-2\mu)}{6E}.(\sigma_1 + \sigma_2 + \sigma_3)^2$$



Distortion Energy per unit volume

Total energy/ Vol = [Volumetric Energy/ Vol^m]
+
[Distortion Energy/Vol^m]



$$\frac{DE}{Vol^m} = \frac{TSE}{Vol^m} - \frac{DE}{Vol^m}$$

$$\frac{DE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right] - \frac{(1-2\mu)}{6E}.\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2$$

$$\frac{DE}{Vol^m} = \frac{(1+\mu)}{6E}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]$$

Factor of safety [FOS]

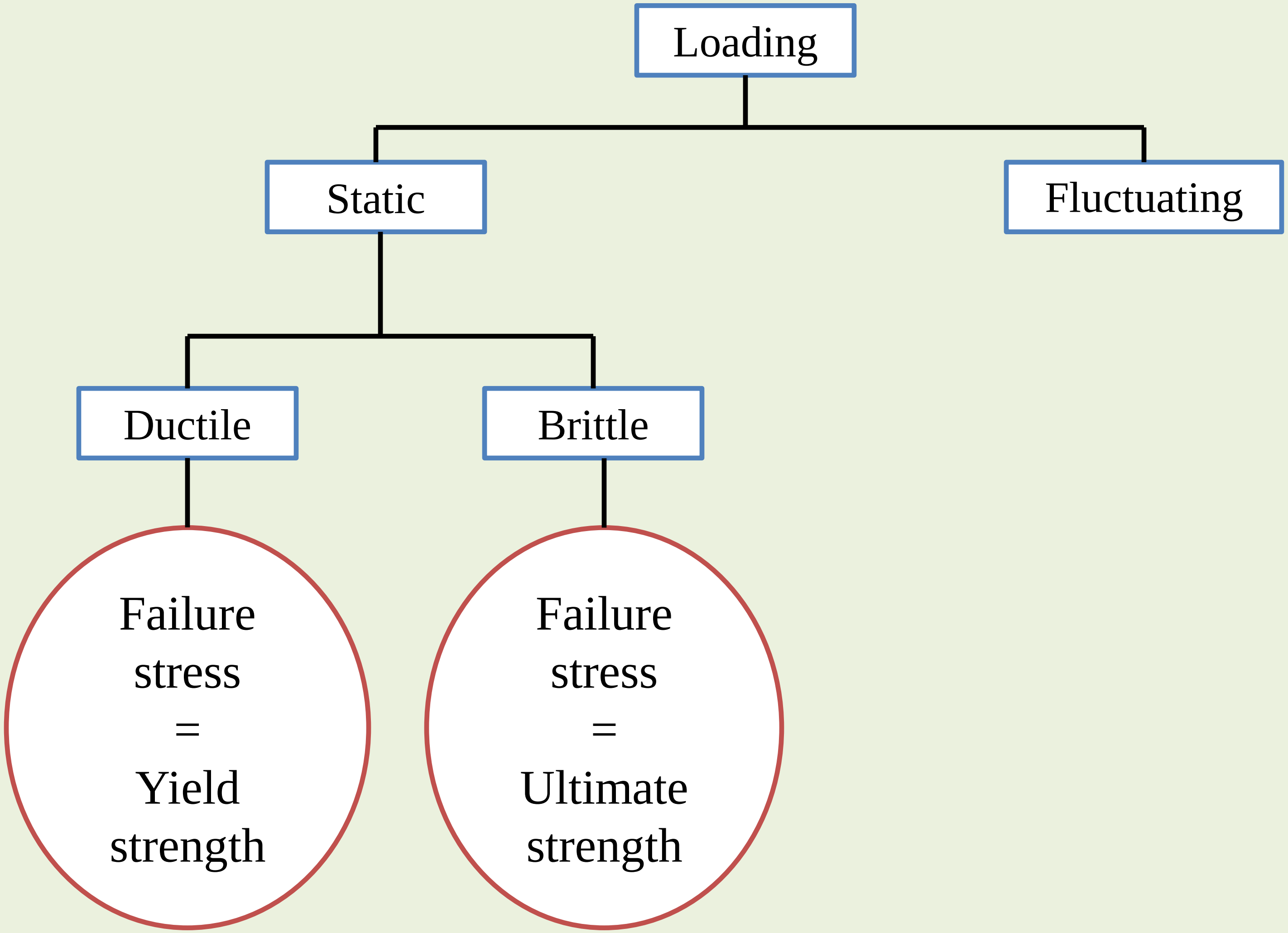
It is the ratio of failure stress to the allowable stress/ permissible stress/ working stress/ Design stress.

$$FOS = \frac{Failure - stress}{Allowable - stress}$$

$$FOS_{\min} = 1$$

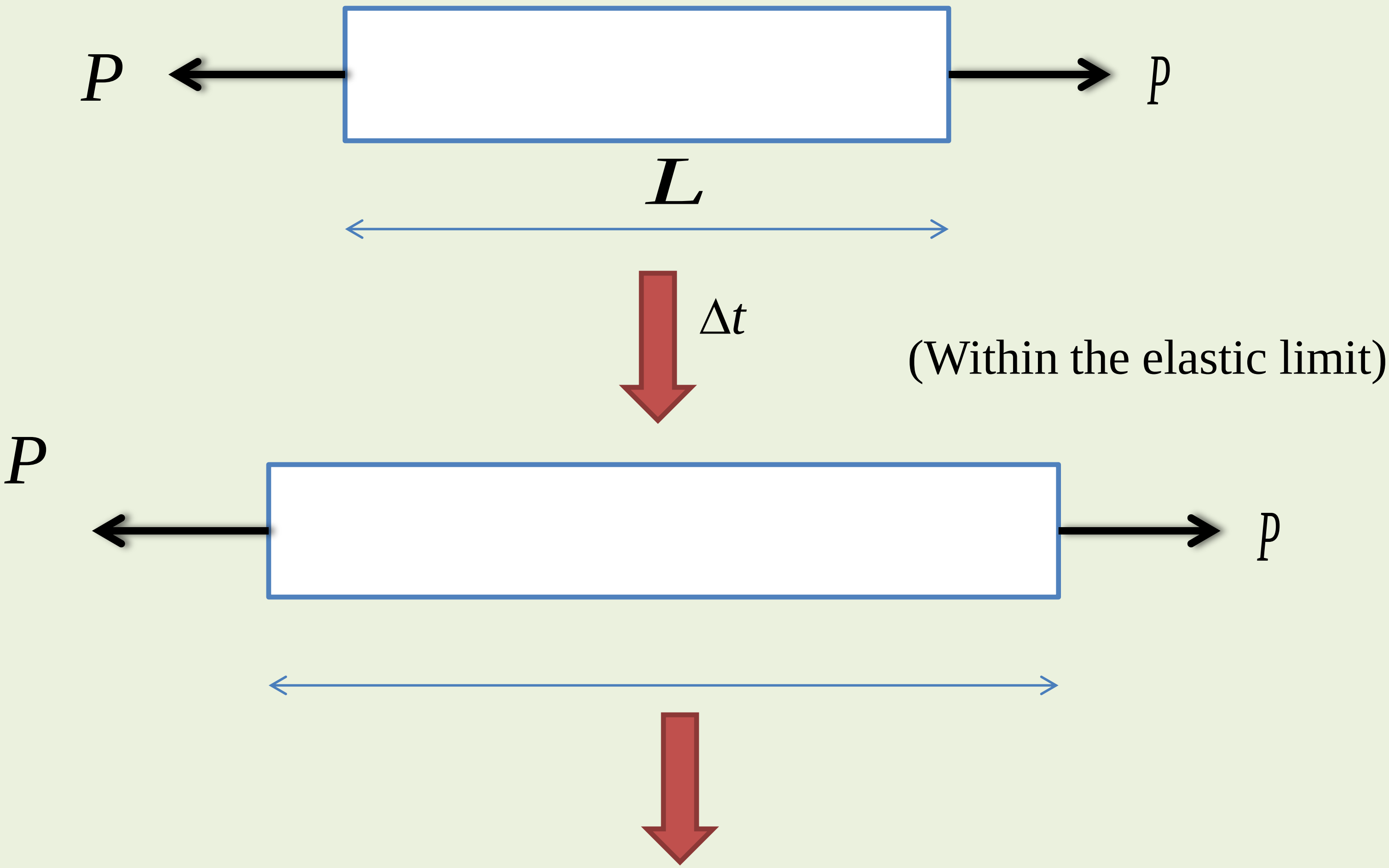
Margin of safety = FOS -1

Important Point



Q. A component is made of brittle material having ultimate strength 350 MPa & yield strength 220 MPa. If the component is subjected a tensile force 150 N & having cross-section area 1 mm * 1 mm. calculate factor of safety and margin of safety.

Strain energy due to principal stresses

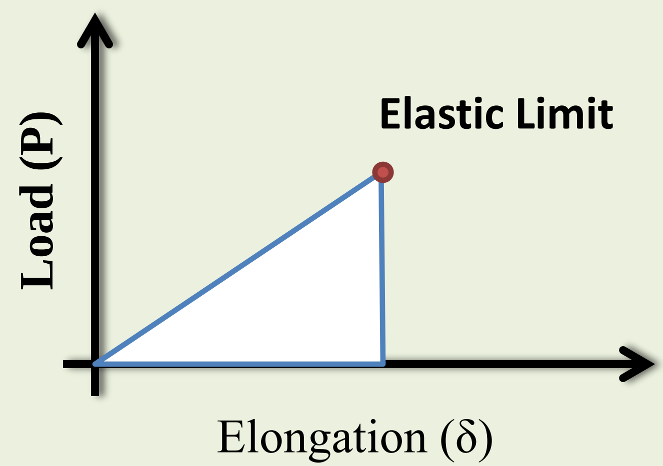


System execute a work

This work is stored within the body in form of internal energy

This energy is known as a strain energy.

- It is denoted by 'U'
- It is also defined as the area under the load-elongation curve (P-δ) within elastic limit.



U = Area under (P-δ) curve

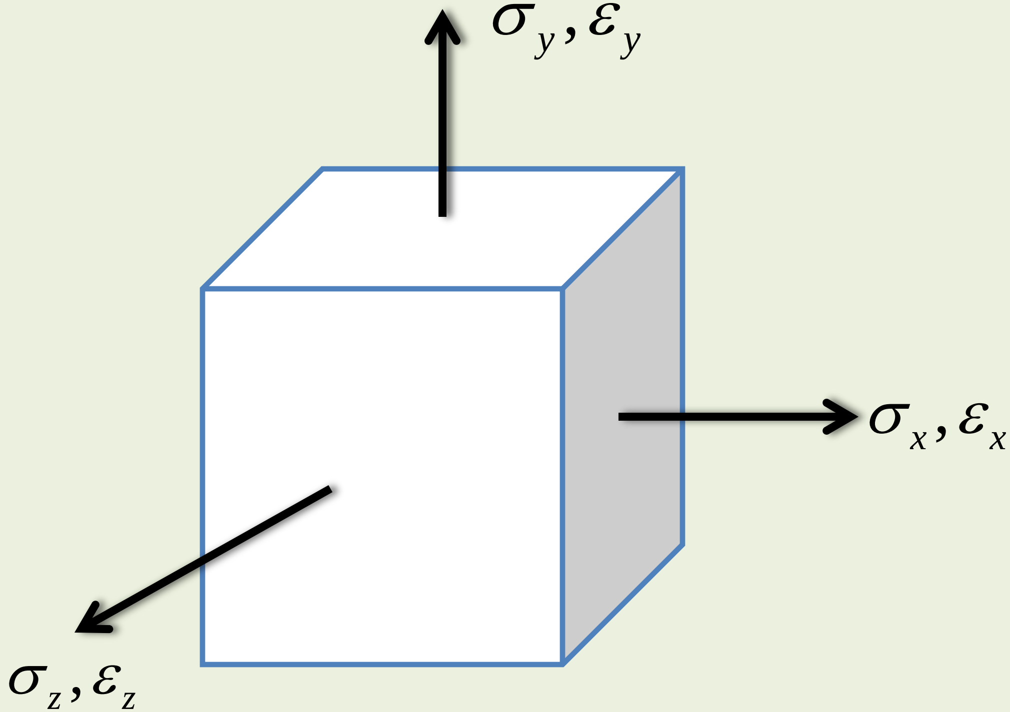
$$U = \frac{1}{2}[P.\delta]$$

$$U = \frac{1}{2}.\sigma.A.\varepsilon.l$$

$$U = \frac{1}{2}(\sigma \times \varepsilon)A.L$$

$$U = \frac{1}{2}(\sigma \times \varepsilon) \times (Volume)$$

$$\frac{Strain - Energy}{Volume} = \frac{1}{2}(\sigma \times \varepsilon)$$



$$\frac{S.E}{Volume} = \frac{1}{2}\left[\left(\sigma_x.\varepsilon_x\right)+\left(\sigma_y.\varepsilon_y\right)+\left(\sigma_z.\varepsilon_z\right)\right]$$

Resilience

It is defined as the energy stored within the elastic limit.

Proof Resilience

It is defined as the energy stored at the elastic limit.

Modulus of Resilience

It is defined as the energy stored per unit volume within the elastic limit.

$$\sigma \alpha \varepsilon$$

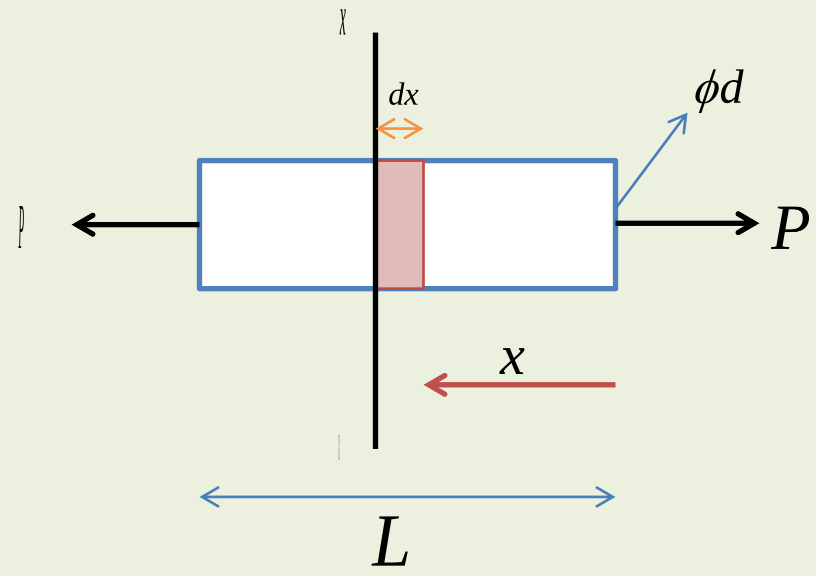
$$\sigma = E\varepsilon$$

$$\frac{P}{A} = E.\frac{\delta}{L}$$

$$\delta = \frac{PL}{AE}$$

$$S.E = \frac{1}{2}\left(P.\delta\right)$$

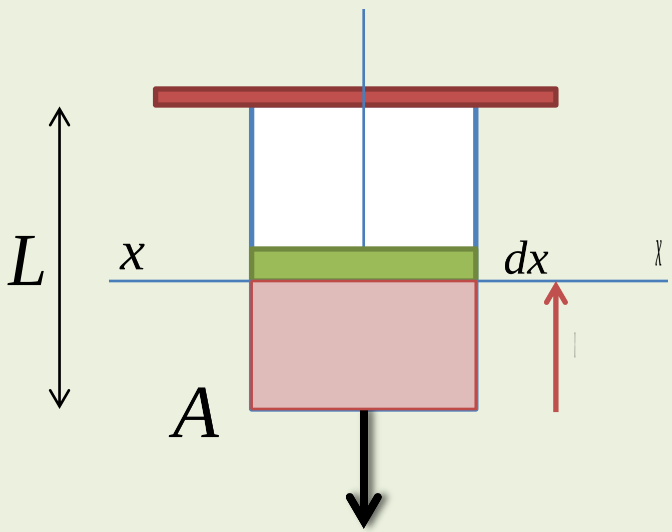
$$S.E = \frac{1}{2}\left(\frac{P^2L}{AE}\right)$$



General equation for strain energy when bar is subjected to pure axial load

$$S.E = \int_0^L \frac{P^2{}_{x-x}dx}{2A_{x-x}E}$$

Strain energy of the bar due to its self weight



$$S.E = \int_0^L \frac{P^2{}_{x-x}\,dx}{2A_{x-x}E}$$

$$= \int_0^L \frac{\gamma^2 A^2 L^2 dx}{2AE}$$

$$= \gamma^2 \frac{A}{E}\int_0^L x^2 dx$$

$$= \frac{1}{2}\gamma^2 \frac{A}{E}.\frac{L^3}{3}$$

$$= \frac{\gamma^2 A L^3}{6E}$$

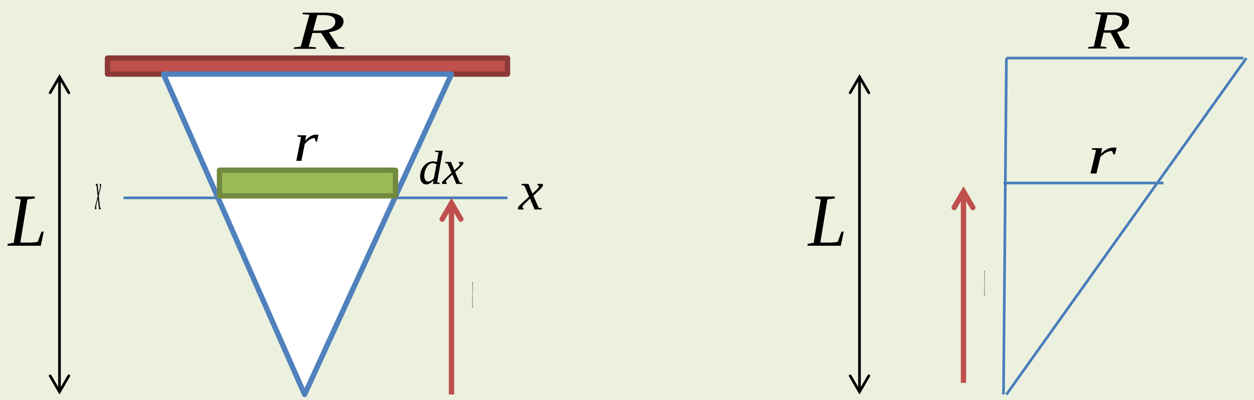
$$= \frac{(\gamma AL)^2 .L}{3\times 2AE}$$

$$SE = \frac{1}{3}\left(\frac{W^2 L}{2AE}\right)$$

Let γ = Weight density (weight per unit volume)

Weight of the member = γAL

Strain energy of conical bar due to its self weight



Let γ = Weight density (weight per unit volume)

Weight of the member

$$W = \gamma . \frac{1}{3} \pi R^2 . L$$

$$P_{x-x} = \gamma \times vol^{m}$$

$$P_{x-x} = \gamma . \frac{1}{3} \pi r^{2} . x$$

$$\frac{R}{r} = \frac{L}{x}$$

$$r = \frac{Rx}{L}$$

$$A_{x-x} = \pi . r^{2.}$$

$$S.E = \int_{0}^{L} \frac{P_{x-x}{}^{2} dx}{2A_{x-x}E}$$

$$S.E=\int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$=\int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$=\int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi r^2.x^2.dx}{2E}$$

$$=\int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{Rx}{L}\right)^2 .x^2.dx}{2E}$$

$$=\int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{R}{L}\right)^2 .x^4.dx}{2E}$$

$$S.E = \frac{\gamma^2 \pi R^2}{18EL^2}\left(\frac{x^5}{5}\right)_0^L$$

$$= \frac{\gamma^2 \pi R^2 L^3}{90E}$$

$$= \frac{\gamma^2 (A.L) L^2}{90E}$$

$$= \frac{\gamma^2 (A^2.L^2) L}{90AE}$$

$$= \frac{1}{10}\frac{\left(\frac{1}{3}\gamma AL\right)^2 .L}{AE}$$

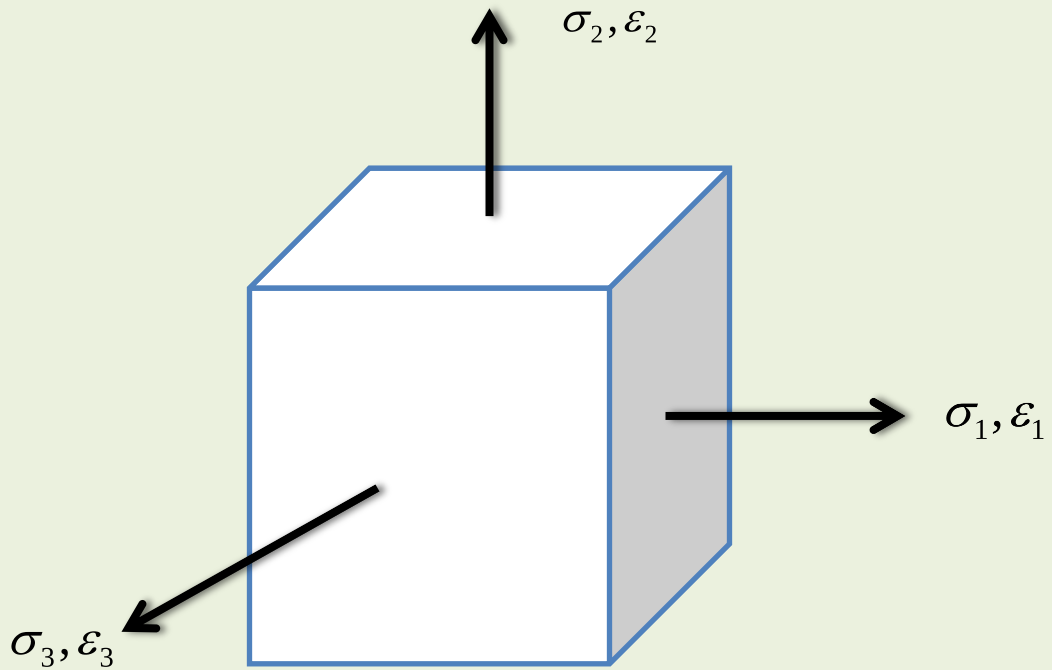
$$= \frac{1}{10}\frac{W^2.L}{AE}$$

Strain energy of bar under axial load

$$U = \frac{1}{K}\left[\frac{P^2 L}{AE}\right]$$

- K = 2, Prismatic bar under pure axial load. [P axial load]
- K = 6, Prismatic bar under self weight. [P is the weight of prismatic bar]
- K = 10, Conical bar under self weight. [P is weight of conical bar]

Strain energy due to principal stress



$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\left(\sigma_1 + \sigma_3\right)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right]$$

$$\frac{SE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

Strain energy per unit volume is also denoted the total strain energy (U).

So,

$$U = \frac{1}{2E}\left[\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

This total strain energy [U] has two components-

1. Volumetric Energy
2. Distortion Energy

Total energy = [Volumetric Energy + Distortion Energy]

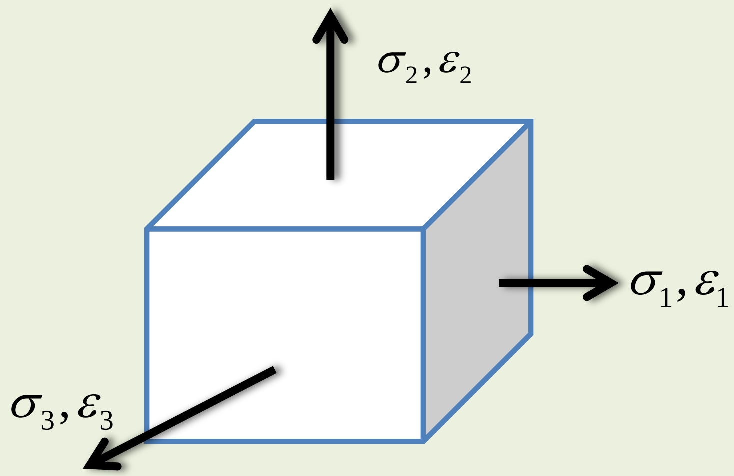
Volumetric Energy per unit volume

$$\frac{Volumetric-Energy}{Volume}=\frac{1}{2}.\sigma_{avg}.\varepsilon_v$$

$$\sigma_{avg}=\frac{\sigma_1+\sigma_2+\sigma_3}{3}$$

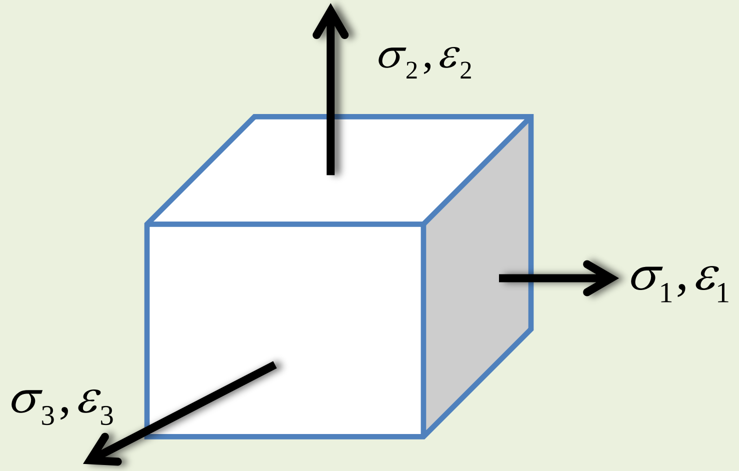
$$\varepsilon_v=\frac{(1-2\mu)}{E}.(\sigma_1+\sigma_2+\sigma_3)$$

$$\frac{Volumetric-Energy}{Volume}=\frac{(1-2\mu)}{6E}.(\sigma_1+\sigma_2+\sigma_3)^2$$



Distortion Energy per unit volume

Total energy/ Vol = [Volumetric Energy/ Vol^m]
+
[Distortion Energy/Vol^m]



$$\frac{DE}{Vol^m} = \frac{TSE}{Vol^m} - \frac{DE}{Vol^m}$$

$$\frac{DE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2}\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right] - \frac{(1-2\mu)}{6E}.\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2$$

$$\frac{DE}{Vol^m} = \frac{(1+\mu)}{6E}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]$$

Factor of safety [FOS]

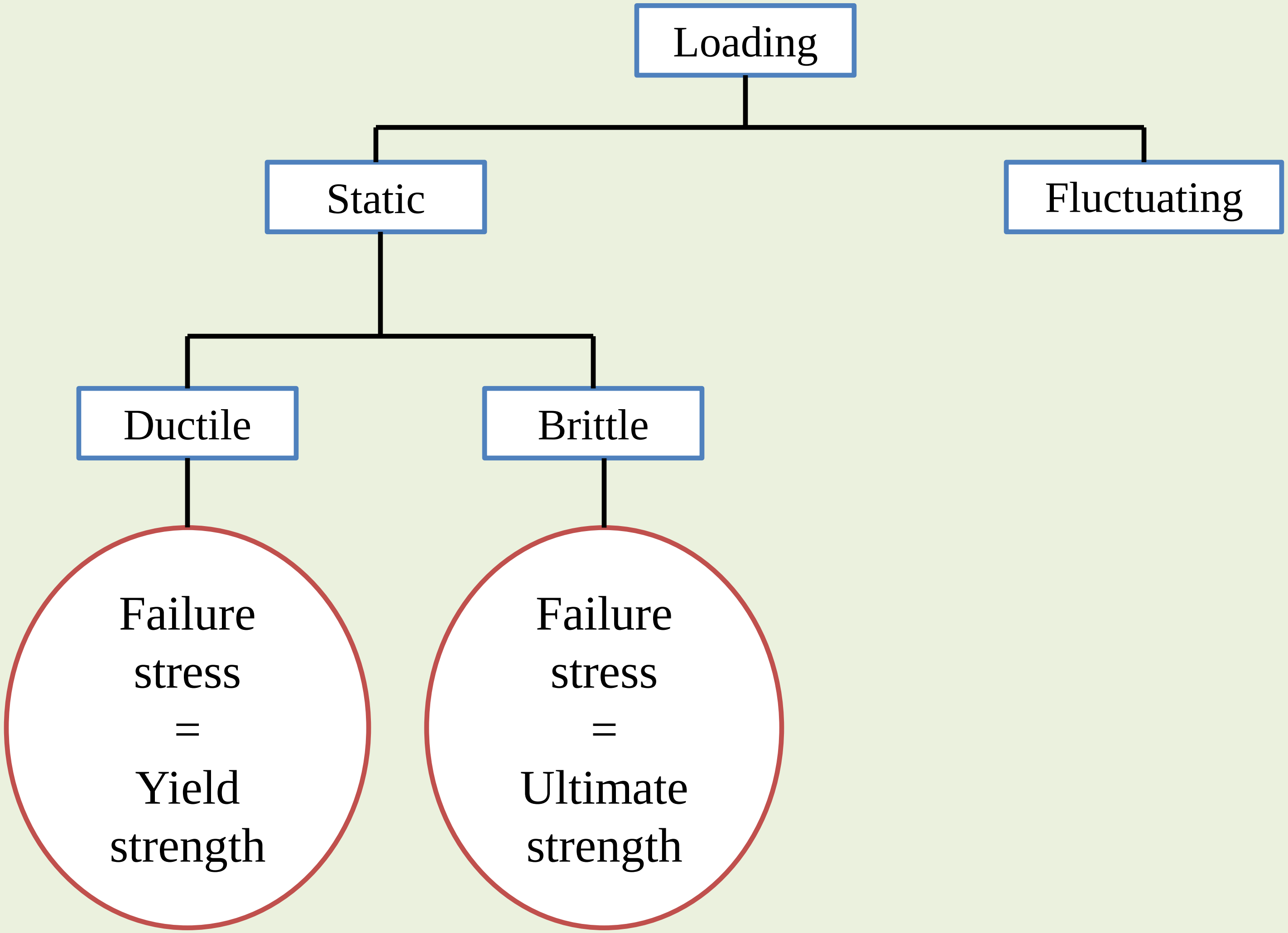
It is the ratio of failure stress to the allowable stress/ permissible stress/ working stress/ Design stress.

$$FOS = \frac{Failure - stress}{Allowable - stress}$$

$$FOS_{\min} = 1$$

Margin of safety = FOS -1

Important Point



Q. A component is made of brittle material having ultimate strength 350 MPa & yield strength 220 MPa. If the component is subjected a tensile force 150 N & having cross-section area 1 mm * 1 mm. calculate factor of safety and margin of safety.

THEORY OF FAILURE

Types of Theory of Failure

1. Maximum principal stress theory [MPST] / Rankine's Theory
2. Maximum shear stress theory [MSST] / Tresca & Guest Theory
3. Maximum principal strain theory [MPSt.T] / St. Venant Theroy
4. Total strain energy theory [TSET] / Haigh's Theory
5. Maximum Distortion Energy Theory [MDET] / Von Mises Theory

Note

1. Ductile materials are strongest in compression and weakest in shear loading condition.

Compression > Tension > Shear

2. Brittle materials are also strongest in compression and these are weakest in tension loading condition.

Compression > Shear > tension

Maximum principal stress theory [MPST]

Design Failure :-

Principal stress > Permissible stress

Design safe :-

Principal stress ≤ Permissible Stress

Major Principal stress, $\sigma_1 \leq \sigma_{per}$

$$\sigma_1 = \frac{S_{yt}}{FOS} \quad \text{(OR)} \quad \sigma_1 = \frac{S_{ut}}{FOS}$$

Ductile Material

Brittle Material

Note

1. Effect of shear stress is not considered.
2. This theory is suitable for brittle material in every state of stress.
3. This theory is not suitable for ductile material because ductile materials are weak in shear.
4. **The graphical representation of the theory of failure is square.**
5. Both, brittle and ductile materials are strongest in compression so yield compressive strength S_{yc} is greater than the yield tensile strength S_{yt} .
6. In most of the cases we consider that yield shear strength is equal to yield tensile strength.

$$S_{yc} = S_{yt}$$

Graphical Representation

S_{yc} = Yield compressive strength

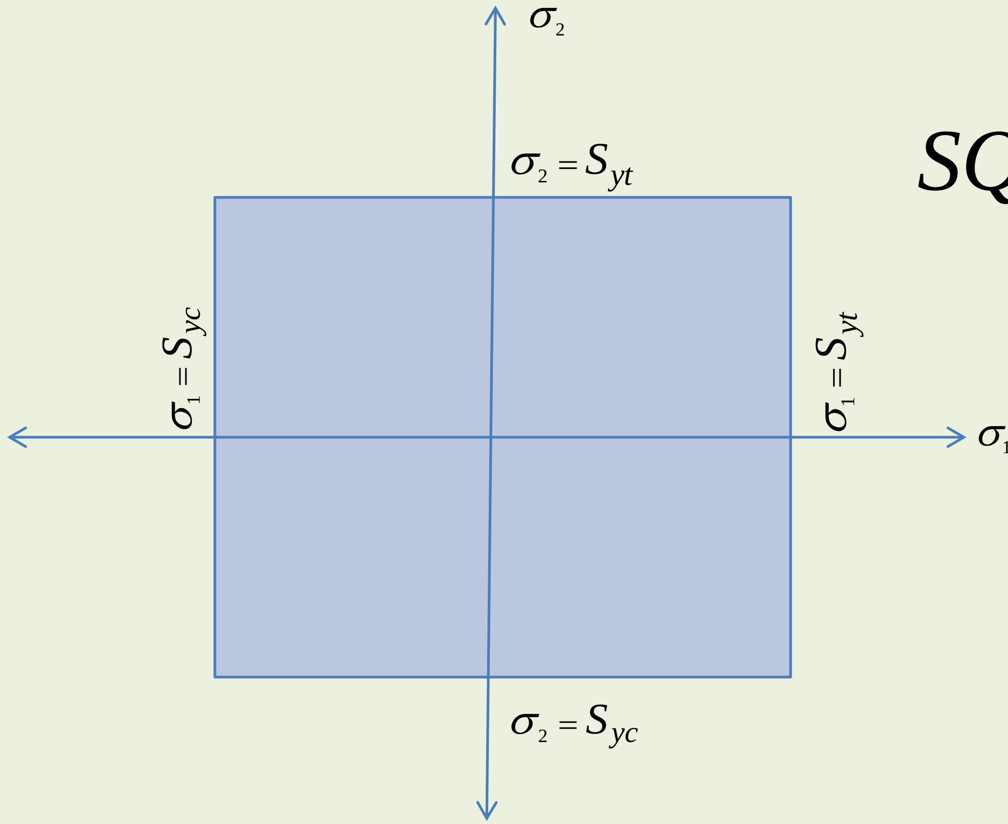
S_{yt} = Yield tensile strength

$$S_{yc} = S_{yt}$$

Assumptions:

- Graphical representation is used for biaxial state of stress.

Graphical Representation of MPST



$SQUARE$

Maximum Shear Stress Theory [MSST] [Tresca's & Guest's Theory]

<u>Design unsafe</u>

$$\text{Absolute}\,\tau_{\max} > \tau_{permissible}$$

<u>Design safe</u>

$$\text{Absolute}\,\tau_{\max} \le \tau_{permissible}$$

$$\tau_{\max} \le \tau_{permissible}$$

$$\tau_{permissible} \le \left(\frac{S_{ys}}{FOS}\right)$$

S_{ys} = Yield shear strength

Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{S_{yt}}{2}$$

$$abs.\tau_{\max} \le \frac{S_{yt}}{2.FOS}$$

$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right|$$

$$Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right| \le \left(\frac{S_{yt}}{2.FOS}\right)$$ {for Tri-axial state od stress}

Note - [For biaxial state of stress, $\sigma_3 = 0$ **]**

$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2}{2}, \frac{\sigma_1}{2}\right|$$

Important Points

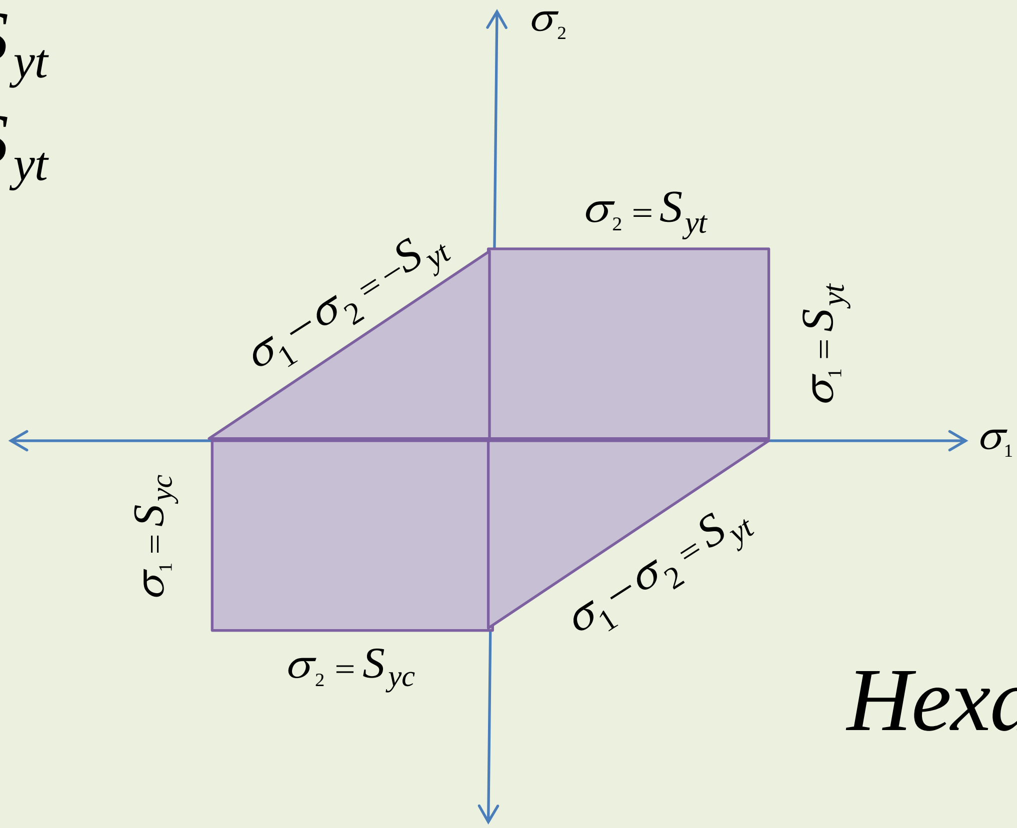
1. This theory of failure is suitable for ductile material.
2. If the principal stresses are like in nature, MPST & MSST will give the same result.
3. Under uniaxial state od stress both MPST & MSST will give the same result.
4. Under the hydrostatic state od stress condition for the ductile material we will use MPST.
5. **The graphical representation of MSST is Hexagon.**
6. **According to MSST,** $S_{ys} = \frac{S_{yt}}{2}$

Graphical Representation of MSST

$$\sigma_1 - \sigma_2 = \pm S_{yt}$$

$$\sigma_1 = \pm S_{yt}$$

$$\sigma_2 = \pm S_{yt}$$



Hexagon

Maximum Principal strain Theory [MPSt.T]

Design Fail $\varepsilon_1 > \left[\left(\varepsilon \right)_{YP} \right]_{TT}$

YP - Yield point & TT - Tension Test

Design Safe

$$\varepsilon_1 \leq \left[\left(\varepsilon \right)_{YP} \right]_{TT}$$

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)]..................(1)$$

$$For, \left[\left(\varepsilon \right)_{YP} \right]_{TT}$$

$$put, \sigma_1 = \frac{S_{yt}}{FOS}$$

$$\sigma_2 = \sigma_3 = 0$$

$$\left[\left(\varepsilon \right)_{YP} \right]_{TT} = \frac{1}{E}\left(\frac{S_{yt}}{FOS} \right)$$

$$\frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] = \frac{1}{E}\left(\frac{S_{yt}}{FOS}\right)$$

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \left(\frac{S_{yt}}{FOS}\right)$$

For Biaxial State of Stress

$$\sigma_1 - \mu\sigma_2 = \left(\frac{S_{yt}}{FOS}\right)$$

Note - The Graphical representation of MPSt.T is Rhombus.

Total Strain Energy Theory [TSET]

Design Fail,

$$\frac{TSE}{Vol^m} > \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{TSE}{Vol^m} \leq \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{TSE}{Vol^m} = \frac{1}{2E}\left[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

$$Now, \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_1 = \frac{S_{yt}}{FOS}$$

$$\left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT} = \frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

For Triaxial state of stress,

$$\frac{1}{2E}\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

For biaxial state of stress,

$$\sigma_3=0$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}-2\mu\left(\sigma_1\sigma_2\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation TSET is Ellipse.**

Semi major axis = $\frac{S_{yt}}{\sqrt{1-\mu}}$ Semi minor axis = $\frac{S_{yt}}{\sqrt{1+\mu}}$

Maximum Distortion Energy Theory [MDET]

Design unsafe,

$$\frac{DE}{Vol^m} > \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{DE}{Vol^m} \leq \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{DE}{Vol^m} = \left(\frac{1+\mu}{6E}\right)\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

$$\left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT} = \left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left(\frac{1+\mu}{6E}\right)\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=2\left(\frac{S_{yt}}{FOS}\right)^2$$

For Biaxial state of stress ,

$$\sigma_1{}^2+\sigma_2{}^2-\sigma_1\sigma_2=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation MDET is Ellipse.**

Semi Major Axis - $\sqrt{2}S_{yt}$

Semi Minor Axis - $\sqrt{\frac{2}{3}}S_{yt}$

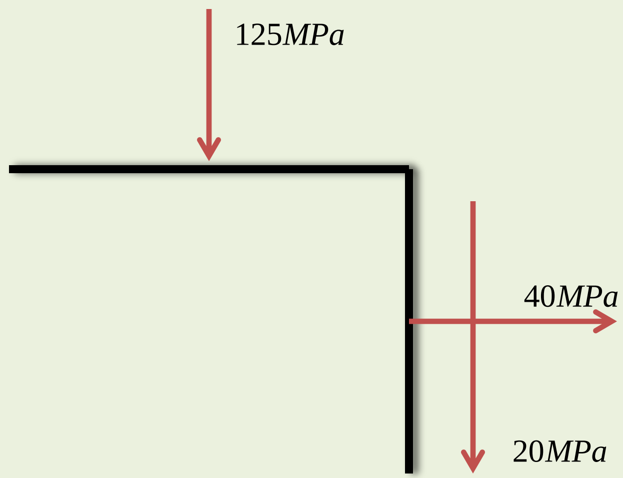
Important Points

- By using MDET, Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{1}{\sqrt{3}} S_{yt} = 0.577 S_{yt}$$

- MSST is most safe but uneconomical theory of failure.
- MDET is most economical and safe theory of failure.
- **Under the uniaxial state of stress, All the five theory of failure will give the same results.**

Q for the given state of stress condition of a point find the factor of safety by using MSST & MDET. [Syt = 400 MPa] Given



Q A rectangular bar is subjected to tensile force of 100 N and shear force 50N. The x-s/c area of the bar is t x t mm^2. Determine the dimension of x-s/c by using MPST & MSST. [Syt = 150 MPa] Given

THEORY OF FAILURE

Types of Theory of Failure

1. Maximum principal stress theory [MPST] / Rankine's Theory
2. Maximum shear stress theory [MSST] / Tresca & Guest Theory
3. Maximum principal strain theory [MPSt.T] / St. Venant Theroy
4. Total strain energy theory [TSET] / Haigh's Theory
5. Maximum Distortion Energy Theory [MDET] / Von Mises Theory

Note

1. Ductile materials are strongest in compression and weakest in shear loading condition.

Compression > Tension > Shear

2. Brittle materials are also strongest in compression and these are weakest in tension loading condition.

Compression > Shear > tension

Maximum principal stress theory [MPST]

Design Failure :-

Principal stress > Permissible stress

Design safe :-

Principal stress ≤ Permissible Stress

Major Principal stress, $\sigma_1 \leq \sigma_{per}$

$$\sigma_1 = \frac{S_{yt}}{FOS} \quad \text{(OR)} \quad \sigma_1 = \frac{S_{ut}}{FOS}$$

Ductile Material

Brittle Material

Note

1. Effect of shear stress is not considered.
2. This theory is suitable for brittle material in every state of stress.
3. This theory is not suitable for ductile material because ductile materials are weak in shear.
4. **The graphical representation of the theory of failure is square.**
5. Both, brittle and ductile materials are strongest in compression so yield compressive strength S_{yc} is greater than the yield tensile strength S_{yt} .
6. In most of the cases we consider that yield shear strength is equal to yield tensile strength.

$$S_{yc} = S_{yt}$$

Graphical Representation

S_{yc} = Yield compressive strength

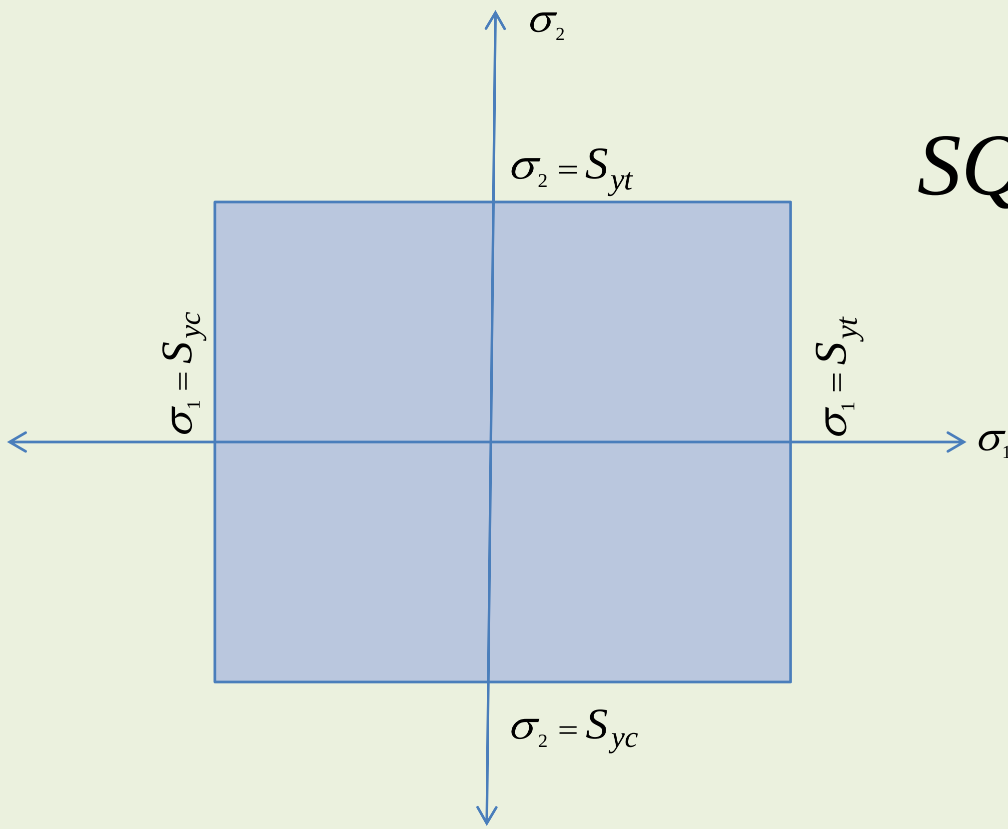
S_{yt} = Yield tensile strength

$$S_{yc} = S_{yt}$$

Assumptions:

- Graphical representation is used for biaxial state of stress.

Graphical Representation of MPST



SQUARE

Maximum Shear Stress Theory [MSST] [Tresca's & Guest's Theory]

<u>Design unsafe</u>

Absolute $\tau_{\max} > \tau_{permissible}$

<u>Design safe</u>

Absolute $\tau_{\max} \leq \tau_{permissible}$

$$\tau_{\max} \leq \tau_{permissible}$$

$$\tau_{permissible} \leq \left(\frac{S_{ys}}{FOS} \right)$$

S_{ys} = Yield shear strength

Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{S_{yt}}{2}$$

$$abs.\tau_{\max} \le \frac{S_{yt}}{2.FOS}$$

$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right|$$

$$Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right| \le \left(\frac{S_{yt}}{2.FOS}\right)$$ {for Tri-axial state od stress}

Note - [For biaxial state of stress, $\sigma_3 = 0$]

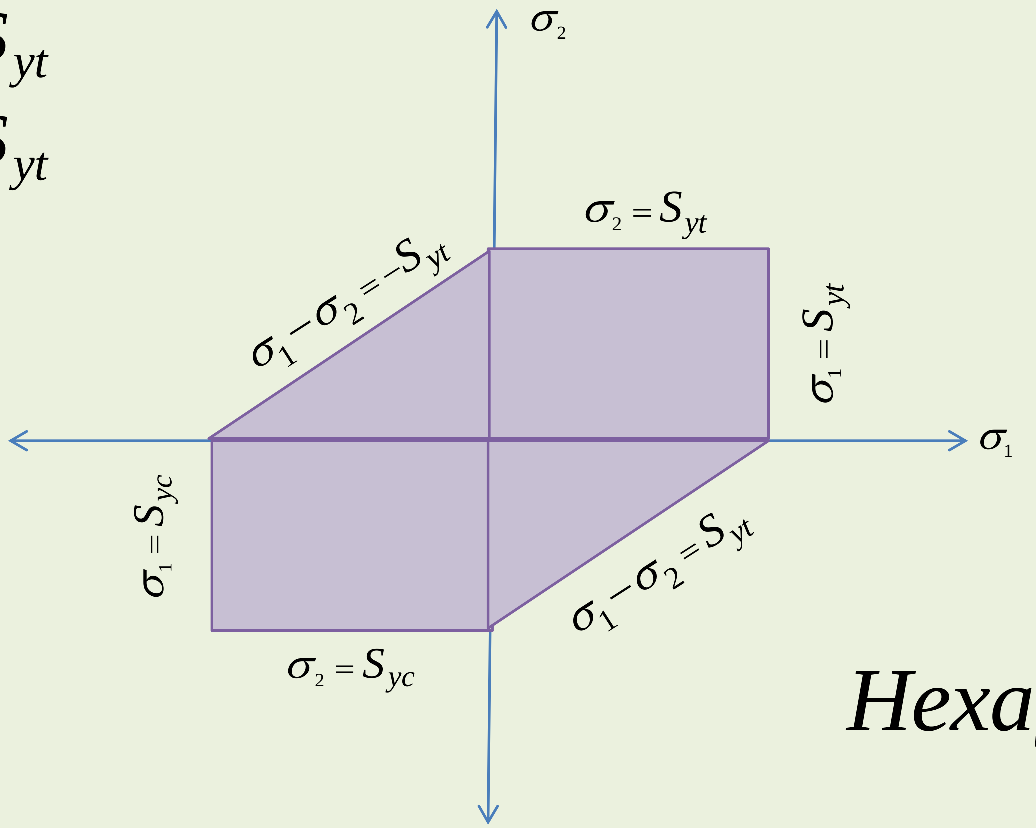
$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2}{2}, \frac{\sigma_1}{2}\right|$$

Important Points

1. This theory of failure is suitable for ductile material.
2. If the principal stresses are like in nature, MPST & MSST will give the same result.
3. Under uniaxial state od stress both MPST & MSST will give the same result.
4. Under the hydrostatic state od stress condition for the ductile material we will use MPST.
5. **The graphical representation of MSST is Hexagon.**
6. **According to MSST,** $S_{ys} = \frac{S_{yt}}{2}$

Graphical Representation of MSST

$$\sigma_1 - \sigma_2 = \pm S_{yt}$$
$$\sigma_1 = \pm S_{yt}$$
$$\sigma_2 = \pm S_{yt}$$



Hexagon

Maximum Principal strain Theory [MPSt.T]

Design Fail $\varepsilon_1 > \left[(\varepsilon)_{YP}\right]_{TT}$

YP - Yield point & TT - Tension Test

Design Safe

$$\varepsilon_1 \leq \left[(\varepsilon)_{YP}\right]_{TT}$$

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] \text{..................(1)}$$

$$For, \left[(\varepsilon)_{YP}\right]_{TT}$$

$$put, \sigma_1 = \frac{S_{yt}}{FOS}$$

$$\sigma_2 = \sigma_3 = 0$$

$$\left[(\varepsilon)_{YP}\right]_{TT} = \frac{1}{E}\left(\frac{S_{yt}}{FOS}\right)$$

$$\frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] = \frac{1}{E}\left(\frac{S_{yt}}{FOS}\right)$$

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \left(\frac{S_{yt}}{FOS}\right)$$

For Biaxial State of Stress

$$\sigma_1 - \mu\sigma_2 = \left(\frac{S_{yt}}{FOS}\right)$$

Note - The Graphical representation of MPSt.T is Rhombus.

Total Strain Energy Theory [TSET]

Design Fail,

$$\frac{TSE}{Vol^m} > \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{TSE}{Vol^m} \le \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{TSE}{Vol^m} = \frac{1}{2E}\left[\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2} - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

$$Now, \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_1 = \frac{S_{yt}}{FOS}$$

$$\left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT} = \frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

For Triaxial state of stress,

$$\frac{1}{2E}\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

For biaxial state of stress,

$$\sigma_3=0$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}-2\mu\left(\sigma_1\sigma_2\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation TSET is Ellipse.**

Semi major axis = $\frac{S_{yt}}{\sqrt{1-\mu}}$ Semi minor axis = $\frac{S_{yt}}{\sqrt{1+\mu}}$

Maximum Distortion Energy Theory [MDET]

Design unsafe,

$$\frac{DE}{Vol^m} > \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{DE}{Vol^m} \leq \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{DE}{Vol^m} = \left(\frac{1+\mu}{6E}\right)\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$\left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT} = \left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left(\frac{1+\mu}{6E}\right)\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]=\left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]=2\left(\frac{S_{yt}}{FOS}\right)^2$$

For Biaxial state of stress ,

$$\sigma_1^{\ 2}+\sigma_2^{\ 2}-\sigma_1\sigma_2=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation MDET is Ellipse.**

Semi Major Axis - $\sqrt{2}S_{yt}$

Semi Minor Axis - $\sqrt{\frac{2}{3}}S_{yt}$

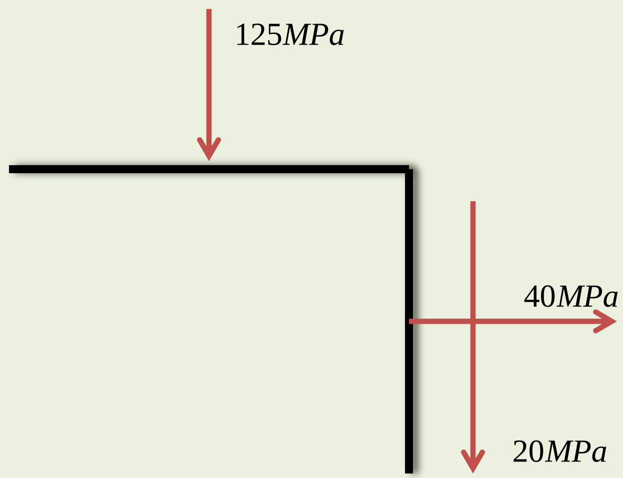
Important Points

- By using MDET, Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{1}{\sqrt{3}} S_{yt} = 0.577 S_{yt}$$

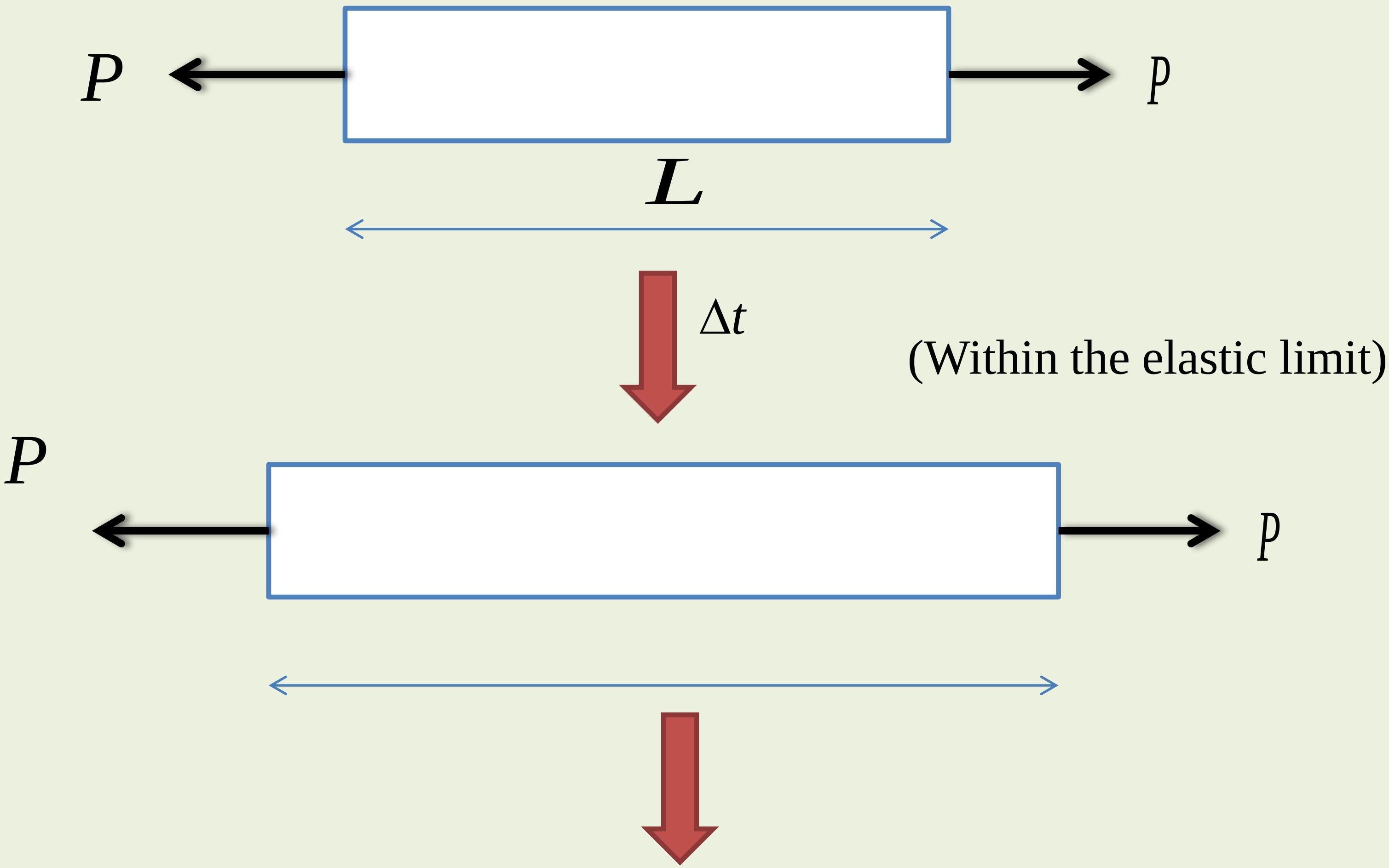
- MSST is most safe but uneconomical theory of failure.
- MDET is most economical and safe theory of failure.
- **Under the uniaxial state of stress, All the five theory of failure will give the same results.**

Q for the given state of stress condition of a point find the factor of safety by using MSST & MDET. [Syt = 400 MPa] Given



Q A rectangular bar is subjected to tensile force of 100 N and shear force 50N. The x-s/c area of the bar is t x t mm^2. Determine the dimension of x-s/c by using MPST & MSST. [Syt = 150 MPa] Given

Strain energy due to principal stresses

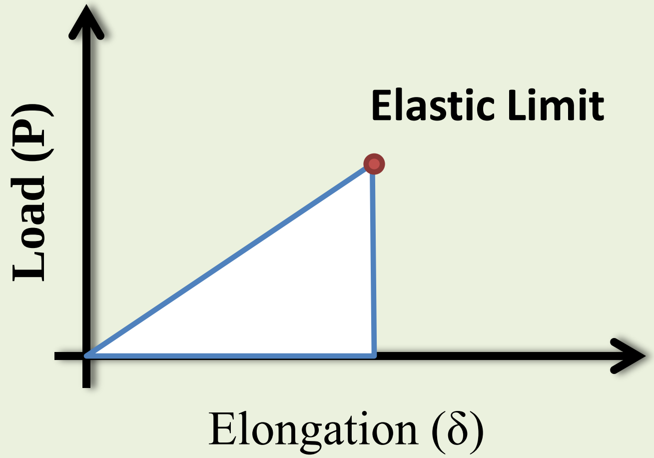


System execute a work

This work is stored within the body in form of internal energy

This energy is known as a strain energy.

- It is denoted by 'U'
- It is also defined as the area under the load-elongation curve (P-δ) within elastic limit.



U = Area under (P-δ) curve

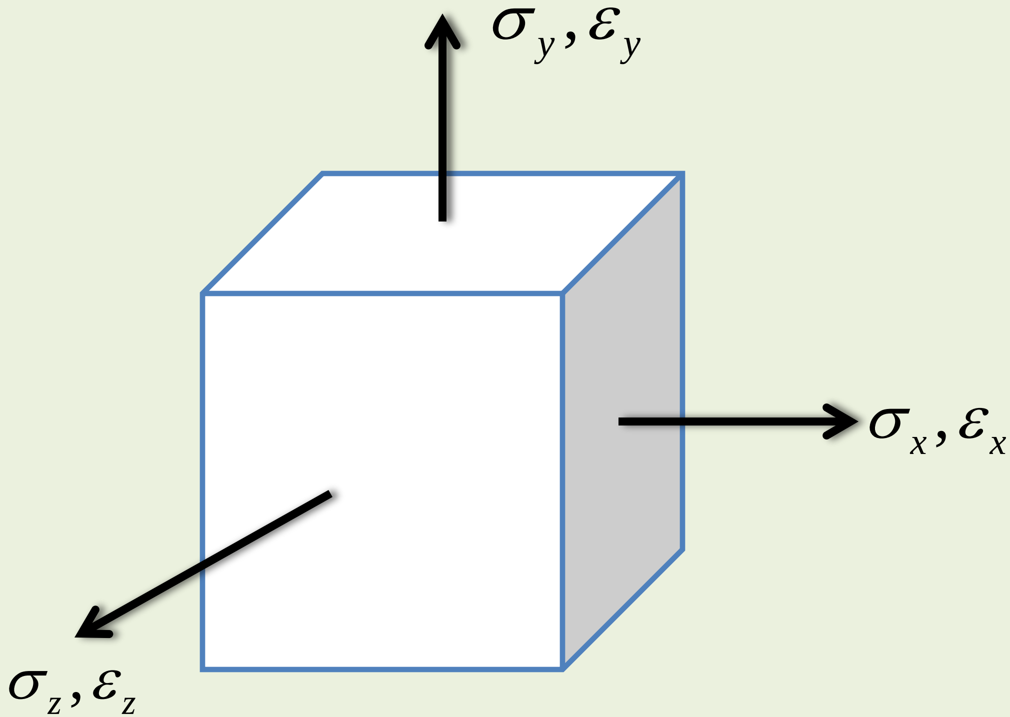
$$U = \frac{1}{2}[P.\delta]$$

$$U = \frac{1}{2}.\sigma.A.\varepsilon.l$$

$$U = \frac{1}{2}(\sigma \times \varepsilon)A.L$$

$$U = \frac{1}{2}(\sigma \times \varepsilon) \times (Volume)$$

$$\frac{Strain - Energy}{Volume} = \frac{1}{2}(\sigma \times \varepsilon)$$



$$\frac{S.E}{Volume} = \frac{1}{2}\left[\left(\sigma_x.\varepsilon_x\right) + \left(\sigma_y.\varepsilon_y\right) + \left(\sigma_z.\varepsilon_z\right)\right]$$

Resilience

It is defined as the energy stored within the elastic limit.

Proof Resilience

It is defined as the energy stored at the elastic limit.

Modulus of Resilience

It is defined as the energy stored per unit volume within the elastic limit.

$$\sigma \alpha \varepsilon$$

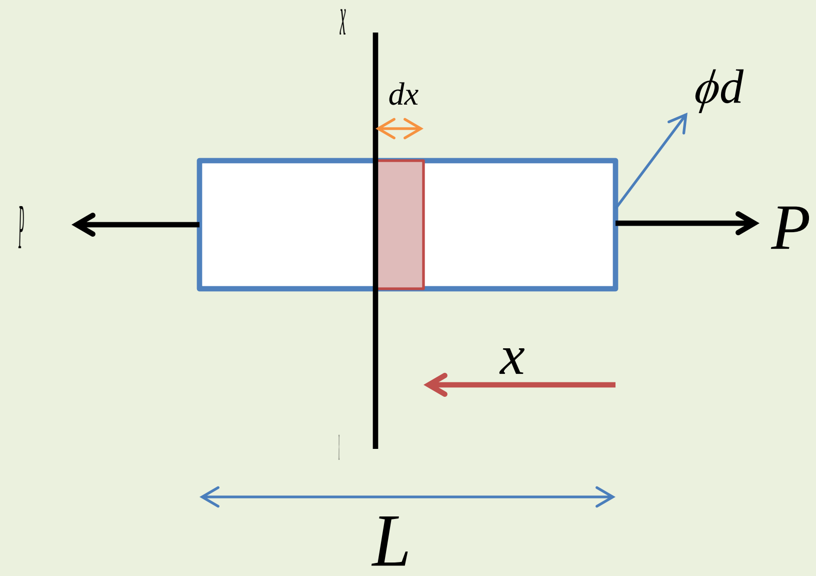
$$\sigma = E\varepsilon$$

$$\frac{P}{A} = E.\frac{\delta}{L}$$

$$\delta = \frac{PL}{AE}$$

$$S.E = \frac{1}{2}\left(P.\delta\right)$$

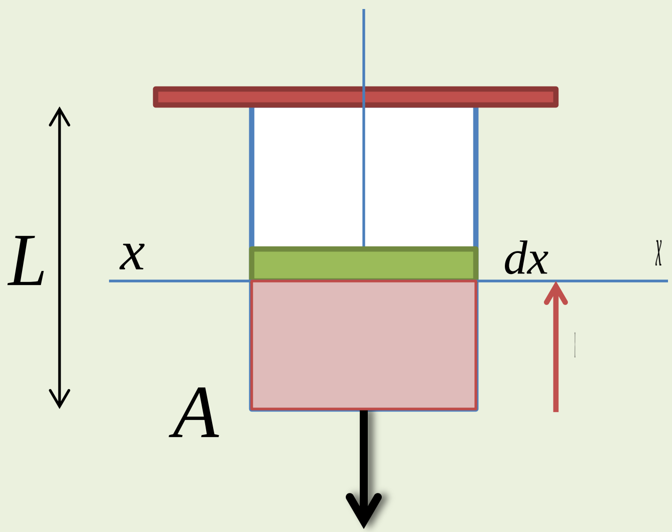
$$S.E = \frac{1}{2}\left(\frac{P^2L}{AE}\right)$$



General equation for strain energy when bar is subjected to pure axial load

$$S.E = \int_0^L \frac{P^2{}_{x-x}dx}{2A_{x-x}E}$$

Strain energy of the bar due to its self weight



Let γ = Weight density (weight per unit volume)

Weight of the member = γAL

$$S.E = \int_0^L \frac{P^2{}_{x-x}\,dx}{2A_{x-x}E}$$

$$= \int_0^L \frac{\gamma^2 A^2 L^2 dx}{2AE}$$

$$= \gamma^2 \frac{A}{E}\int_0^L x^2 dx$$

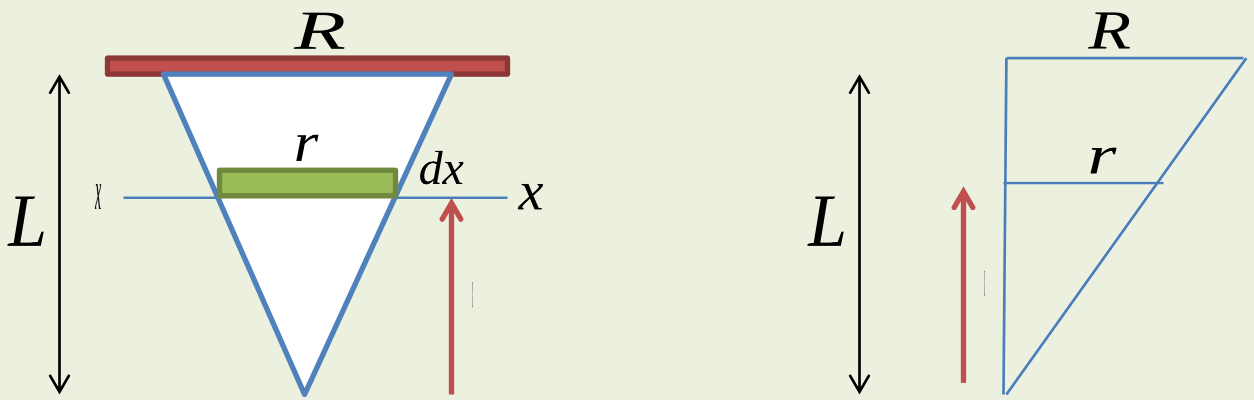
$$= \frac{1}{2}\gamma^2 \frac{A}{E}\cdot\frac{L^3}{3}$$

$$= \frac{\gamma^2 A L^3}{6E}$$

$$= \frac{(\gamma AL)^2 .L}{3\times 2AE}$$

$$SE = \frac{1}{3}\left(\frac{W^2 L}{2AE}\right)$$

Strain energy of conical bar due to its self weight



Let γ = Weight density (weight per unit volume)

Weight of the member

$$W = \gamma . \frac{1}{3} \pi R^2 . L$$

$$P_{x-x} = \gamma \times vol^{m}$$

$$P_{x-x} = \gamma.\frac{1}{3}\pi r^{2}.x$$

$$\frac{R}{r} = \frac{L}{x}$$

$$r = \frac{Rx}{L}$$

$$A_{x-x} = \pi.r^{2.}$$

$$S.E = \int_{0}^{L} \frac{P_{x-x}{}^{2}dx}{2A_{x-x}E}$$

$$S.E = \int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$= \int_0^L \frac{\left(\gamma.\frac{1}{3}\pi r^2.x\right)^2 dx}{2\left(\pi.r^2\right)E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi r^2.x^2.dx}{2E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{Rx}{L}\right)^2 .x^2.dx}{2E}$$

$$= \int_0^L \frac{\left(\frac{\gamma}{3}\right)^2 \pi \left(\frac{R}{L}\right)^2 .x^4.dx}{2E}$$

$$S.E = \frac{\gamma^2 \pi R^2}{18EL^2}\left(\frac{x^5}{5}\right)_0^L$$

$$= \frac{\gamma^2 \pi R^2 L^3}{90E}$$

$$= \frac{\gamma^2 \left(A.L\right) L^2}{90E}$$

$$= \frac{\gamma^2 \left(A^2.L^2\right) L}{90AE}$$

$$= \frac{1}{10}\frac{\left(\frac{1}{3}\gamma AL\right)^2 .L}{AE}$$

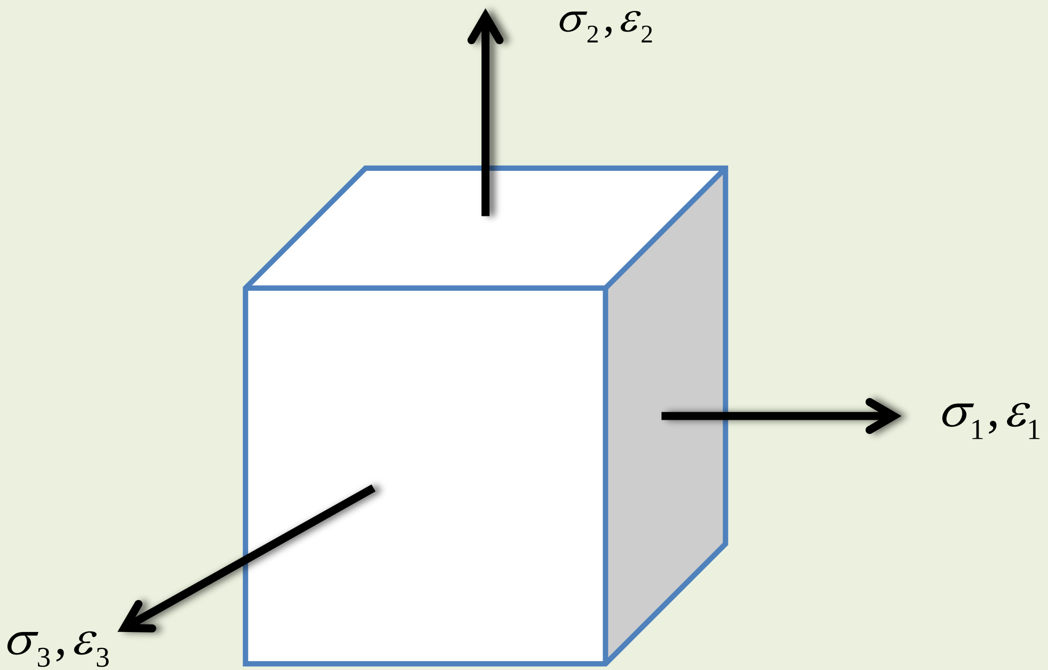
$$= \frac{1}{10}\frac{W^2.L}{AE}$$

Strain energy of bar under axial load

$$U = \frac{1}{K}\left[\frac{P^2 L}{AE}\right]$$

- K = 2, Prismatic bar under pure axial load. [P axial load]
- K = 6, Prismatic bar under self weight. [P is the weight of prismatic bar]
- K = 10, Conical bar under self weight. [P is weight of conical bar]

Strain energy due to principal stress



$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\frac{SE}{Vol^m} = \frac{1}{2}\sigma_1.\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3.\varepsilon_3$$

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right]$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\left(\sigma_1 + \sigma_3\right)\right]$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right]$$

$$\frac{SE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1^{\ 2} + \sigma_2^{\ 2} + \sigma_3^{\ 2}\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

Strain energy per unit volume is also denoted the total strain energy (U).

So,

$$U = \frac{1}{2E}\left[\left(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right]$$

This total strain energy [U] has two components-

1. Volumetric Energy
2. Distortion Energy

Total energy = [Volumetric Energy + Distortion Energy]

ANKIT SAXENA (saxena01ankit@gmail.com)

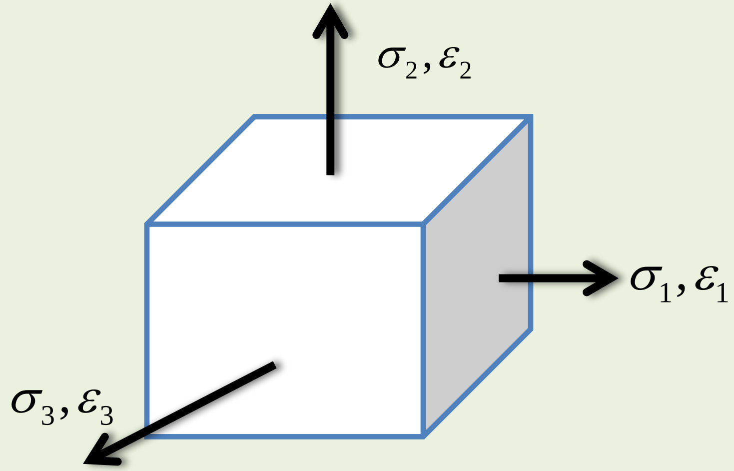
Volumetric Energy per unit volume

$$\frac{Volumetric-Energy}{Volume}=\frac{1}{2}.\sigma_{avg}.\varepsilon_v$$

$$\sigma_{avg}=\frac{\sigma_1+\sigma_2+\sigma_3}{3}$$

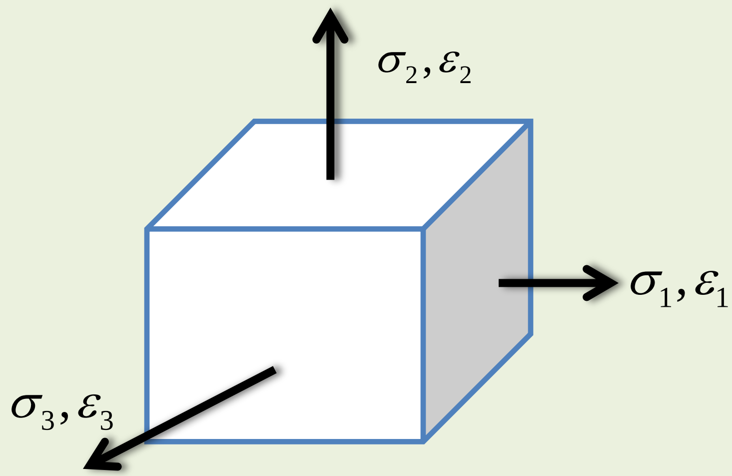
$$\varepsilon_v=\frac{(1-2\mu)}{E}.(\sigma_1+\sigma_2+\sigma_3)$$

$$\frac{Volumetric-Energy}{Volume}=\frac{(1-2\mu)}{6E}.(\sigma_1+\sigma_2+\sigma_3)^2$$



Distortion Energy per unit volume

Total energy/ Vol = [Volumetric Energy/ Vol^m]
+
[Distortion Energy/Vol^m]



$$\frac{DE}{Vol^m} = \frac{TSE}{Vol^m} - \frac{DE}{Vol^m}$$

$$\frac{DE}{Vol^m} = \frac{1}{2E}\left[\left(\sigma_1^{\,2} + \sigma_2^{\,2} + \sigma_3^{\,2}\right) - 2\mu\left(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1\right)\right] - \frac{(1-2\mu)}{6E}.\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2$$

$$\frac{DE}{Vol^m} = \frac{(1+\mu)}{6E}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]$$

Factor of safety [FOS]

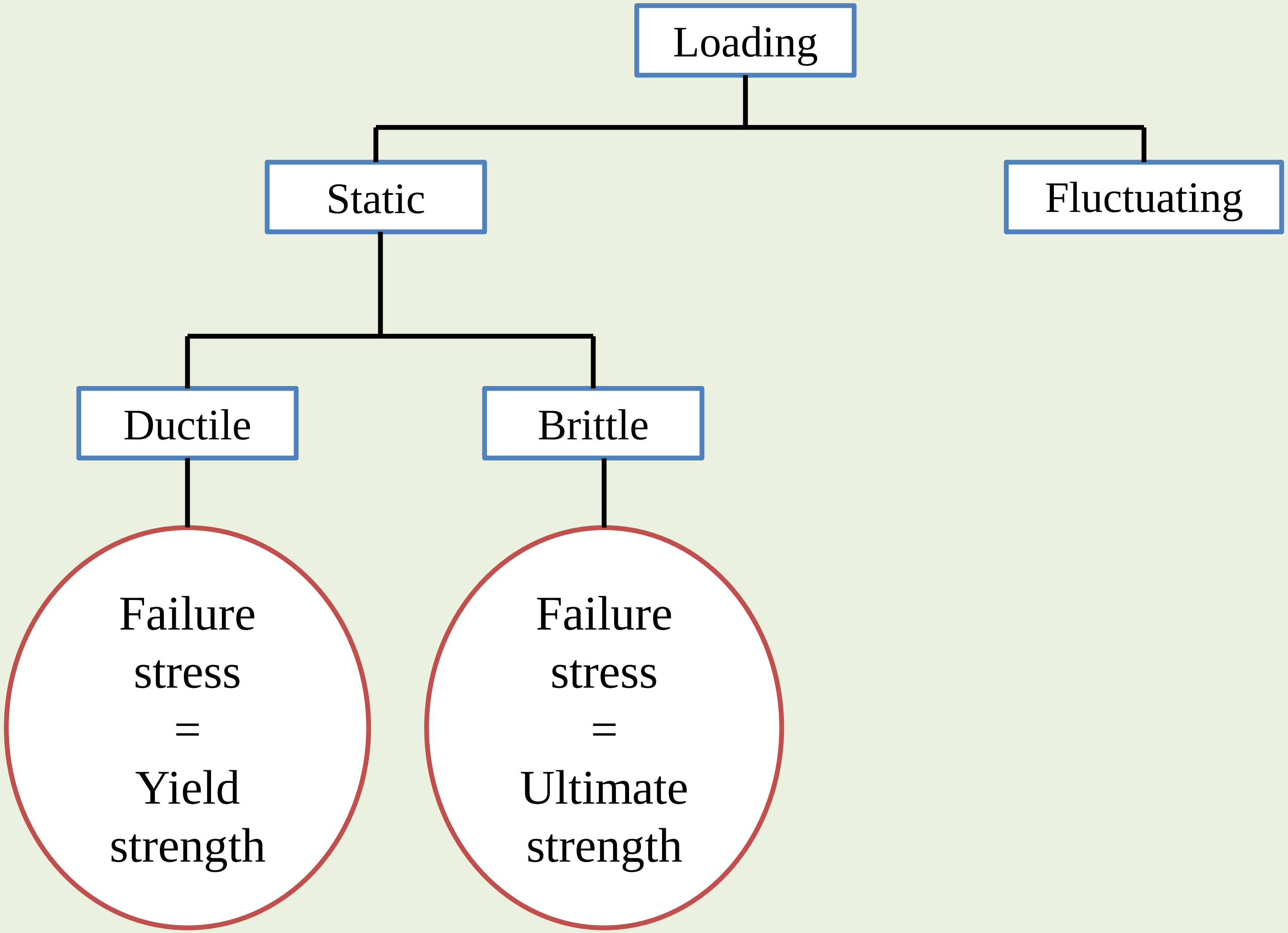
It is the ratio of failure stress to the allowable stress/ permissible stress/ working stress/ Design stress.

$$FOS = \frac{Failure - stress}{Allowable - stress}$$

$$FOS_{\min} = 1$$

Margin of safety = FOS -1

Important Point



Q. A component is made of brittle material having ultimate strength 350 MPa & yield strength 220 MPa. If the component is subjected a tensile force 150 N & having cross-section area 1 mm * 1 mm. calculate factor of safety and margin of safety.

THEORY OF FAILURE

Types of Theory of Failure

1. Maximum principal stress theory [MPST] / Rankine's Theory
2. Maximum shear stress theory [MSST] / Tresca & Guest Theory
3. Maximum principal strain theory [MPSt.T] / St. Venant Theroy
4. Total strain energy theory [TSET] / Haigh's Theory
5. Maximum Distortion Energy Theory [MDET] / Von Mises Theory

Note

1. Ductile materials are strongest in compression and weakest in shear loading condition.

Compression > Tension > Shear

2. Brittle materials are also strongest in compression and these are weakest in tension loading condition.

Compression > Shear > tension

Maximum principal stress theory [MPST]

Design Failure :-

Principal stress > Permissible stress

Design safe :-

Principal stress ≤ Permissible Stress

Major Principal stress, $\sigma_1 \leq \sigma_{per}$

$$\sigma_1 = \frac{S_{yt}}{FOS} \quad \text{(OR)} \quad \sigma_1 = \frac{S_{ut}}{FOS}$$

↓ Ductile Material

↓ Brittle Material

Note

1. Effect of shear stress is not considered.
2. This theory is suitable for brittle material in every state of stress.
3. This theory is not suitable for ductile material because ductile materials are weak in shear.
4. **The graphical representation of the theory of failure is square.**
5. Both, brittle and ductile materials are strongest in compression so yield compressive strength S_{yc} is greater than the yield tensile strength S_{yt} .
6. In most of the cases we consider that yield shear strength is equal to yield tensile strength.

$$S_{yc} = S_{yt}$$

Graphical Representation

S_{yc} = Yield compressive strength

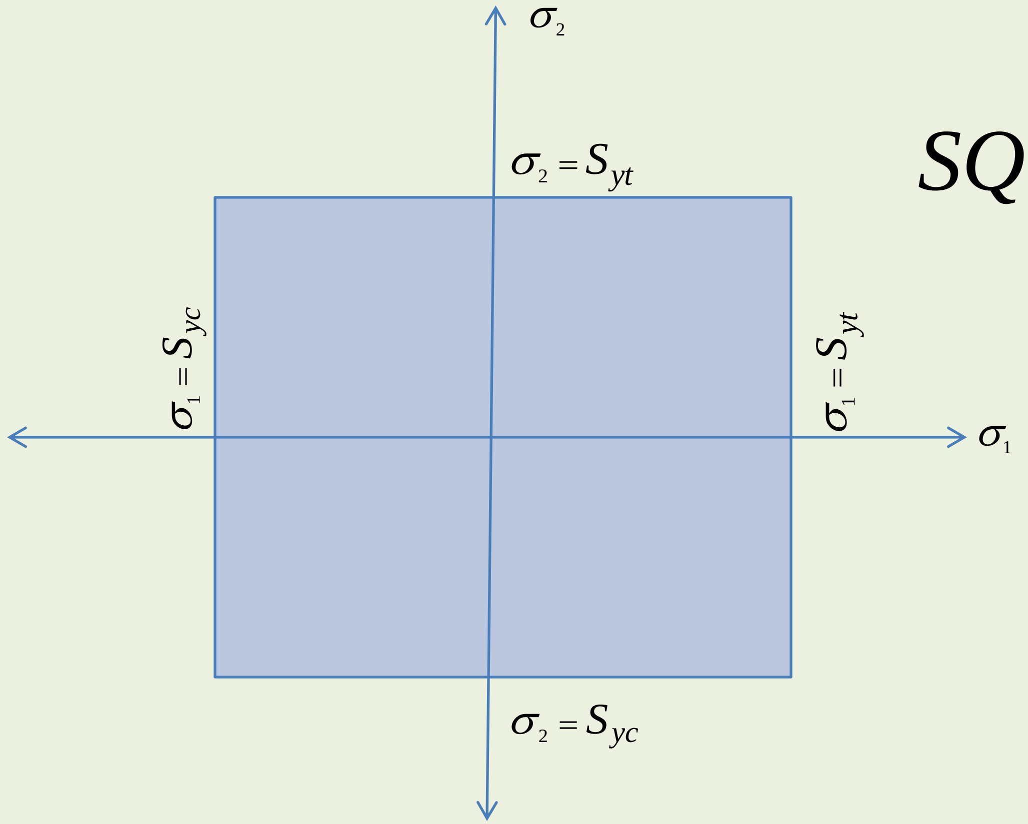
S_{yt} = Yield tensile strength

$$S_{yc} = S_{yt}$$

Assumptions:

- Graphical representation is used for biaxial state of stress.

Graphical Representation of MPST



SQUARE

Maximum Shear Stress Theory [MSST] [Tresca's & Guest's Theory]

<u>Design unsafe</u>

$$\text{Absolute}\, \tau_{\max} > \tau_{permissible}$$

<u>Design safe</u>

$$\text{Absolute}\, \tau_{\max} \leq \tau_{permissible}$$

$$\tau_{\max} \leq \tau_{permissible}$$

$$\tau_{permissible} \leq \left(\frac{S_{ys}}{FOS} \right)$$

S_{ys} = Yield shear strength

Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{S_{yt}}{2}$$

$$abs.\tau_{\max} \le \frac{S_{yt}}{2.FOS}$$

$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right|$$

$$Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right| \le \left(\frac{S_{yt}}{2.FOS}\right)$$ {for Tri-axial state od stress}

Note - [For biaxial state of stress, $\sigma_3 = 0$]

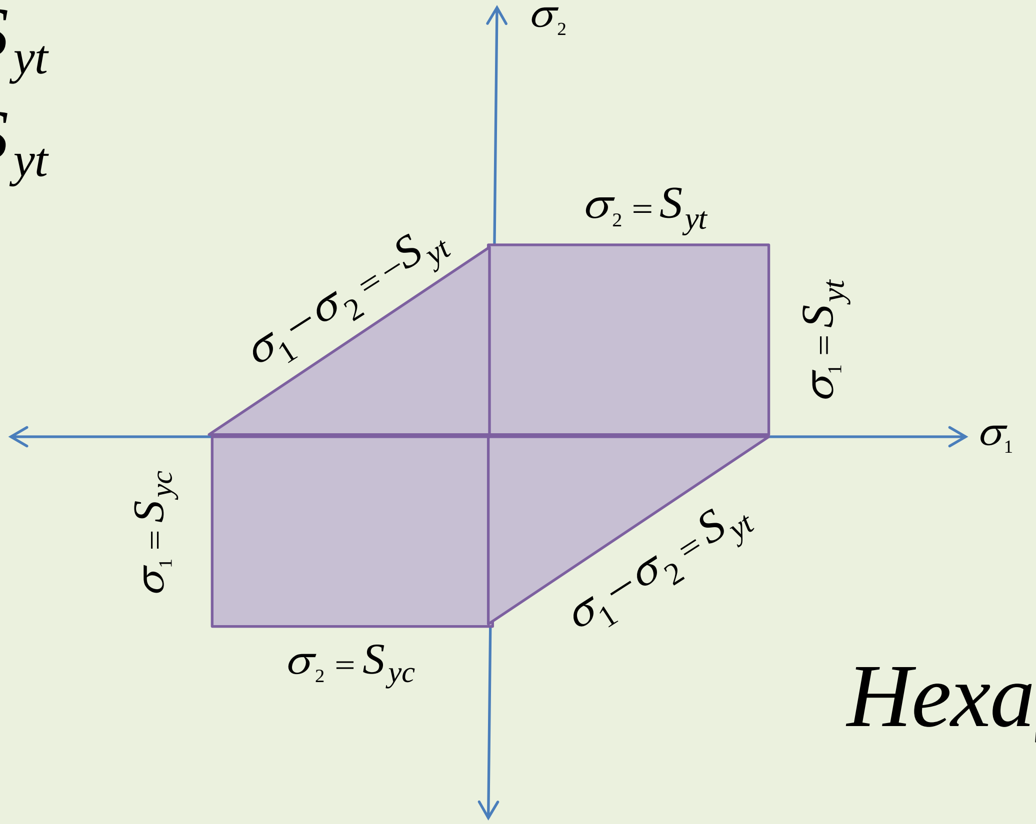
$$abs.\tau_{\max} = Max\left|\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2}{2}, \frac{\sigma_1}{2}\right|$$

Important Points

1. This theory of failure is suitable for ductile material.
2. If the principal stresses are like in nature, MPST & MSST will give the same result.
3. Under uniaxial state od stress both MPST & MSST will give the same result.
4. Under the hydrostatic state od stress condition for the ductile material we will use MPST.
5. **The graphical representation of MSST is Hexagon.**
6. **According to MSST,** $S_{ys} = \frac{S_{yt}}{2}$

Graphical Representation of MSST

$$\sigma_1 - \sigma_2 = \pm S_{yt}$$
$$\sigma_1 = \pm S_{yt}$$
$$\sigma_2 = \pm S_{yt}$$



Hexagon

Maximum Principal strain Theory [MPSt.T]

<u>Design Fail</u> $\varepsilon_1 > \left[(\varepsilon)_{YP} \right]_{TT}$

YP - Yield point & TT - Tension Test

<u>Design Safe</u>

$$\varepsilon_1 \le \left[(\varepsilon)_{YP} \right]_{TT}$$

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$For, \left[(\varepsilon)_{YP} \right]_{TT}$$

$$put, \sigma_1 = \frac{S_{yt}}{FOS}$$

$$\sigma_2 = \sigma_3 = 0$$

$$\left[(\varepsilon)_{YP} \right]_{TT} = \frac{1}{E}\left(\frac{S_{yt}}{FOS} \right)$$

$$\frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] = \frac{1}{E}\left(\frac{S_{yt}}{FOS}\right)$$

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \left(\frac{S_{yt}}{FOS}\right)$$

For Biaxial State of Stress

$$\sigma_1 - \mu\sigma_2 = \left(\frac{S_{yt}}{FOS}\right)$$

Note - The Graphical representation of MPSt.T is Rhombus.

Total Strain Energy Theory [TSET]

Design Fail,

$$\frac{TSE}{Vol^m} > \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{TSE}{Vol^m} \le \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{TSE}{Vol^m} = \frac{1}{2E}\left[\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]$$

$$Now, \left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_1 = \frac{S_{yt}}{FOS}$$

$$\left[\left(\frac{TSE}{Vol^m}\right)_{YP}\right]_{TT} = \frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

For Triaxial state of stress,

$$\frac{1}{2E}\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\frac{1}{2E}\left(\frac{S_{yt}}{FOS}\right)^2$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}+\sigma_3^{\ 2}-2\mu\left(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

For biaxial state of stress,

$$\sigma_3=0$$

$$\left[\sigma_1^{\ 2}+\sigma_2^{\ 2}-2\mu\left(\sigma_1\sigma_2\right)\right]=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation TSET is Ellipse.**

Semi major axis = $\frac{S_{yt}}{\sqrt{1-\mu}}$ Semi minor axis = $\frac{S_{yt}}{\sqrt{1+\mu}}$

Maximum Distortion Energy Theory [MDET]

Design unsafe,

$$\frac{DE}{Vol^m} > \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

Design Safe,

$$\frac{DE}{Vol^m} \leq \left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT}$$

$$\frac{DE}{Vol^m} = \left(\frac{1+\mu}{6E}\right)\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]$$

$$\left[\left(\frac{DE}{Vol^m}\right)_{YP}\right]_{TT} = \left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left(\frac{1+\mu}{6E}\right)\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]=\left(\frac{1+\mu}{6E}\right)\left[2\left(\frac{S_{yt}}{FOS}\right)^2\right]$$

$$\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]=2\left(\frac{S_{yt}}{FOS}\right)^2$$

For Biaxial state of stress ,

$$\sigma_1{}^2+\sigma_2{}^2-\sigma_1\sigma_2=\left(\frac{S_{yt}}{FOS}\right)^2$$

This is a equation of ellipse. So the **Graphical representation MDET is Ellipse.**

Semi Major Axis - $\sqrt{2}S_{yt}$

Semi Minor Axis - $\sqrt{\frac{2}{3}}S_{yt}$

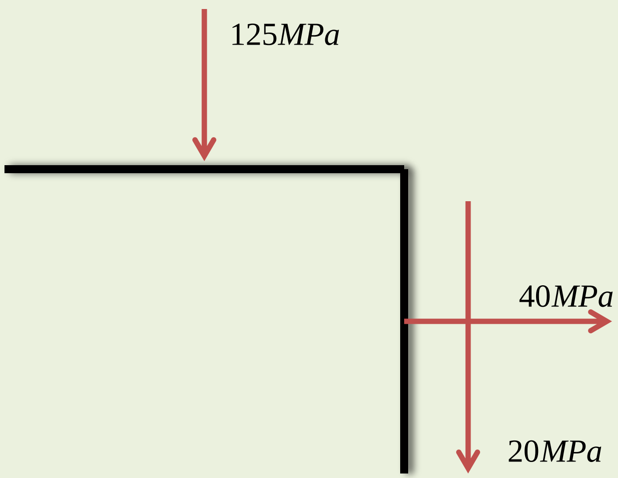
Important Points

- By using MDET, Relationship between S_{ys} & S_{yt}

$$S_{ys} = \frac{1}{\sqrt{3}} S_{yt} = 0.577 S_{yt}$$

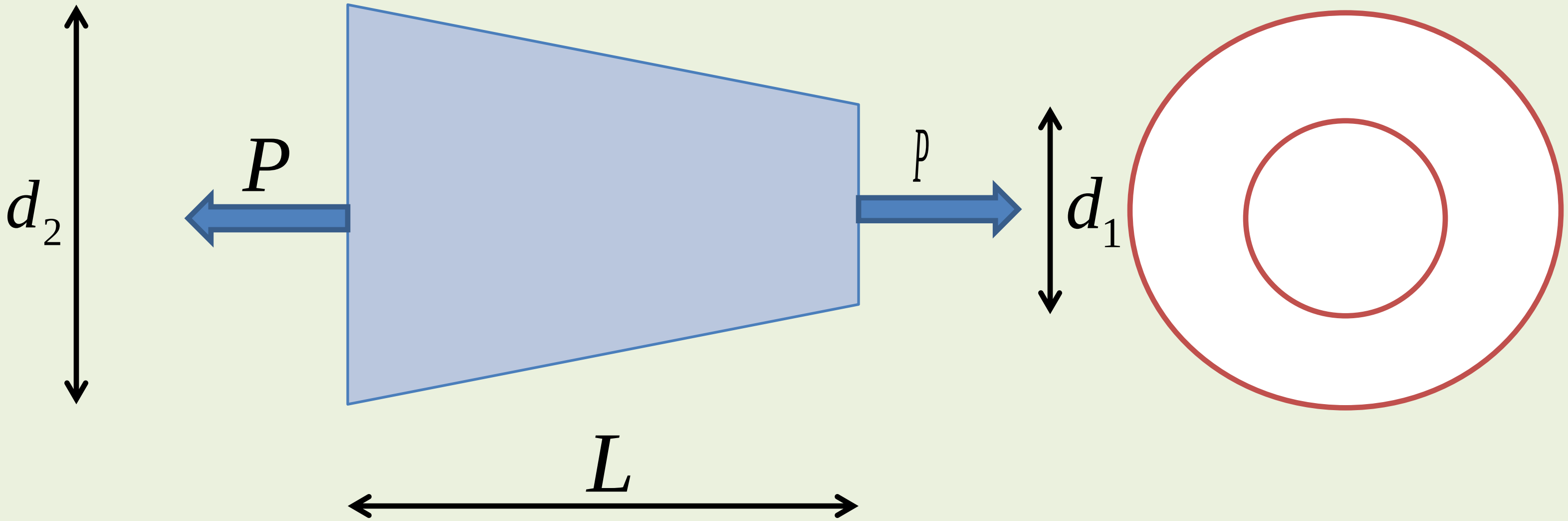
- MSST is most safe but uneconomical theory of failure.
- MDET is most economical and safe theory of failure.
- **Under the uniaxial state of stress, All the five theory of failure will give the same results.**

Q for the given state of stress condition of a point find the factor of safety by using MSST & MDET. [Syt = 400 MPa] Given

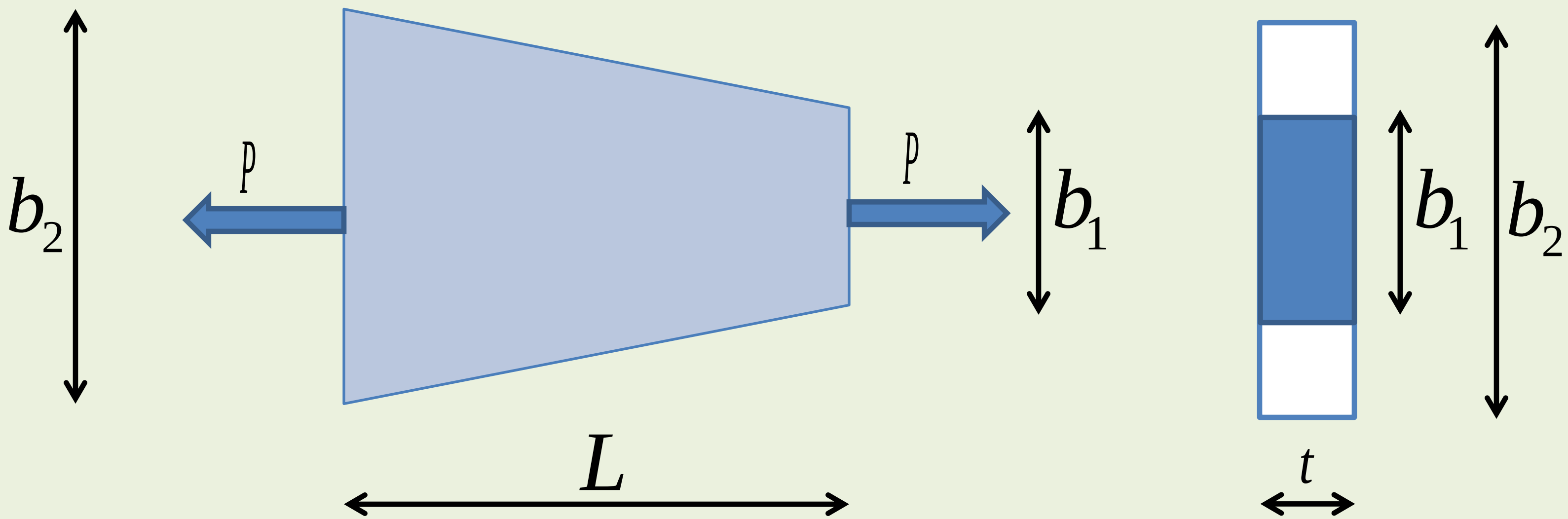


Q A rectangular bar is subjected to tensile force of 100 N and shear force 50N. The x-s/c area of the bar is t x t mm^2. Determine the dimension of x-s/c by using MPST & MSST. [Syt = 150 MPa] Given

Elongation in Taper Conical Bar due to Pure Axial Load


P
P
d_2
d_1
L

ELONGATION IN TRAPEZOID BAR DUE TO PURE AXIAL LOAD



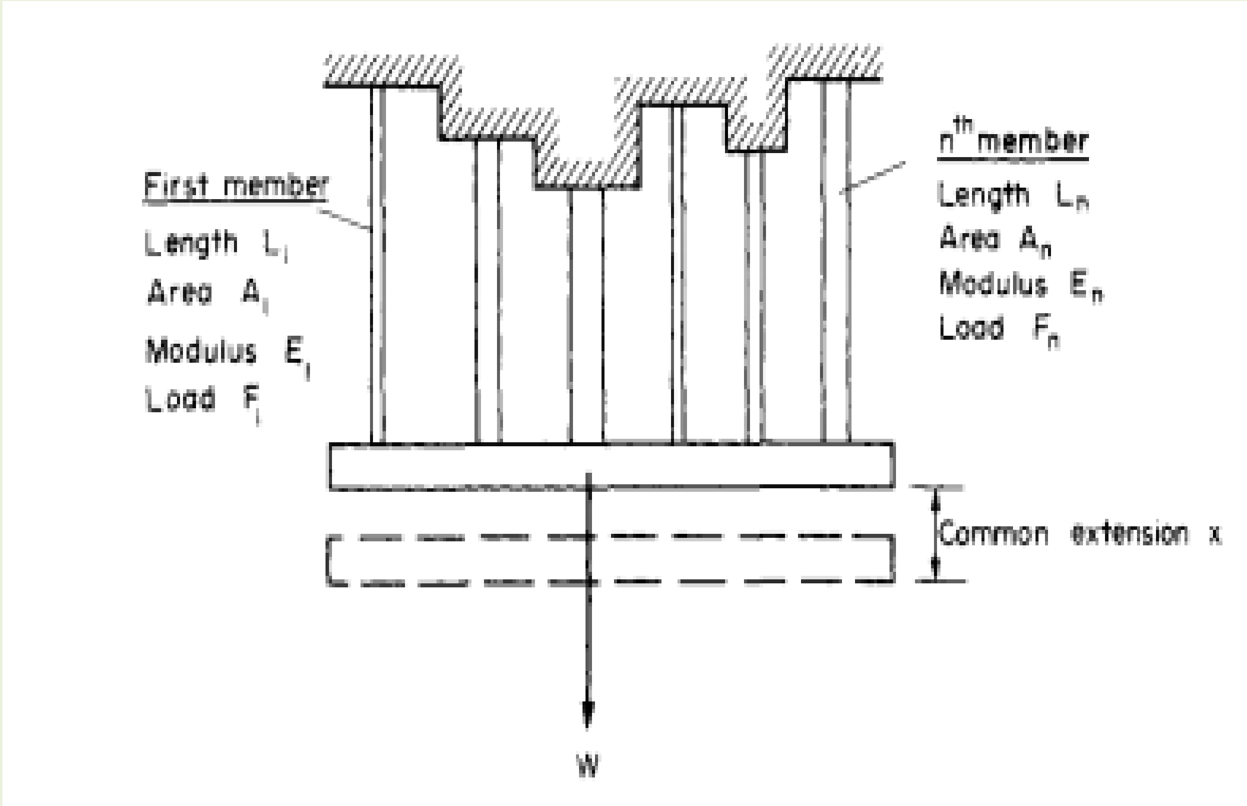
b_2
P
P
b_1
L
b_1
b_2
t

COMPOUND BAR SUBJECTED TO EXTERNAL LOAD

Compound Bar

1. In certain applications it is necessary to use a combination of elements or bars made from different materials, each material performs a different functions.
2. These combinations of materials are termed as compound bar.

Assumptions: Due to external loading in compound bar there is no bending.



First member
Length L_1
Area A_1
Modulus E_1
Load F_1
n^{th} member
Length L_n
Area A_n
Modulus E_n
Load F_n
Common extension x
W

EQUIVALENT OR COMBINED MODULUS

Equivalent or Combined Modulus

1. In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an equivalent or combined modulus E_c .

Assumptions:

1. Extension and the original lengths of the individual members of the compound bar are the same.
2. So the strain in all members will be equal.
3. Area of equivalent bar is the summation of all the area of individual members.

THERMAL STRESSES

Thermal stress

Thermal stress always normal tensile or normal compressive stress.

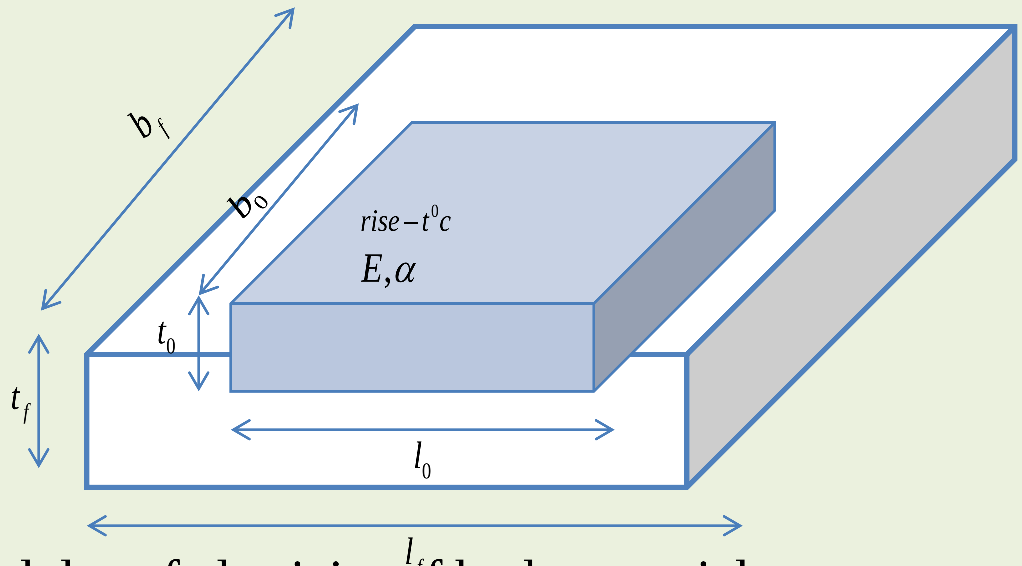
Conditions

1. There should be always temperature variations.

2. Thermal deformation due to above temp. variation should be restricted either completely or partially.

Note - If condition 2 is not satisfied, thermal stress developed in the component is equal to zero. [Free expansion]

Thermal Expansion



- temp rise

E - Young's Modulus of elasticity of body material

t^0c α - Thermal linear expansion co-efficient for body material

- Initial Dimensions of body
- Final Dimensions of body

l_0, t_0, b_0

l_f, t_f, b_f

Thermal Strain (ε_{th})

$$\varepsilon_{th} = \alpha.T$$

In the above case thermal strain developed in all the 3 directions, which are given as -

$$\left|\varepsilon_{th}\right|_x = \left|\varepsilon_{th}\right|_y = \left|\varepsilon_{th}\right|_z = \alpha.T$$

Note - This is a case of free expansion so the thermal stress produced in the body is zero.

Case-1
Completely restricted expansion in one direction

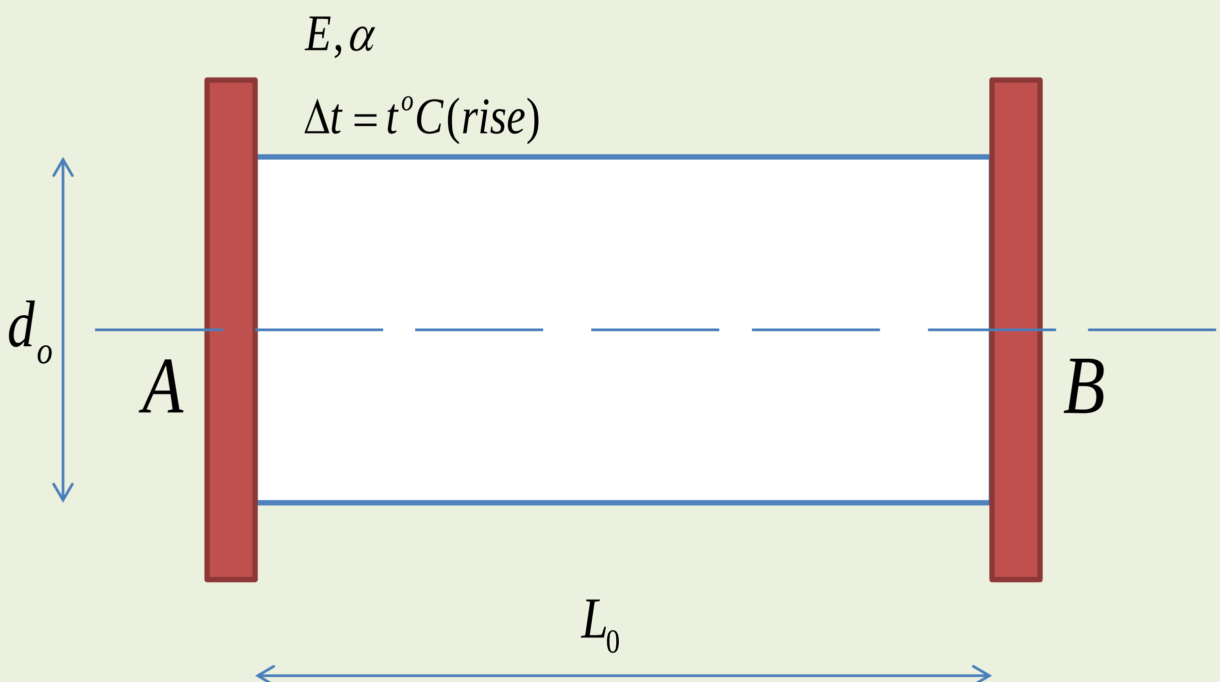


Fig. - A prismatic bar slightly held between two supports

$$\delta L = 0$$

$$(\delta_{th}) + (\delta_{compressive}) = 0$$

$$\alpha T L_0 + \left(\frac{-R L_0}{AE} \right) = 0$$

$$R = \alpha TEA(compressive)$$

$$(\sigma_{th}) = \frac{R}{A} = \alpha TE\,(compressive)$$

$$(\sigma_{th}) = \pm \alpha TE$$

When temp rise occurs thermal stress is compressive in nature.	When temp drop occurs thermal stress is tensile in nature.

Note

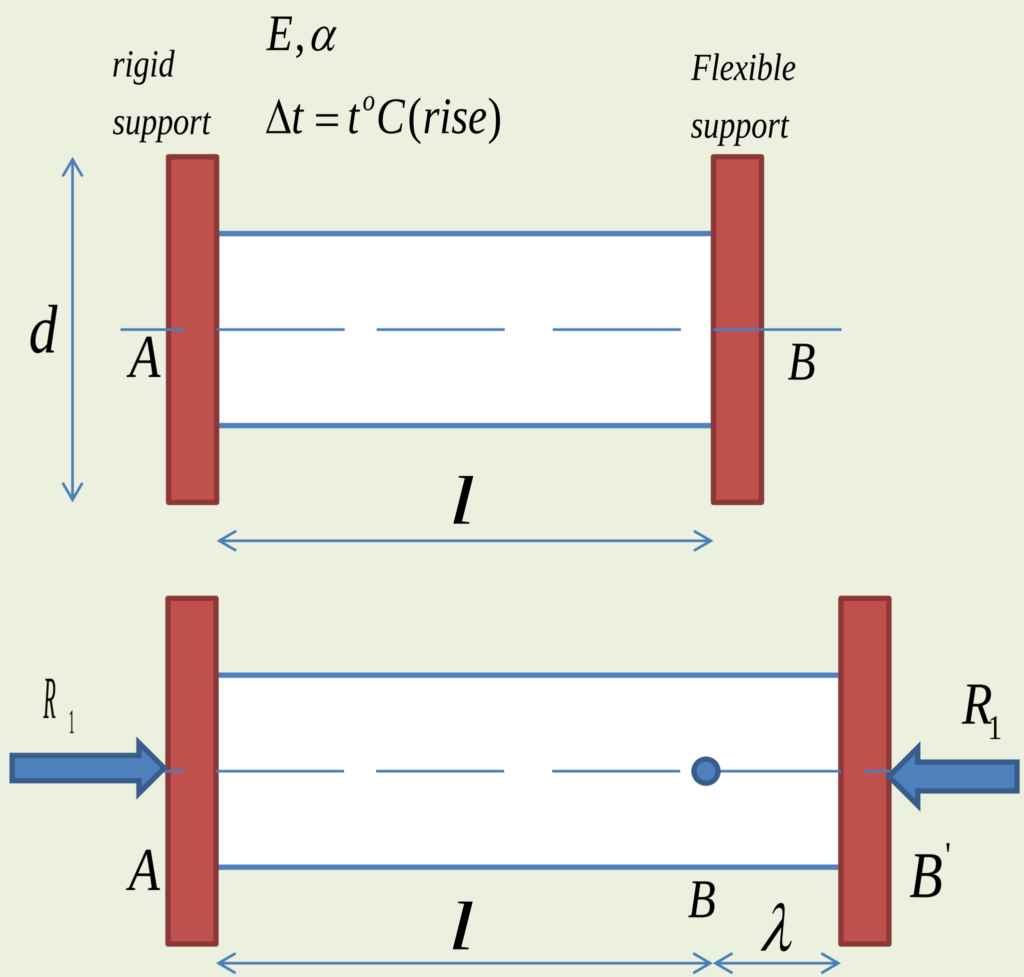
$$\left(\sigma_{th}\right) = \pm\alpha TE$$

$$or$$

$$\left(\sigma_{th}\right)\alpha f\left(E,\alpha,T\right)$$

- Thermal stress is independent from the member length and x-s/c dimensions.

Case-2
Partially restricted expansion in one direction



R_1 = Reaction offered by flexible support like spring.

λ = Expansion permitted by flexible support/ Gap between rails/ yielding of supports/ deflection of spring.

$\delta_{th} - \lambda$ = restricted expansion by flexible support.

$$\delta L = \lambda$$

$$(\delta_{th}) + (\delta_{compressive}) = \lambda$$

$$\alpha TL + \left(\frac{-R_1 L}{AE} \right) = \lambda$$

$$\alpha TL - \frac{\sigma_{th} L}{E} = \lambda$$

$$\sigma_{th} = \left(\frac{\alpha TL - \lambda}{L} \right).E = \left(\frac{\delta_{th} - \lambda}{L} \right).E$$

Note

$$\sigma_{th} = \left(\frac{\alpha TL - \lambda}{L} \right).E = \left(\frac{\delta_{th} - \lambda}{L} \right).E$$

- For free expansion,

$$\delta_{th} = \lambda, \sigma_{th} = 0$$

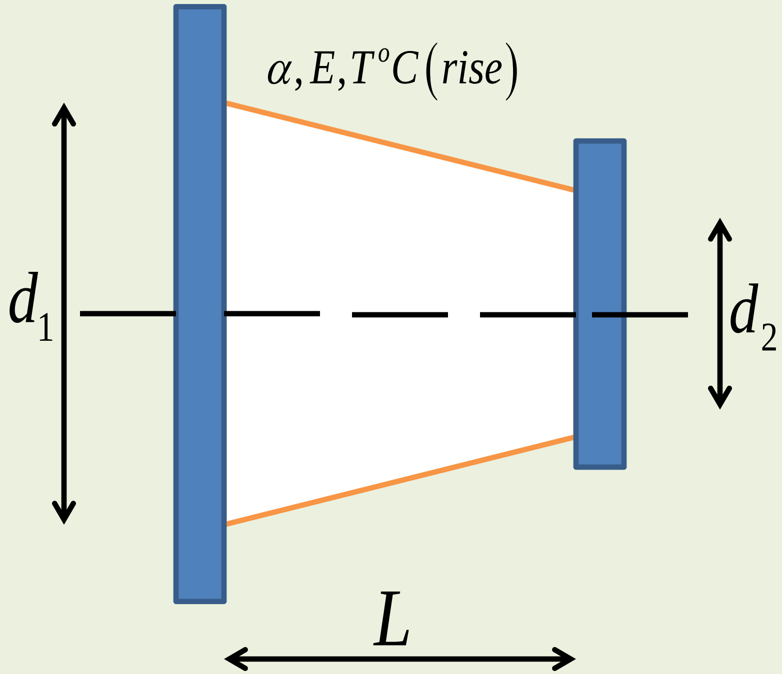
- Completely restricted expansion/compression

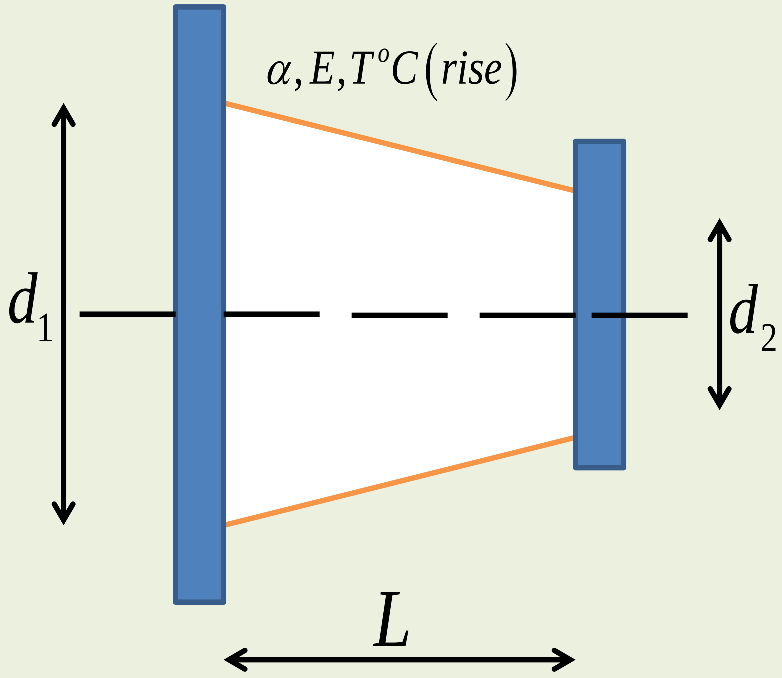
$$\lambda = 0, \sigma_{th} = \alpha TL$$

Q. For the tapered bas as shown in the figure. Determine- (i) Maximum thermal stress developed on the x-s/c of the tapered bar.

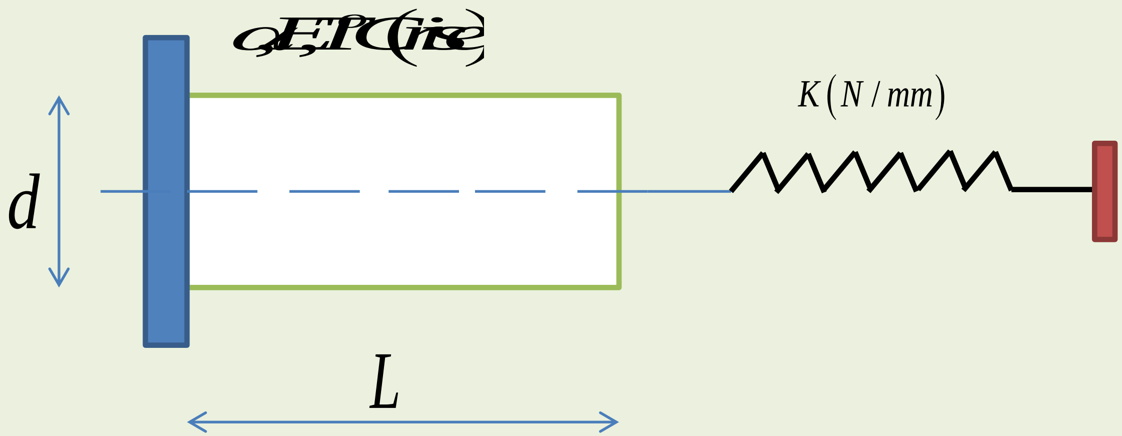
(ii) Ratio of maximum and minimum thermal stress developed on x-s/c of the bar.

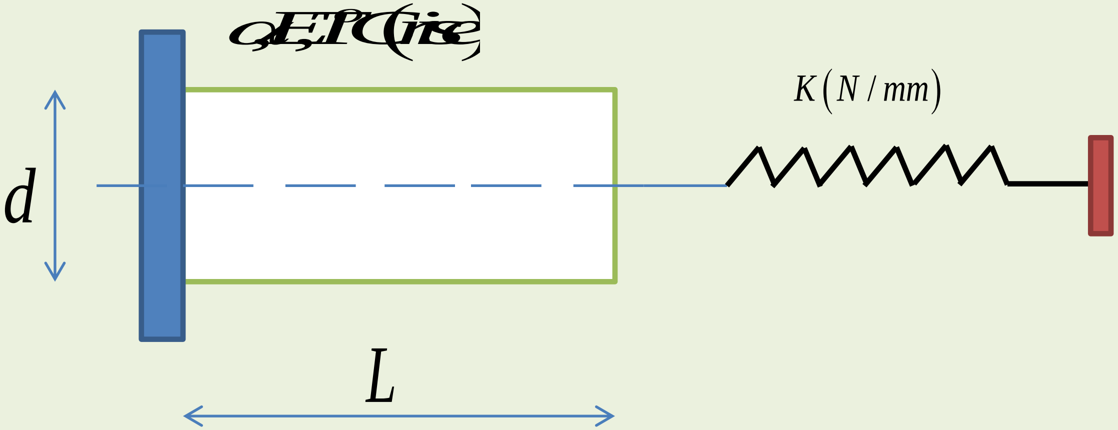
(iii) Reactions offered by the supports.



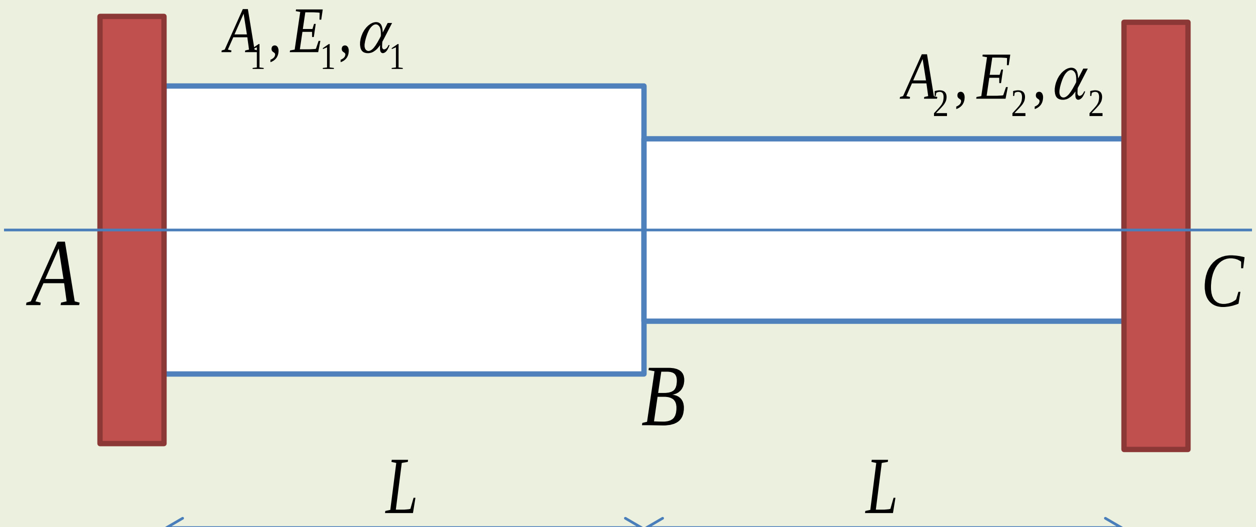

$\alpha, E, T^{o}C\,(rise)$
d_1
d_2
L

Q For the bar as shown in the fig. derive the expression for thermal stress.




K (N / mm)
d
L

Thermal stress in compound Bars (Bars in series)



$$\left(\delta_{Total}\right)_{C.B} = 0$$

Note

Sum of thermal deformation = Sum of axial deformation

$$\alpha_1 L_1 T + \alpha_2 L_2 T = \left(\frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} \right)$$

Reactions offered by the supports are equal.

$$R_A = R_c$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$\frac{\sigma_1}{\sigma_2} = \frac{A_2}{A_1}$$

Q. For the compound bar as shown in the figure determine the following-

(i) Thermal stresses developed in both the bar due to temp. rise of T°C.

(ii) Reaction offered by the support by assuming A = 100 mm^2.

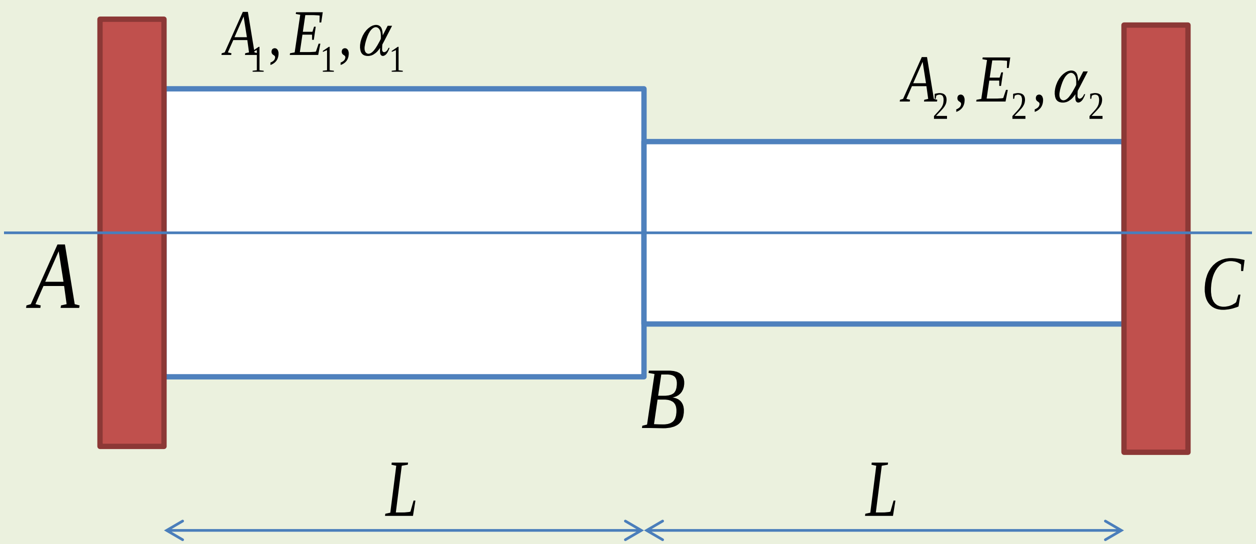
(iii) Deformation at B by assuming L=500 mm

Assume :-

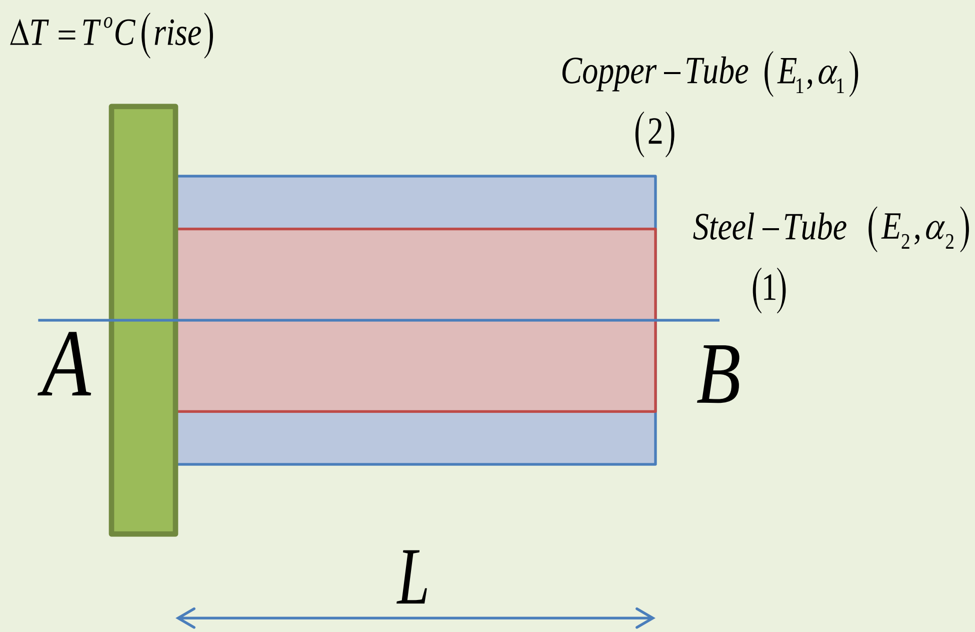
$$E = 200 GPa$$

$$\alpha = 12 \times 10^{-6} /^{o} C$$

$$T^{o}C = 100^{o} C \left(rise \right)$$



Thermal stress in composite Bars (Bars in parallel)



$$\delta_1 = \delta_2$$

Note

Difference in thermal deformation = Sum of axial deformation

$$\alpha_1 LT - \alpha_2 LT = \left(\frac{\sigma_1 L}{E_1} + \frac{\sigma_2 L}{E_2} \right)$$

Reactions offered by the supports are equal.

$$R_A = R_c$$

$$\sigma_1 A_1 = \sigma_2 A_2$$

$$\frac{\sigma_1}{\sigma_2} = \frac{A_2}{A_1}$$