UNIT-4

FOURIER SERIES.

Some Important Definitions:

(*) **Periodic function** - A function $f: \mathbb{R} \to \mathbb{R}$ is said to be periodic if there exists a positive number T such that

$$f(x+T) = f(x) \quad \text{for all real number } x$$

i.e. $f(x) = f(x+T) = f(x+2T) \cdots$

then T is called period of $f(x)$, T is Smallest positive number.

Exp: Period of $\sin x$.

Since, $\sin(x) = \sin(x+2\pi) = \sin(x+4\pi) = \cdots$

$T = 2\pi$ (Smallest)

so, Period of $\sin x$ is 2π

Similarly, Period of $\cos x$, $\sec x$, $\operatorname{cosec} x$ is 2π

$\tan x$ and $\cot x$ is π.

(*) **Even and odd function:** $f: \mathbb{R} \to \mathbb{R}$.

A function $f(x)$ is known as an even function if $f(-x) = f(x)$

or. $f: [-a, a] \to \mathbb{R}$.

domain must be Symmetric, if $a \in$ Domain then $-a \in$ domain (must).

The graph of an even function, is symmetrical about y-axis.

Exp= $f(x) = x^2$, $\cos x$, $|x|$. etc.

A function $f(x)$ is known as an <u>odd function</u> if $f(-x) = -f(x)$.

The graph of an odd function is symmetrical about the origin.

<u>Exp</u> - $\sin x$, x etc.

<u>Fourier Series of a function</u>

Let $f(x)$ be periodic function with period 2π.

So, $f(x)$ can be represented as a trignometric series in interval $c < x < c+2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \cos(nx)$$

and,
$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cdot \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cdot \sin(nx)\, dx$$

a_0, a_n, b_n is Euler's Constant ~~Constant~~. (Formula) (Fourier Coefficients)

$n = 1, 2, 3, 4, \ldots$ i.e. $n \in \mathbb{N}$.

This is Euler's Formula

Dirichlet's Conditions for fourier Series :-

A function f(x) of period 2π defined on $(c, c+2\pi)$ can be expanded in Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Provided -

1) f(x) is Singled Valued and bounded in $(c, c+2\pi)$

2) f(x) has atmost a finite number of maxima and minima in $(c, c+2\pi)$

3) f(x) is piece wise Continuous with finite number of discontinuties in $(c, c+2\pi)$

These Condition on f(x) are sufficient but not necessary.

Convergence of Fourier Series :

If $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin(nx))$$

Converges to -

1) $\Rightarrow f(x)$

If f(x) is Continuous on $(c, c+2\pi)$.

(2) $$\frac{1}{2}\left[f(a^+) + f(a^-)\right]$$

If $x = a \in (c, c+2\pi)$ is point of discontinuty.

(3) $$\frac{1}{2}\left[f(c^+) + f(c+2\pi)^-\right]$$

Point of discontinuity at end point of $(c, c+2\pi)$

i.e. at $x = c$ a $x = c+2\pi$.

here,

$$f(a^+) = \lim_{h\to 0} f(a+h) \quad \text{Right hand Limit}$$

$$f(a^{-}) = \lim_{h\to 0} f(a-h) \quad \text{Left hand limit}$$

Some Special Cases:

$c=0$ then,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin(nx)\,dx$$

If,

$C = -\pi$

then,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 1, 2, 3, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Note.

(1) (odd function) $\times$ (even function) = (odd function)

(odd function) $\times$ (odd function) = (even function)

(even function) $\times$ (even function) = (even function).

(2)
$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & f(x) \text{ is odd function} \\ 2\int_{0}^{a} f(x)\,dx & f(x) \text{ is even function.} \end{cases}$$

*
$$\boxed{\int u.v\,dx = u.v_1 - u'v_2 + u''.v_3 - u'''.v_4 + \cdots}$$

where, $u' = \frac{du}{dx}$ and $v_1 = \int v\,dx$

$u'' = \frac{d^2u}{dx^2}$ $\qquad v_2 = \int v_1\,dx$

and so on.

Exp:- $f(x) = \frac{\pi - x}{2}$; Find Fourier Series in $0 < x < 2\pi$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sol^n.

$$f(2\pi - x) = \frac{\pi}{2} - \frac{(2\pi - x)}{2} = \frac{\pi}{2} - \pi + \frac{x}{2} = -\frac{\pi}{2} + \frac{x}{2}$$

or $$f(2\pi - x) = -\frac{(\pi - x)}{2} = -f(x)$$

So, fourier Series -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Now, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

Using Property -

$$\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

and, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(2\pi - x)\,dx = \frac{1}{\pi}\int_0^{2\pi} -f(x)\,dx.$$

on adding

$$2a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx - \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 0$$

$$\Rightarrow \boxed{a_0 = 0.}$$

Since, $f(2\pi - x) = -f(x)$

$\cos(2\pi - x) = \cos x.$

So, $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \quad —(1)$$

also,
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(2\pi - x)\cdot \cos n(2\pi - x)\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} -f(x)\cdot \cos(nx)\, dx \quad \text{---(2)}$$

so, on adding (1)+(2)

$$2a_n = 0 \quad \Rightarrow \boxed{a_n = 0}$$

Now,
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \sin(nx)\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)}{2} \sin(nx)\, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x) \sin(nx)\, dx$$

$$b_n = \frac{1}{2\pi}\left[(\pi - x)\frac{(-\cos nx)}{n} - (-1)\cdot\left(\frac{-\sin(nx)}{n^2}\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[\frac{-(\pi - x)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_0^{2\pi}$$

$\cos 2\pi = 1$

$$b_n = \frac{1}{2\pi}\left[\frac{-(\pi - 2\pi)}{n} + \frac{(\pi)}{n}\right] = \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$\boxed{b_n = \frac{1}{n}}$$

Thus, Fourier series -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

or $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ as, $a_0 = 0$

$a_n = 0$

$b_n = \frac{1}{n}$

or $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$

or, $\boxed{\frac{\pi - x}{2} = \sin(x) + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \frac{\sin(4x)}{4} + \cdots}$

Required fourier series.

at $x = \frac{\pi}{2}$.

$$\frac{\pi - \pi/2}{2} = \sin(\pi/2) + \frac{(\sin\pi)}{2} + \frac{\sin(3\pi/2)}{3} + \frac{\sin(4\pi/2)}{4} + \cdots$$

or $\frac{\pi}{4} = 1 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

$\sin(n\pi) = 0$.

$\sin\left(\frac{\pi}{2}\right) = 1$

$\sin\left(\frac{3\pi}{2}\right) = -1$.

or $\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}}$

(2) Fourier series of 2π-periodic function

$f(x) = x + x^2 \quad -\pi < x < \pi$ at $x = \pi$

Converges to ?

Sol^n.

$$f(x) = x + x^2 \qquad x \in (-\pi, \pi)$$

at $x = \pi$, function has discontinuity

since, π is not in interval.

So, Fourier series converge to at $x = \pi$

$$= \frac{1}{2}\left[f(-\pi^+) + f(\pi^-)\right]$$

$$= \frac{1}{2}\lim_{h\to 0} f(-\pi+h) + \frac{1}{2}\lim_{h\to 0} f(\pi-h)$$

$$= \frac{1}{2}\left[(-\pi+h) + (-\pi+h)^2\right] + \frac{1}{2}\left[(\pi-h) + (\pi-h)^2\right]$$

$$= \frac{1}{2}\left[-\pi + \pi^2\right] + \frac{1}{2}\left[\pi + \pi^2\right]$$

$$= \frac{1}{2}\left[2\pi^2\right] = \underline{\pi^2}.$$

Functions having arbitrary Periods
(Change of Interval)

Let $f(x)$ be given periodic function in interval

$$c < x < c+2l.$$

i.e. $f(x)$ is periodic function of period $2l$.
in $(c, c+2l)$

where, l is any positive integer.

Now, using substitution-

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or,} \quad x = \frac{zl}{\pi}$$

when $x = c \Rightarrow z = \frac{\pi c}{l} = (d)$ (let)

$x = c+2l$

$$\Rightarrow z = \frac{\pi(c+2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi$$

hence the function $f(x)$ of period $2l$ in $(c, c+2\pi)$

is transformed into

$f\left(\frac{lz}{\pi}\right) = F(z)$ of Period 2π in
$(d, d+2\pi)$.

Then, $F(z)$ can be

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nz) + \sum_{n=1}^{\infty} b_n \sin(nz)$$

where, $$a_0 = \frac{1}{\pi}\int_{d}^{d+2\pi} F(z)\,dz$$

$$a_n = \frac{1}{\pi}\int_{d}^{d+2\pi} F(z)\cos(nz)\,dz$$

$$n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{d}^{d+2\pi} F(z)\cdot\sin(nz)\,dz$$

Now, Putting. $z = \frac{\pi x}{l} \Rightarrow dz = \frac{\pi}{l}\,dx$.

when, $z = d \Rightarrow x = c$

$z = d+2\pi \Rightarrow x = c+2l$

So, above equation reduce into –

$$F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where,

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx.$$

Exp: Find Fourier Series for f(x) defined by -

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1-x & 1 < x < 2 \end{cases}$$

then deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol:. Here, f(x) is periodic function of Period = 2 and, (0, 2) = (c

then, Fourier's series of f(x) for Period = 2l is on (c, c+2l)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot \sin\left(\frac{n\pi x}{l}\right) dx.$$

here, $(c, c+2l) = (0, 2)$ $\Rightarrow 2l = 2$

$c = 0, \quad l = 1.$

so,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \cdot \sin(n\pi x).$$

$$a_0 = \int_0^2 f(x)\,dx, \quad a_n = \int_0^2 f(x)\cdot \cos(n\pi x)\,dx$$

$$b_n = \int_0^2 f(x)\cdot \sin(n\pi x)\,dx.$$

Now,

$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

or $$a_0 = \int_0^1 x\,dx + \int_1^2 (1-x)\,dx$$

or $$a_0 = \left[\frac{x^2}{2}\right]_0^1 + \left[x - \frac{x^2}{2}\right]_1^2$$

or $$a_0 = \frac{1}{2} + \left[2 - \frac{4}{2} - 1 + \frac{1}{2}\right] = \frac{1}{2} + 2 - 2 - 1 + \frac{1}{2} = 0$$

Now,

Now,

$$a_n = \int_0^2 f(x)\cdot\cos(n\pi x)\,dx$$

or $$a_n = \int_0^1 f(x)\cdot\cos(n\pi x)\,dx + \int_1^2 f(x)\cdot\cos(n\pi x)\,dx$$

or $$a_n = \int_0^1 x\cdot\cos(n\pi x)\,dx + \int_1^2 (1-x)\cos(n\pi x)\,dx$$

or $$a_n = \left[x\left(\frac{\sin n\pi x}{n\pi}\right) + \left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$+ \left[(1-x)\left(\frac{\sin(n\pi x)}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$\text{or, } a_n = \left[1 \cdot \frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2} - 0 - \frac{\cos(0)}{n^2\pi^2}\right]$$

$$+ \left[(1-x)\frac{\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{n^2\pi^2}\right]_1^2$$

$$\text{or } a_n = \left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] + \left[0 - \frac{\cos(2n\pi)}{n^2\pi^2} + 0 + \frac{\cos(n\pi)}{n^2\pi^2}\right]$$

$$\text{or } a_n = \left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] + \left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$\cos(n\pi) = (-1)^n$$

$$\text{or } a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 & n \text{ is Even} \\ \frac{-4}{n^2\pi^2} & n \text{ is odd.} \end{cases}$$

Now,

$$b_n = \int_0^2 f(x) \cdot \sin(n\pi x)\, dx$$

$$\text{or } b_n = \int_0^1 f(x) \cdot \sin(n\pi x)\, dx + \int_1^2 f(x) \sin(n\pi x)\, dx$$

$$\text{or } b_n = \int_0^1 x \cdot \sin(n\pi x)\, dx + \int_1^2 (1-x) \cdot \sin(n\pi x)\, dx.$$

$$bn = \left[(x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin(n\pi x)}{n^2\pi^2}\right)\right]_0^1$$
$$+\left[(1-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$\sin(n\pi) = 0$

or $$bn = \left[-\frac{\cos(n\pi)}{n\pi} + 0 + 0 - 0\right] + \left[\frac{\cos(2n\pi)}{n\pi} - 0 - 0 - 0\right]$$

$$\therefore\ bn = -\frac{(-1)^n}{n\pi} + \frac{1}{n\pi} = \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore\ bn = \begin{cases} 0 & n \text{ is even} \\ \frac{2}{n\pi} & n \text{ is odd.} \end{cases}$$

So,
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

So $$f(x) = 0 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{(2n-1)}.$$

as, n replace by (2n-1)

only for odd. , as for even, $a_n = 0$ & $b_n = 0$.

$$\boxed{f(x) = -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{(2n-1)}}$$

For deduction.

Put $x=1$

Since, $x=1$ is point of discontinuity so,

sum of series at $x=1$

$$= \frac{1}{2}\left[f(1^-) + f(1^+)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} f(1-h) + \lim_{h\to 0} f(1+h)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} (1-h) + \lim_{h\to 0} [1-(1+h)\right]$$

$$= \frac{1}{2}[1+0] = \frac{1}{2}$$

So,

$$\frac{1}{2} = -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\cdot\pi}{(2n-1)^2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi}{(2n-1)}$$

or $\displaystyle \frac{1}{2} = \frac{-4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{(2n-1)^2} + 0.$ $\sin(2n-1)\pi = 0$

$\because \cos(n\pi) = (-1)^n$

or $\displaystyle \frac{1}{2} = \frac{4}{\pi^2}\cdot\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ $(-1)^{2n-1} = -1.$

or $\displaystyle \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

So, $$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}$$

HALF RANGE SERIES :-

Half Range sine series :-

Let $f(x)$ be defined on $(0, l)$.

and in $(-l, l)$

$f(-x) = -f(x)$ i.e. $f(x)$ is an odd function.

So, Fourier series on $(-l, l)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx.$$

$$a_0 = \frac{1}{\pi}\int_0^l f(x)\,dx = 0 \qquad \text{as } f(x) \text{ is odd.}$$

$$a_n = \frac{1}{\pi}\int_0^l \cos\left(\frac{n\pi x}{l}\right).f(x)\,dx = 0. \qquad f(x)\cos\left(\frac{n\pi x}{l}\right) \text{ is odd.}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right).f(x)\,dx$$

So, the series is - for the interval $(-l, l)$.

$$f(x) = \sum_{n=1}^{\infty} b_n . \cos \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).dx$$

So, Half range sine series of $f(x)$ on $(0, l)$.

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n . \sin\left(\frac{n\pi x}{l}\right)}$$

Fourier series containe only sine term.

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx}$$

Particular Case :-

If $f(x)$ is defined in $(0, \pi)$ i.e. $l = \pi$ then required Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx) \qquad \text{where} \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin(nx)\,dx .$$

Half - Range Cosine Series :-

if $f(x) = f(-x)$ i.e. $f(x)$ is even function in $(-l, l)$

$b_n = 0$ as.

so, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right)$$

where, $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot \cos\left(\frac{n x \pi}{l}\right)dx .$$

Thus, Half Range Cosine Series of $f(x)$ in $(0, l)$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Contain only cosine terms.

where, $$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx ,$$

$$a_n = \frac{2}{2}\int_0^{l} f(x)\cdot \cos\left(\frac{n\pi x}{l}\right)dx .$$

Particular Case :

$l = \pi$.

If $f(x)$ is defined in $(0, \pi)$ then required half range cosine series is given by -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos(nx)\,dx.$$

Exp : $f(x) = x\sin x$ in $(0, \pi)$

Find half Range Cosine series.

Sol^n.

$$f(-x) = (-x)(\sin(-x)) = +x\sin x = f(x)$$

even function.

So,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

So,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[-\pi\cos(\pi) + \sin(\pi) + 0 - \sin(0)\right]$$

or $$a_0 = \frac{2}{\pi}\cdot\pi = 2.$$

Now,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\,dx$$

or $$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\cos(nx)\,dx$$

or $$a_n = \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cdot\cos nx)\,dx$$

or $$a_n = \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

or $$a_n = \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

or $$a_n = \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - 1\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi}$$
$$- \frac{1}{\pi}\left[x\left(-\frac{\cos(n-1)x}{(n-1)}\right) - \left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

or $$a_n = \frac{1}{\pi}\left[\frac{-\pi\cdot(-1)^{n+1}}{(n+1)}\right] - \frac{1}{\pi}\left[\frac{-\pi(-1)^{n-1}}{n-1}\right] \qquad n \neq 1$$

or, $$a_n = \frac{-1\cdot(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}$$

or, $a_n = \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{n-1} = (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$

a $a_n = (-1)^n\left[\frac{n-1-n-1}{(n+1)(n-1)}\right] = \frac{-2(-1)^n}{(n+1)(n-1)}$ $n \neq 1$

or $a_n = \frac{2(-1)^{n+1}}{(n-1)(n+1)}$ $n \neq 1$

If $n=1$,

then, $a_1 = \frac{2}{\pi}\int_0^\pi x(\sin\cos x)\,dx$

$= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$

$= \frac{1}{\pi}\left[x\frac{(-\cos 2x)}{2} - (1)\frac{(-\sin 2x)}{4}\right]_0^\pi$

$= \frac{1}{\pi}\left[\frac{-\pi}{2} + 0\right] = \frac{-1}{2}$

So, Half Range Cosine Series -

$f(x) = x\sin x = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos(nx)$

or, $x\sin x = 1 - \frac{\cos x}{2} - 2\left[\frac{\cos 2x}{1\times 3} - \frac{\cos 3x}{2\times 4} + \frac{\cos 4x}{3\times 5} - \cdots\right]$

So, half Range cosine series of $x \sin x$ in $(0,\pi)$

$$x\sin x = 1 - \frac{\cos x}{2} - \frac{2\cos 2x}{1\times 3} + \frac{2\cos 3x}{2\times 4} - \frac{2\cos 4x}{3\times 5} + \cdots$$

Now, deduce –

$$\frac{\pi}{2} = 1 + \frac{2}{1\times 3} - \frac{2}{3\times 5} + \frac{2}{5\times 7} - \cdots$$

Put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 1 - \frac{\cos(\pi/2)}{2} - \frac{2\cos(\pi)}{1\times 3} + \frac{2\cos(3\pi/2)}{2\times 4} - \cdots$$

$$\boxed{\frac{\pi}{2} = 1 + \frac{2}{1\times 3} - \frac{2}{3\times 5} + \frac{2}{5\times 7} - \cdots}$$

$$\cos\left(n\cdot\frac{\pi}{2}\right) = \begin{cases} 0 & n \text{ is odd.} \\ (-1)^{n/2} & n \text{ is even.} \end{cases}$$

i.e.

$$\cos(2\pi) = \cos(4\pi) = \cos(6\pi) = 1$$
$$\cos(\pi) = \cos(3\pi) = \cos(5\pi) = -1$$

Some time we don't need to find only even function or odd function of half Range sine series of half Range cosine series.

i.e. we find half Range Sine Series or cosine series of any function.

Exp: Find half Range Sine Series for e^x on $0<x<1$

sol^n:
$$f(x) = e^x \qquad x \in (0,1)$$

So, Required half Range sine Series of this function -

$$f(x) = e^x = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad \text{as, } l=1$$

where,
$$b_n = 2\int_0^1 e^x \sin(n\pi x)\,dx.$$

or,
$$b_n = 2\left[\frac{e^x(\sin(n\pi x) - n\pi\cos n\pi x)}{1+n^2\pi^2}\right]_0^1$$

as -
$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}[a\sin bx - b\cos(bx)]}{a^2+b^2}$$

$$\therefore b_n = 2\left[\frac{e(-n\pi\cos n\pi)}{1+n^2\pi^2} - \frac{(-n\pi\cos(0))}{1+n^2\pi^2}\right]$$

$$\sin(n\pi) = 0. \quad n \in \mathbb{Z}.$$

$$b_n = 2\left[\frac{e(-n\pi(-1)^n)}{1+n^2\pi^2} + \frac{n\pi}{1+n^2\pi^2}\right]$$

$$\Rightarrow \quad b_n = \frac{2}{n^2\pi^2+1}\left[n\pi - en\pi(-1)^n\right]$$

$$\Rightarrow \quad b_n = \frac{2n\pi}{1+n^2\pi^2}\left[1-e(-1)^n\right]$$

So,

$$\boxed{f(x) = e^x = \sum_{n=1}^{\infty} \frac{2n\pi\,(1-e(-1)^n)}{1+n^2\pi^2} \sin(n\pi x)}$$

PARSEVAL's theorem.

(Parseval's identity)

Let $f(x)$ be a periodic function with period $2l$ defined in the interval $(c, c+2l)$

then,

$$\boxed{\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)}$$

where, a_0, a_n, b_n are the fourier coefficient of $f(x)$.

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx \quad , \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\cdot \sin\left(\frac{n\pi x}{l}\right) dx.$$

<u>Exp</u>: Apply Parseval's identity to the function

$$f(x) = x \quad ; \quad -\pi < x < \pi$$

and deduce $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Sol^n: Parseval's identity of given function -

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Since, $f(x) = x \Rightarrow f(-x) = -x = -f(x)$
odd function.

So, $a_0 = 0$ and $a_n = 0$.

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x \sin(nx)\,dx$.

or $b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$

or $b_n = \frac{2}{\pi}\left[\frac{+\pi.(-\cos\pi n)}{n} - 0 + 0\right]$

or $b_n = \frac{-2.(-1)^n}{n} = \frac{(-1)^{n+1}.2}{n}$

Again, $\int_{-\pi}^{\pi}(f(x))^2 dx = \int_{-\pi}^{\pi} x^2 dx = 2\int_{0}^{\pi} x^2 dx = \frac{2\pi^3}{3}$.

Now, using Parseval's identity -

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{2\pi}\cdot\left[\frac{2\pi^3}{3}\right] = 0 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2$$

$$\Rightarrow \frac{\pi^2}{3} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{4}{n^2} \qquad \left[\frac{2(-1)^{n+1}}{n}\right]^2 = \frac{4}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

Exp: Using the Fourier Series for

$$f(x) = |x| \qquad -\pi < x < \pi$$

Find Sum of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ assuming that Fourier series of function f at $x=0$ Converge to $f(0)$

Again using Parseval's identity to calculate -

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

Sol^n:

$$f(x) = |x|$$

$$f(-x) = |-x| = |x| = f(x)$$

so, $f(x)$ is even function on $(-\pi,\pi)$

So, Fourier's series -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Since, f(x) is even function -

then, $$f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$ as, $\underline{b_n = 0}$

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$ f(x) is even.

or, $$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

Now,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

or $$a_n = \frac{2}{\pi}\left[\frac{\pi\cdot\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2} - 0 - \frac{\cos(0)}{n^2}\right]$$

or $$a_n = \frac{2}{\pi}\left[\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

Scanned with CamScanner

so, $a_n = \frac{-2}{n^2\pi}\left[1-(-1)^n\right]$

or, $a_n = \frac{-2}{n^2\pi} \times 2$ when n is even

$a_n = 0$ when n is odd.

So, the Fourier's Series -

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

or, $|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right]$

Now, Given in question -

Fourier series converge to $f(0)$ at $x=0$

so,

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

or, $\frac{\pi}{2} = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$

or $\boxed{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}}$

To find Sum -

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$ Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{4}+\sum_{n=1}^{\infty}\frac{(a_n^2+b_n^2)}{2}$$

as, $a_0=\pi$, $b_n=0$, $a_n=0$ for n is even

$a_n=\frac{-4}{n^2\pi}$ for n is odd.

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx = \frac{\pi^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{(2n-1)^4\pi^2}$$

$$\text{or. } \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{4}+\frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4\pi^2}$$

$$\text{or, } \frac{\pi^2}{3} = \frac{\pi^2}{4}+8\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4\cdot\pi^2}$$

$$\text{or } \frac{\pi^2}{3}-\frac{\pi^2}{4} = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\text{or, } \frac{\pi^2\cdot\pi^2}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\Rightarrow \boxed{\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots = \frac{\pi^4}{96}}$$

PARTIAL DIFFERENTIAL Equations with Constant Coefficient

There are two type- of PDE with constant coefficient.

1) Homogenous linear partial differential equation (PDE)
2) Non-homogenous linear PDE

1)- Homogenous linear partial differential equation.

Let $z = f(x, y)$ i.e. z is depend on two variable x and y.

Partial derivative with respect to 'x'

$$D_x z = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y)$$ treat 'y' as Constant

Similarly, with respect to 'y'

$$D_y z = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y)$$ treat 'x' as Constant

or, we use notation-

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$D_x = D$

$D_y = D'$

Now Consider -
The General form

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} . \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y} = f(x, y)$$

or, $(A_0 D^n + A_1 D^{n-1} . D' + A_2 D^{n-2} D'^2 + \cdots + A_n . D'^n) z = f(x, y)$

This equation is known as linear partial differential equation (homogenous) with constant coefficient.

where, $A_0, A_1, A_2, \cdots, A_n$ are Constant.

also, $\boxed{F(D, D') z = f(x, y)}$ ——

(*) Solution of a homogenous linear partial differential equation with constant coefficient?

Let the General solution of

$$F(D, D') z = f(x, y)$$

$$z(x, y) = C.F + P.I.$$

C.F = Complementry function which are obtained from -

$$F(D, D') z = 0.$$

and, P.I = Particular Integral of linear PDE.

and P.I is obtained by -

$$P.I = \frac{1}{F(D,D')} \cdot f(x,y)$$

Working rule for finding C.F -

Step I - Put the given equation in standard form -

$$(A_0 D^n + A_1 D^{n-1} \cdot D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n) z = f(x,y)$$

Step - 2 Replacing D by m and D' by 1

we obtain auxiliary equation is -

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0 \quad \text{—(2)}$$

Step - (3) Solve (2) For m -

Case (i) Let $m = m_1, m_2, m_3, \cdots, m_n$ are different roots.

then -

$$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

If $m_1 = \frac{a_1}{b_1}$, $m_2 = \frac{a_2}{b_2}$, $\cdots$ $m_n = \frac{a_n}{b_n}$ types.

then.

$$C.F = \phi_1(b_1 y + a_1 x) + \phi_2(b_2 y + a_2 x) + \cdots + \phi_n(b_n y + a_n x).$$

Case (ii) If all the roots are same.

$$m_1 = m_2 = m_3 = \cdots m_n = m =$$

then $$C.F = \phi_1(y+mx) + x\,\phi_2(y+mx) + x^2\phi_3(y+mx) + \cdots + x^{n-1}\phi_n(y+mx).$$

where, $\phi_1, \phi_2, \cdots, \phi_n$ are arbitrary function.

Exp: ① $(2D+D')(D+2D')(D-D')z=0$

Put $D=m$, $D'=1$

$$(2m+1)(m+2)(m-1)=0$$

So, roots are, $m=-\frac{1}{2}$, $m=-2$, $m=1$

then $$C.F = \phi_1\left(y-\frac{x}{2}\right) + \phi_2(y-2x) + \phi_3(y-x)$$

or $$\boxed{C.F = \phi_1(2y-x) + \phi_2(y-2x) + \phi_3(y-x)}$$

② $D^4 z = 0$

$D = 0, 0, 0, 0$. (repeated root.

$m = 0, 0, 0, 0$.

So, $C.F = \phi_1(y-0.x) + x\,\phi_2(y-0.x) + x^2\phi_3(y-0.x) + x^3\phi_4(y-0.x)$

or $$\boxed{C.F = \phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + x^3\phi_4(y).}$$

(3) $D'^4 - D^2D'^2 = 0$.

here we assume $D' = m$ and $D = 1$.

and solution is type $\phi(x+my)$.

So,

$$m^4 - m^2 = 0$$

$$m^2(m^2-1) = 0$$

$$m = 0, 0, 1, -1$$

here, if we assume $D = m$, $D' = 1$
then, $1 - m^2 = 0$
only two roots.
So, two arbitrary function
and order $= 4$, 4

so, $C \cdot F = \phi_1(x - 0y) + y \cdot \phi_2(x - 0 \cdot y) + \phi_3(x-y) + \phi_4(x+y)$

or,

$$\boxed{C \cdot F = \phi_1(x) + y\,\phi_2(x) + \phi_3(x-y) + \phi_4(x+y)}$$

Finding Particular Integral (Short Method).

(First Type)

When $f(x,y)$ is of the form $f(ax+by)$

If $F(D,D')$ be homogenous function of D and D' of degree n, then-

$$PI = \frac{1}{F(D,D')} f(ax+by) = \frac{1}{F(a,b)} \underbrace{\int\int\int \cdots \int}_{n\text{-time}} f(v) \underbrace{dv\,dv\,dv \cdot dv}_{n\text{-times}}.$$

where, $v = ax + by$.

Provided- $F(a,b) \neq 0$.

Exp:- $(D^2+3DD'+2D'^2)z = (x+y)$

Particular Integral

$$P.I = \frac{1}{F(D,D')} \cdot (x+y)$$

or $$PI = \frac{1}{(D^2+3DD'+2D'^2)}(x+y)$$

homogenous of order = 2.

or $$PI = \frac{1}{1+3+2} \cdot \iint v \, dv dv$$

$v = x+y$.

or $$PI = \frac{1}{6}\int \frac{v^2}{2} dv = \frac{1}{36} v^3$$

here, $F(a,b) = 6 \neq 0$

or $$\boxed{PI = \frac{1}{36}(x+y)^3}$$

Exceptional Case when $F(a,b) = 0$.

$F(D.D') = 0$

$F(a,b) = 0$ if and only if $(bD - aD')$ is a Factor of $F(D,D')$

$$\frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{b^n.n!}\phi(ax+by)$$

or, If $f(a,b)=0$

$$P.I = \frac{x}{\frac{\partial}{\partial D}[f(D,D')]_{(a,b)}} \cdot f[(ax+by)]$$

or,

$$P.I = \frac{y}{\frac{\partial}{\partial D'}[F(D,D')]_{(a,b)}} \cdot f(ax+by)$$

⇒ If there is $(bD-aD')^{r}$ Factor. $r<n$.

then, $$\frac{1}{(bD-aD')^{r} f(D,D')} f(ax+by)$$

order of $F(D,D') = n-r$.

then,

$$P.I = \frac{x^{r}}{b^{r}.r!} \underbrace{\iint\int..\int}_{(n-r)\text{ times}} f(u)\underbrace{du\,du}_{(n-r)\text{ times}}$$

$u = ax+by$.

$$P.I. = \frac{y^{r}}{(-a)^{r}.r!} \underbrace{\iint..\int}_{n-r\text{ times}} f(u)\cdot \underbrace{du\,du\cdot dy}_{(n-r)\text{ times}}$$

Example. $(2D^2-5DD'+2D'^2)z = 5\sin(2x+y)$

here, Auxiliary equation-

$$D=m,\ D'=1$$

$$2m^2-5m+2=0$$

$$\Rightarrow 2m^2-4m-m+2=0$$

$$\Rightarrow 2m(m-2)-(m-2)=0$$

$$\Rightarrow (2m-1)(m-2)=0$$

$$\Rightarrow m=2,\ \frac{1}{2}$$

So, $C.F = \phi_1(y+2x)+\phi_2(2y+x)$

Now. $$P.I. = \frac{1}{(2D^2-5DD'+2D'^2)}\cdot 5\sin(2x+y)$$

$ax+by=2x+y$

$$= \frac{1\times 5}{(2\times 4-5\times 2\times 1+2\times 1)}\iint \sin(u)\,du\,du.$$

$ax+by=u.$

$\Rightarrow 2x+y=u.$

$$= \frac{5}{10-10}\cdot \underline{5\sin(2x+y)}\ F(a,b)=0.$$

Now, $$P.I. = \frac{1}{(D-2D')(2D-D')}\cdot 5\sin(2x+y)$$

$$\therefore P.I = \frac{5}{(D-2D')}\left[\frac{1}{(2D-D')}\cdot \sin(2x+y)\right]$$

$$P.I = \frac{5}{(D-2D')}\left[\frac{1}{2\times2-1}\int \sin u \, du\right] \qquad u = 2x+y$$

$$= \frac{5}{3(D-2D')} \cdot (-\cos u)$$

$$= -\frac{5}{3} \cdot \frac{1}{(D-2D')} \cos(2x+y)$$

$$= -\frac{5}{3} \cdot \frac{x}{\frac{d}{dD}(D-2D')} \cdot \cos(2x+y)$$

$$P.I = -\frac{5}{3} \cdot \frac{x}{1} \cos(2x+y)$$

Second Method:
By formula -

$$\rightarrow -\frac{5}{3} \cdot \frac{x^1}{1^1 \times 1!} \cos(2x+y)$$

$$= -\frac{5x}{3} \cos(2x+y)$$

So, Complete Solution $z = C.F + P.I$

or, $$\boxed{z = \phi_2(x+2y) + \phi_1(2x+y) - \frac{5x}{3}\cos(2x+y)}$$

There, is another P.I.

$$P.I = -\frac{5}{3} \frac{1}{(D-2D')} \cos(2x+y)$$

$$= -\frac{5}{3} \cdot \frac{y}{\frac{d}{dD'}(D-2D')} \cos(2x+y)$$

$$\Rightarrow P.I. = \frac{-5y}{3\cdot(-2)} \cos(2x+y) = \frac{5y}{6} \cos(2x+y)$$

$$\boxed{z = \phi_1(2x+y) + \phi_2(x+2y) + \frac{5y}{6}\cos(2x+y)}$$

Second Type.

When $f(x,y) = x^m y^n$

$$P.I = \frac{1}{F(D,D')} x^m y^n$$

If $n<m$, then $\frac{1}{F(D,D')}$ should expanded in power of $\frac{D'}{D}$.

$m<n$ then $\frac{1}{F(D,D')}$ should Expanded in powor $\frac{D}{D'}$

Exp: Solve -

$$(D^2-6DD'+9D'^2)z = 12x^2+36xy$$

Sol:

Auxiliary Equation - of

$$(D^2-6DD'+9D'^2)z=0$$

$D=m$, $D'=1$

is $\quad m^2-6m+9=0$

or $\quad (m-3)^2=0$

$\quad m=3, 3.$

thus, $C.F = \phi_1(y+3x) + x\,\phi_2(y+3x)$

ϕ_1 and ϕ_2 arbitrary function.

Now, $P.I = \frac{1}{(D-3D')^2}\left[12x^2+36xy\right]$

$$P.I = \frac{1}{D^2\left(1-\frac{3D'}{D}\right)^2}\left[12x^2+36xy\right]$$

$\underline{n<m}$ as power of y is less than power of x.

or $$= \frac{12}{D^2}\left[1-\frac{3D'}{D}\right]^{-2}(x^2+3xy)$$

or $$= \frac{12}{D^2}\left[1+\frac{6D'}{D}+\cdots\right](x^2+3xy)$$

as $\underline{D'^2 y = 0}$.

or $$= \frac{12}{D^2}\left[(x^2+3xy)+\frac{6}{D}\cdot D'(x^2+3yx)+0\cdots\right]$$

D' as Maximum power of y in (x^2+3xy) is one

$\because$ Binomial Expansion of $(1-x)^{-2}$
$= 1+2x+3x^2+\cdots$

or $$= \frac{12}{D^2}\left[(x^2+3xy)+\frac{6}{D}\cdot(3x)\right]$$

or $$= \frac{12}{D^2}\left[(x^2+3xy)+18\cdot\frac{x^2}{2}\right]$$

$$\frac{1}{D}x = \int x\,dx = \frac{x^2}{2}$$

or $$= \frac{12}{D^2}\left[10x^2+3xy\right]$$

or $$= \frac{12}{D}\left[\frac{10x^3}{3}+\frac{3x^2y}{2}\right] = 12\left[\frac{10x^4}{12}+\frac{3x^3y}{6}\right]$$

or, $$P.I = 10x^4+6x^3y$$

hence General solution -

$$\boxed{z = \phi_1(y+3x)+x\,\phi_2(y+3x)+10x^4+6x^3y.}$$

Non-Homogenous linear PDE with Constant Coefficient

Let the PDE of the form (1).

$$(D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2)\cdots(D-m_nD'-\alpha_n)z = F(x,y)$$

then,

$$CF = e^{\alpha_1 x}\phi_1(y+m_1x)+e^{\alpha_2 x}\phi_2(y+m_2x)+\cdots+e^{\alpha_n x}\phi_n(y+m_nx)$$

where, $m_1, m_2, \cdots m_n$ are distinct.

If $m_1 = m_2 = m$ are repeated. & $\alpha_1 = \alpha_2$

i.e. $(D-m_1D'-\alpha_1)^2(D-m_2D'-\alpha_2)z = F(x,y)$

then, $C.F = e^{\alpha_1 x}\left[\phi_1(y+m_1x)+x\,\phi_2(y+m_1x)\right]+e^{\alpha_2 x}\phi_3(y+m_2x).$

Form-(2) PDE-

$$(D'-m_1D-\beta_1)(D'-m_2D-\beta_2)\cdots(D'-m_nD-\beta_n)z = F(x,y)$$

then, C.F

$$= e^{\beta_1 y}\left[\phi_1(x+m_1y)\right]+e^{\beta_2 y}\phi_2(x+m_2y)$$

$$+\cdots+e^{\beta_n y}\phi_n(x+m_ny).$$

This is for Distinct Factor.

For Repeated Factor we use as above Form(1) type.

For, Particular Integral.

1) When, $f(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$.

then,

$$P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}.$$

2). When $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I = \frac{1}{F(D,D')} \sin(ax+by) \text{ or } \cos(ax+by)$$

Just by Replacing $D^2 = -a^2$, $DD' = -ab$

$D'^2 = -b^2$

3) When $f(x,y) = Ve^{ax+by}$.

$$P.I. = \frac{1}{F(D,D')} \cdot Ve^{ax+by} = e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} \cdot V$$

4) When $f(x,y) = x^m y^n$

$$P.I = \frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} x^m y^n$$

Exp
$$(D^2+DD'+D'-1)z = e^{-x+y}\cdot\sin(x+2y)$$

Soln.
$$(D+1)(D+D'-1)z = e^{-x+y}\sin(x+2y)$$

or, $$(D-0\cdot D'+1)(D+D'-1)z = e^{-x+y}\sin(x+2y)$$

$$\Rightarrow (D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2)z = e^{-x+y}\sin(x+2y)$$

$$C\cdot F = e^{\alpha_1 x}\phi_1(y+m_1x) + e^{\alpha_2 x}\phi_2(y+m_2x)$$

or, $$C\cdot F = e^{-x}\phi_1(y+0x) + e^{x}\phi_2(y-x)$$

or $$C\cdot F = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$$

Now,
$$P.I = \frac{1}{F(D,D')}e^{-x+y}\sin(x+2y)$$

or $$PI = e^{-x+y}\frac{1}{F(D-1,D'+1)}\cdot\sin(x+2y)$$

or $$PI = e^{-x+y}\frac{1}{D(D-1+D'+1-1)}\cdot\sin(x+2y)$$

or $$PI = e^{-x+y}\frac{1}{D(D+D'-1)}\sin(x+2y)$$

or $$P.I = e^{-x+y} \cdot \frac{1}{(D^2+DD'-D)} \sin(x+2y)$$

Replace, $D^2 = -1^2$, $DD' = -2$

$$P.I = e^{-x+y} \cdot \frac{1}{-1-2-D} \sin(x+2y)$$

$$P.I = -e^{-x+y} \frac{1}{(D+3)} \sin(x+2y)$$

$$P.I = -e^{-x+y} \frac{(D-3)}{(D^2-9)} \sin(x+2y)$$

$D^2 = -1$

$$P.I = \frac{e^{-x+y}}{10} (D-3) \sin(x+2y)$$

$\because D \sin(x+2y) = \frac{\partial}{\partial x} \sin(x+2y)$

$$P.I = \frac{e^{-x+y}}{10} \left[\cos(x+2y) - 3\sin(x+2y)\right]$$

so, General solution -

$$z = C.F + P.I$$

or, $$\boxed{z = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x) + \frac{e^{-x+y}}{10}\left[\cos(x+2y) - 3\sin(x+2y)\right]}$$