Control Systems

Subject Code: BEC-26

Third Year ECE

Unit-II

Shadab A. Siddique
Assistant Professor

Maj. G. S. Tripathi
Associate Professor

Lecture 4

Department of Electronics & Communication Engineering,
Madan Mohan Malaviya University of Technology, Gorakhpur

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, C = [0 \quad 1] \; and \; D = [0]$$

State Space Model from Transfer Function

Consider the two types of transfer functions based on the type of terms present in the numerator.

- Transfer function having constant term in Numerator.
- Transfer function having polynomial function of 's' in Numerator.

Transfer function having constant term in Numerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} = \dot{x}_n$$

and $u(t) = u$

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1x_2 + a_0x_1 = b_0u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + b_0u$$

The output equation is -

$$y(t) = y = x_1$$

The state space model is -

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here, $D = [0]$.

Example

Find the state space model for the system having transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Rearrange, the above equation as,

$$(s^2 + s + 1)Y(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_2 = -x_1 - x_2 + u$$

The output equation is

$$y(t) = y = x_1$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer function having polynomial function of 's' in Numerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \left(\frac{1}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}\right)(b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0)$$

The above equation is in the form of product of transfer functions of two blocks, which are cascaded.

$$\frac{Y(s)}{U(s)} = \left(\frac{V(s)}{U(s)}\right)\left(\frac{Y(s)}{V(s)}\right)$$

Here,

$$\frac{V(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1} s^{n-1} + \dots + a_0)V(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} + \dots + a_1 \frac{\mathrm{d}v(t)}{\mathrm{d}t} + a_0 v(t) = u(t)$$

Let

$$v(t) = x_1$$

$$\frac{\mathrm{d}v((t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} = \dot{x}_n$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u$$

Consider,

$$\frac{Y(s)}{V(s)} = b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0$$

Rearrange, the above equation as

$$Y(s) = (b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0) V(s)$$

Apply inverse Laplace transform on both the sides.

$$y(t) = b_n \frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + b_{n-1} \frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} + \ldots + b_1 \frac{\mathrm{d}v(t)}{\mathrm{d}t} + b_0 v(t)$$

By substituting the state variables and $y(t) = y$ in the above equation, will get the output equation as,

$$y = b_n \dot{x}_n + b_{n-1} x_n + \ldots + b_1 x_2 + b_0 x_1$$

Substitute, $\dot{x}_n$ value in the above equation.

$$y = b_n(-a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u) + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

$$y = (b_0 - b_na_0)x_1 + (b_1 - b_na_1)x_2 + \ldots + (b_{n-1} - b_na_{n-1})x_n + b_nu$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = [b_0 - b_na_0 \quad b_1 - b_na_1 \quad \ldots \quad b_{n-2} - b_na_{n-2} \quad b_{n-1} - b_na_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If $b_n = 0$, then,

$$Y = [b_0 \quad b_1 \quad \dots \quad b_{n-2} \quad b_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Transfer Function from State Space Model

We know the state space model of a Linear Time-Invariant (LTI) system is -

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

Apply Laplace Transform on both sides of the state equation.

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Apply Laplace Transform on both sides of the output equation.

$$Y(s) = CX(s) + DU(s)$$

Substitute, X(s) value in the above equation.