LaPlace Transform in Circuit Analysis

Objectives:

- Calculate the Laplace transform of common functions using the definition and the Laplace transform tables
- Laplace-transform a circuit, including components with non-zero initial conditions.
- Analyze a circuit in the s-domain
- Check your s-domain answers using the initial value theorem (IVT) and final value theorem (FVT)
- Inverse Laplace-transform the result to get the time-domain solutions; be able to identify the forced and natural response components of the time-domain solution.

(Note – this material is covered in Chapter 12 and Sections 13.1 – 13.3)

LaPlace Transform in Circuit Analysis

What types of circuits can we analyze?

- Circuits with any number and type of DC sources and any number of resistors.
- First-order (RL and RC) circuits with no source and with a DC source.
- Second-order (series and parallel RLC) circuits with no source and with a DC source.
- Circuits with sinusoidal sources and any number of resistors, inductors, capacitors (and a transformer or op amp), but can generate only the <u>steady-state</u> response.

LaPlace Transform in Circuit Analysis

What types of circuits will Laplace methods allow us to analyze?

- Circuits with any type of source (so long as the function describing the source has a Laplace transform), resistors, inductors, capacitors, transformers, and/or op amps; the Laplace methods produce the <u>complete</u> response!

LaPlace Transform in Circuit Analysis

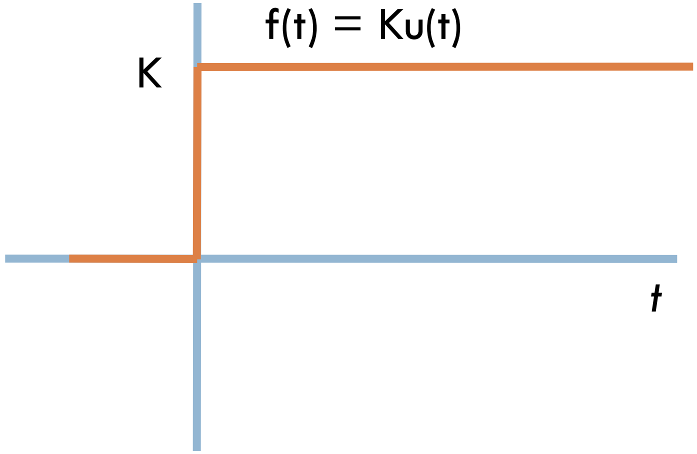
Definition of the Laplace transform:

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt}$$

Note that there are limitations on the types of functions for which a Laplace transform exists, but those functions are "pathological", and not generally of interest to engineers!

LaPlace Transform in Circuit Analysis

Aside – formally define the "step function", which is often modeled in a circuit by a voltage source in series with a switch.



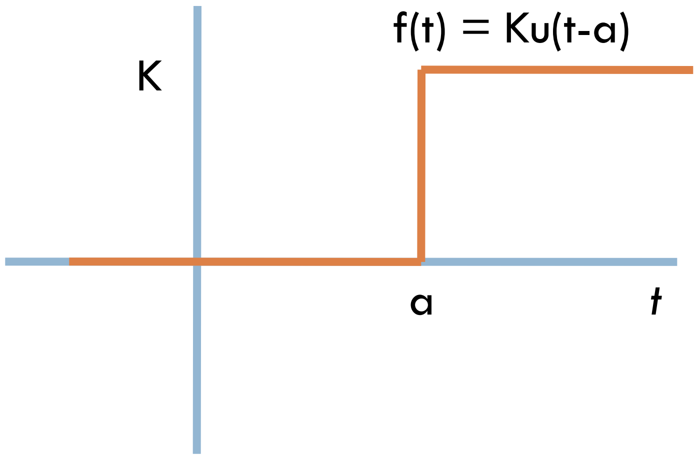
$$f(t) = 0, \qquad t < 0$$
$$= K, \qquad t > 0$$

When K = 1, f(t) = u(t), which we call the unit step function

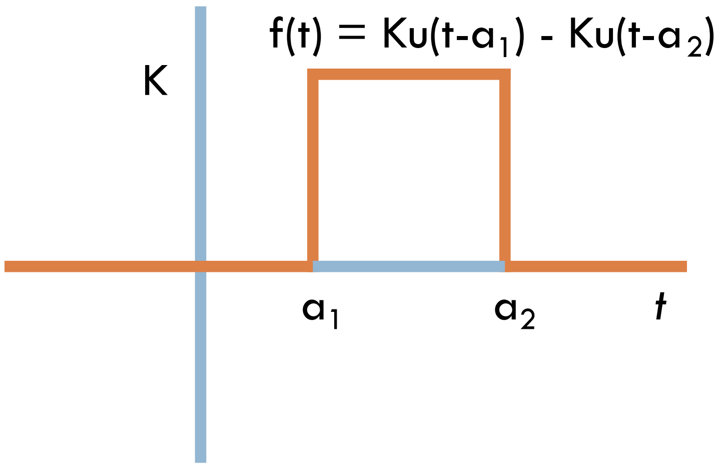
LaPlace Transform in Circuit Analysis

More step functions:

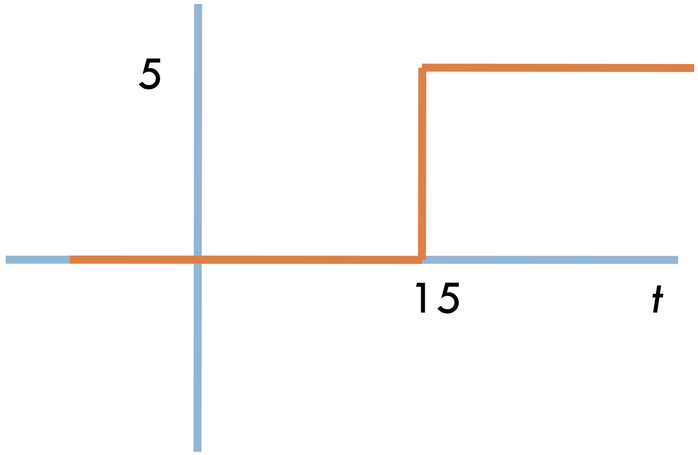
The step function shifted in time



The "window" function



Which of these expressions describes the function plotted here?



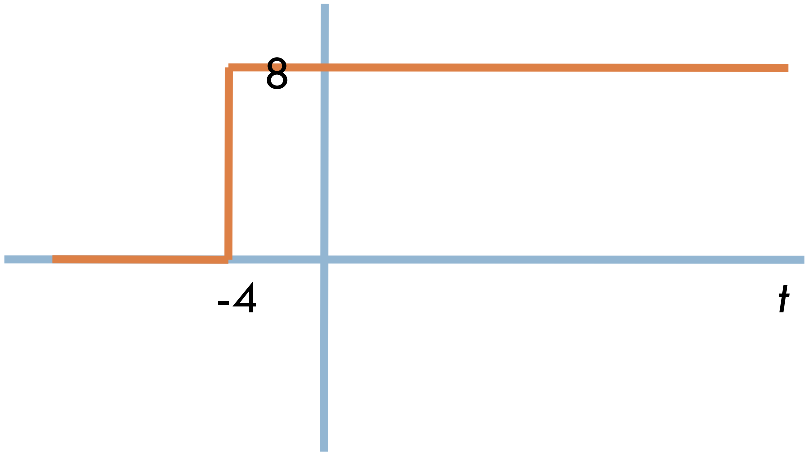
- A. $u(t - 5)$
- B. $5u(t + 15)$
- C. $5u(t - 15)$
- D. $15u(t - 5)$

Which of these expressions describes the function plotted here?

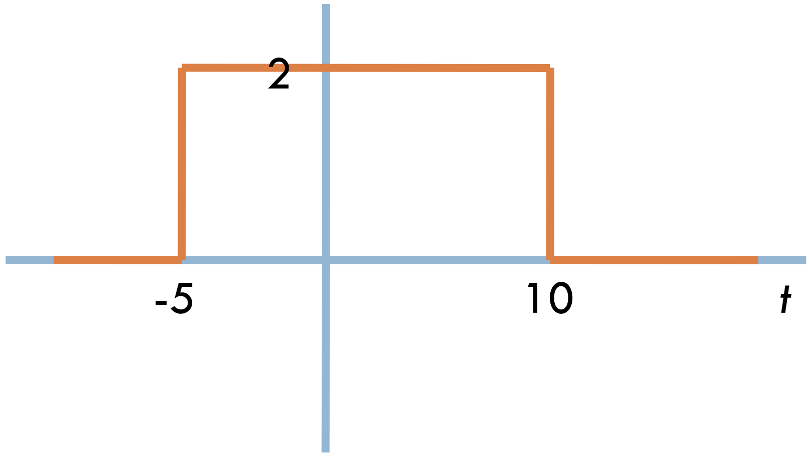
A. $8u(t + 4)$

B. $4u(t - 8)$

C. $8u(t - 4)$



Which of these expressions describes the function plotted here?



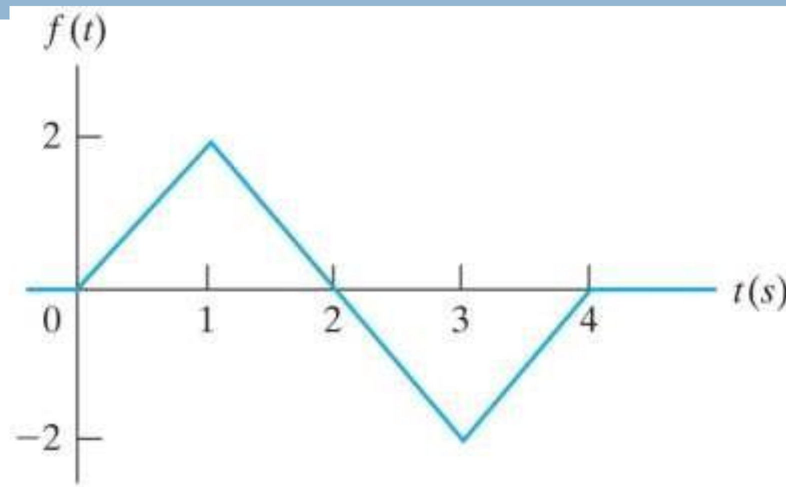
A. $2u(t + 5) + 2u(t - 10)$

B. $2u(t - 5) + 2u(t + 10)$

C. $2u(t + 5) - 2u(t - 10)$

LaPlace Transform in Circuit Analysis

Use "window" functions to express this piecewise linear function as a single function valid for all time.



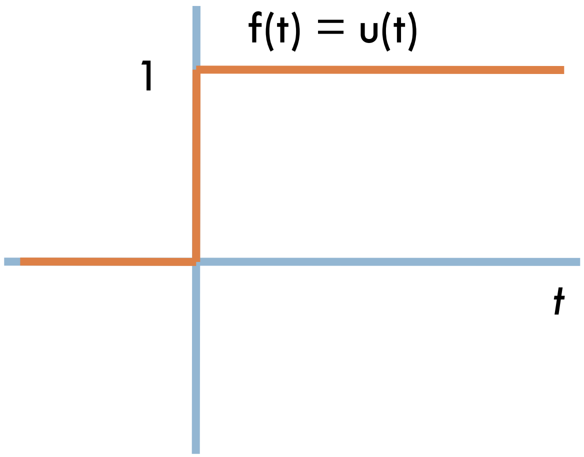
$$
f(t) = \begin{array}{lll}
0, & t < 0 & \\
2t, & 0 \le t \le 1\text{s} & [u(t) - u(t-1)] \\
-2t + 4, & 0 \le t \le 1\text{s} & [u(t-1) - u(t-3)] \\
2t - 8, & 0 \le t \le 1\text{s} & [u(t-3) - u(t-4)] \\
0, & t > 4\,\text{s} &
\end{array}
$$

$$
\begin{aligned}
f(t) &= 2t[u(t) - u(t-1)] + (-2t+4)[u(t-1) - u(t-3)] \\
&\quad + (2t-8)[u(t-3) - u(t-4)] \\
&= 2tu(t) - 4(t-1)u(t-1) + 4(t-3)u(t-3) - 2(t-4)u(t-4)
\end{aligned}
$$

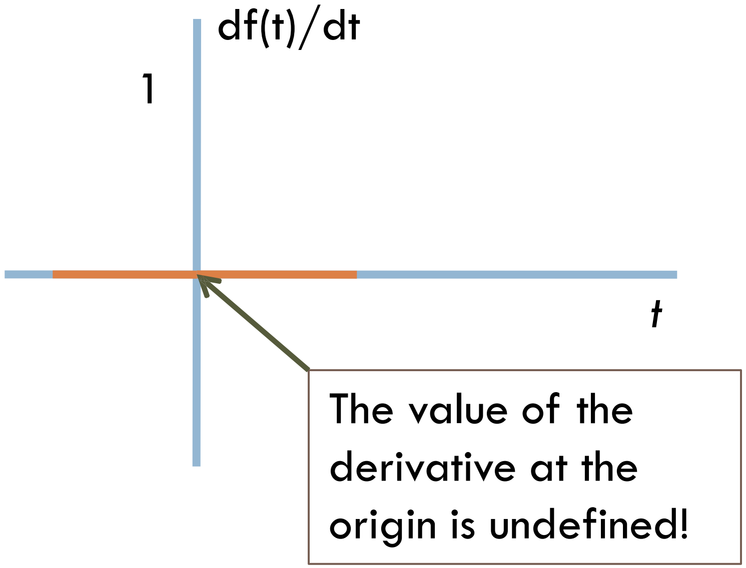
LaPlace Transform in Circuit Analysis

The impulse function, created so that the step function's derivative is defined for all time:

The step function



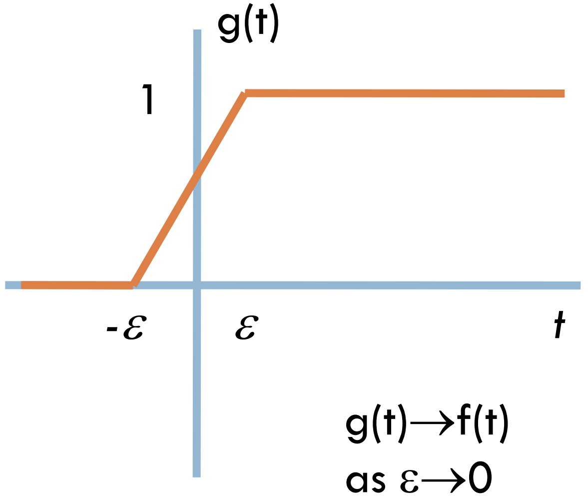
The first derivative of the step function



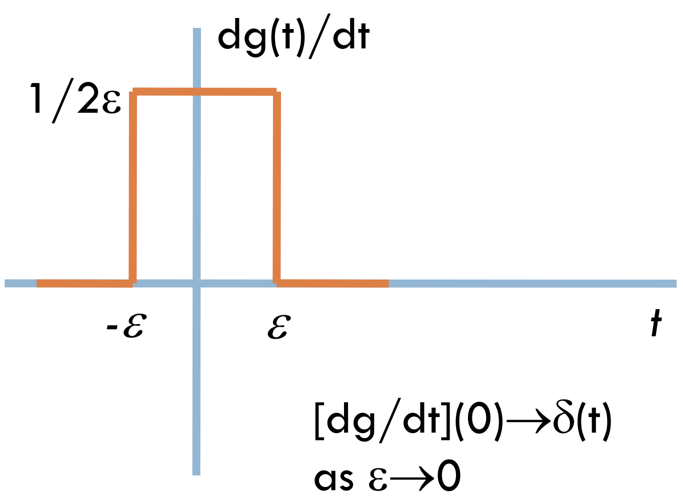
LaPlace Transform in Circuit Analysis

Use a limiting function to define the step function and its first derivative!

The step function



The first derivative of the step function



LaPlace Transform in Circuit Analysis

The unit impulse function is represented symbolically as $\delta(t)$.

Definition:

$$\delta(t) = 0 \qquad \text{for} \qquad t \neq 0$$

$$\text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

(Note that thearea under the $g(t)$ function is

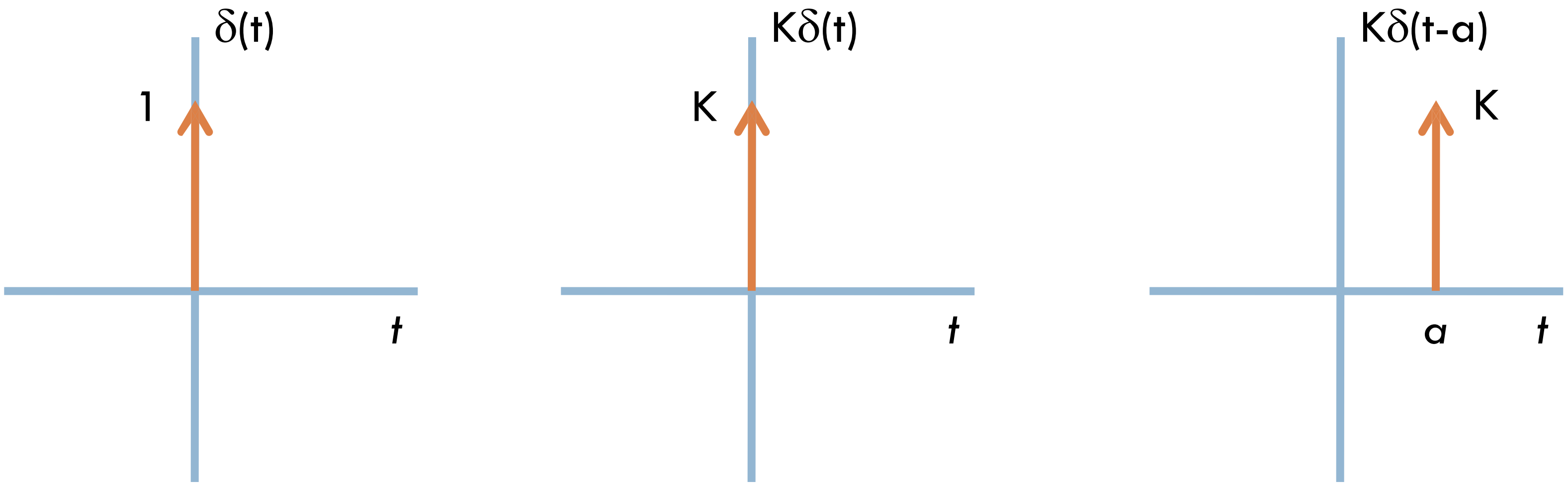
$$\frac{1}{2\varepsilon}(\varepsilon + \varepsilon), \text{ which approaches} 1 \text{ as } \varepsilon \to 0)$$

Note also that any limiting function with the following characteristics can be used to generate the unit impulse function:

- Height $\to \infty$ as $\varepsilon \to 0$
- Width $\to 0$ as $\varepsilon \to 0$
- Area is constant for all values of ε

LaPlace Transform in Circuit Analysis

Another definition: $\delta(t) = \dfrac{du(t)}{dt}$



The sifting property is an important property of the impulse function:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

Evaluate the following integral, using the sifting property of the impulse function.

$$\int_{-10}^{10} (6t^2 + 3)\delta(t - 2)\,dt$$

A. 24

B. 27

C. 3

$$6(2)^2 + 3 = 27$$

LaPlace Transform in Circuit Analysis

Use the definition of Laplace transform to calculate the Laplace transforms of some functions of interest:

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t)e^{-st}dt = \int_0^\infty \delta(t-0)e^{-st}dt = e^{-s(0)} = 1$$

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt = \frac{1}{-s}e^{-st}\Big|_0^\infty = 0 - \frac{1}{-s} = \frac{1}{s}$$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}\ dt = \int_0^\infty e^{-(s+a)t}dt = \frac{1}{-(s+a)}e^{-(s+a)t}\Big|_0^\infty = 0 - \frac{1}{-(s+a)} = \frac{1}{(s+a)}$$

$$\mathcal{L}\{\sin\omega t\} = \int_0^\infty \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]e^{-st}dt = \tfrac{1}{j2}\int_0^\infty [e^{-(s-j\omega)t} - e^{-(s+j\omega)t}]dt$$

$$= \frac{1}{j2}\left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right]_0^\infty - \frac{1}{j2}\left[\frac{e^{-(s+j\omega)t}}{-(s+j\omega)}\right]_0^\infty = \frac{1}{j2}\left[\frac{1}{(s-j\omega)} - \frac{1}{(s+j\omega)}\right] = \frac{\omega}{s^2+\omega^2}$$

Look at the Functional Transforms table. Based on the pattern that exists relating the step and ramp transforms, and the exponential and damped-ramp transforms, what do you predict the Laplace transform of t^2 is?

A. $1/(s + a)$
B. s
C. $1/s^3$

LaPlace Transform in Circuit Analysis

Using the definition of the Laplace transform, determine the effect of various operations on time-domain functions when the result is Laplace-transformed. These are collected in the Operational Transform table.

$$\mathcal{L}\{K_1 f_1(t) + K_2 f_2(t) - K_3 f_3(t)\} = \int_0^\infty [K_1 f_1(t)e^{-st} + K_2 f_2(t)e^{-st} - K_3 f_3(t)e^{-st}]dt$$

$$= \int_0^\infty K_1 f_1(t)e^{-st}dt + \int_0^\infty K_2 f_2(t)e^{-st}dt - \int_0^\infty K_3 f_3(t)e^{-st}dt$$

$$= K_1\int_0^\infty f_1(t)e^{-st}dt + K_2\int_0^\infty f_2(t)e^{-st}dt - K_3\int_0^\infty f_3(t)e^{-st}dt$$

$$= K_1F_1(s) + K_2F_2(s) - K_2F_2(s)$$

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)[-se^{-st}]dt \qquad \text{(integration by parts!)}$$

$$= -f(0) + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Now lets use the operational transform table to find the correct value of the Laplace transform of t^2, given that

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

A. $1/s^3$
B. $2/s^3$
C. $-2/s^3$

LaPlace Transform in Circuit Analysis

Example – Find the Laplace transform of t^2e^{-at}.

Use the operational transform: $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n F(s)}{ds^n}$

Use the functional transform: $\mathcal{L}\{e^{-at}\} = \dfrac{1}{(s+a)}$

$$\mathcal{L}\{t^2e^{-at}\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s+a}\right] = \frac{d}{ds}\left[\frac{-1}{(s+a)^2}\right] = \frac{2}{(s+a)^3}$$

Alternatively,
Use the operational transform: $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

Use the functional transform: $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

$$\mathcal{L}\{t^2e^{-at}\} = \frac{2}{(s+a)^3}$$

LaPlace Transform in Circuit Analysis

How can we use the Laplace transform to solve circuit problems?

- Option 1:
  - Write the set of differential equations in the time domain that describe the relationship between voltage and current for the circuit.
  - Use KVL, KCL, and the laws governing voltage and current for resistors, inductors (and coupled coils) and capacitors.
  - Laplace transform the equations to eliminate the integrals and derivatives, and solve these equations for V(s) and I(s).
  - Inverse-Laplace transform to get v(t) and i(t).

LaPlace Transform in Circuit Analysis

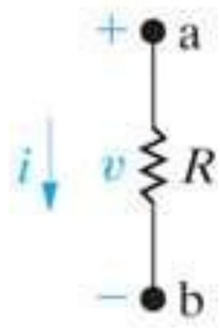
How can we use the Laplace transform to solve circuit problems?

- Option 2:
  - Laplace transform the circuit (following the process we used in the phasor transform) and use DC circuit analysis to find V(s) and I(s).
  - Inverse-Laplace transform to get v(t) and i(t).

LaPlace Transform in Circuit Analysis

Laplace transform – resistors:

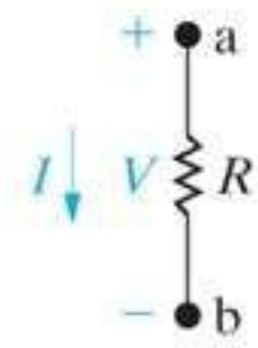
Time-domain



$$v(t) = Ri(t)$$

$$\xrightarrow{\mathcal{L}}$$

s-domain (Laplace)

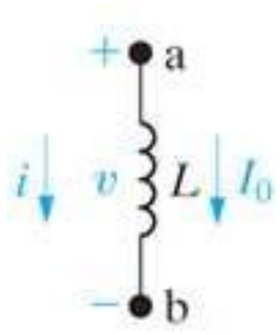


$$V(s) = RI(s)$$

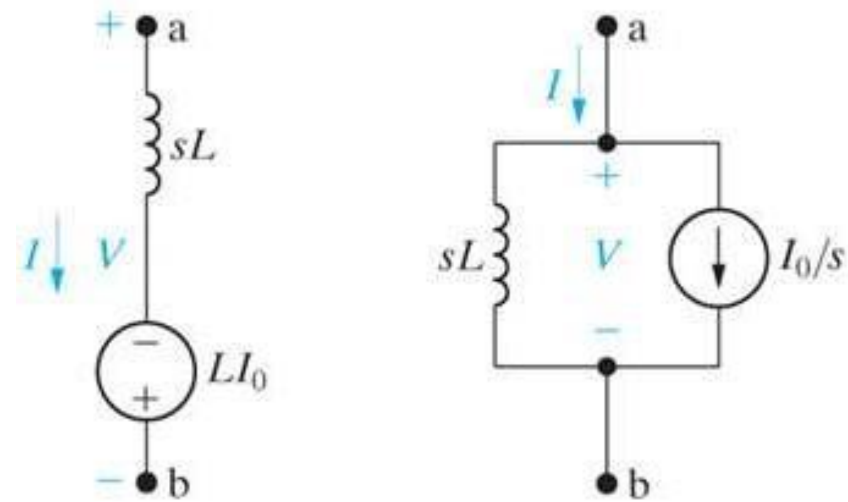
LaPlace Transform in Circuit Analysis

Laplace transform – inductors:

Time-domain



s-domain (Laplace)



$$v(t) = L\frac{di(t)}{dt}$$

$$i(0) = I_0$$

$$\xrightarrow{\mathcal{L}}$$

$$V(s) = sLI(s) - LI_0$$

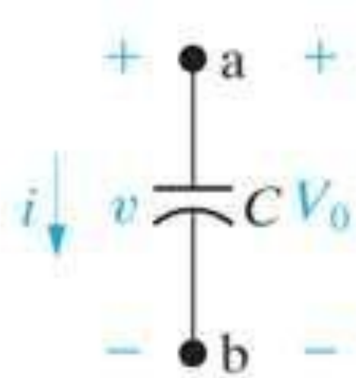
$$I(s) = \frac{V(s)}{sL} + \frac{I_0}{s}$$

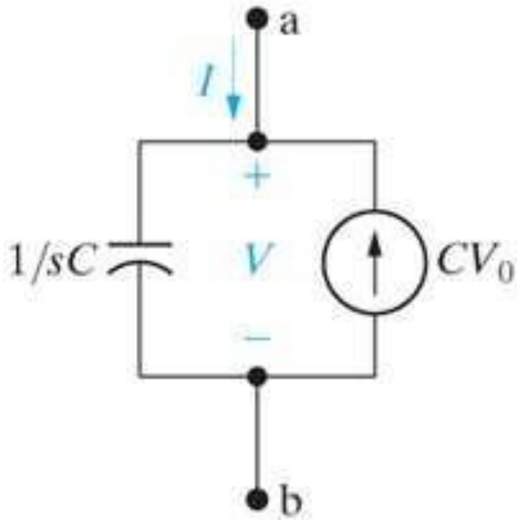
LaPlace Transform in Circuit Analysis

Laplace transform – resistors:

Time-domain

s-domain (Laplace)





$$i(t) = C\frac{dv(t)}{dt}$$

$$v(0) = V_0$$

$$\xrightarrow{\mathcal{L}}$$

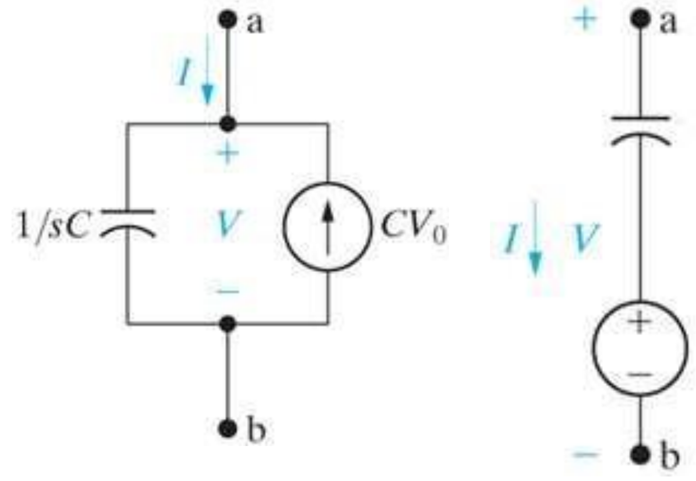
$$I(s) = sCV(s) - CV_0$$

Find the value of the complex impedance and the series-connected voltage source, representing the Laplace transform of a capacitor.

A. sC, V_0/s

B. $1/sC$, V_0/s

C. $1/sC$, $-V_0/s$



$$I(s) = sCV(s) - CV_0$$

LaPlace Transform in Circuit Analysis

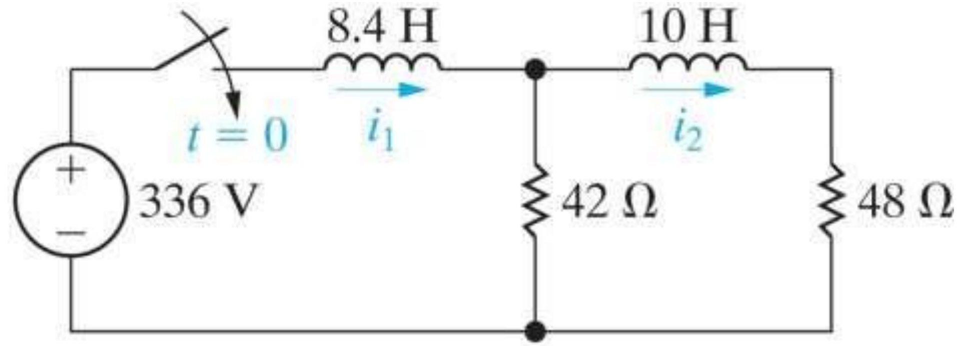
Recipe for Laplace transform circuit analysis:

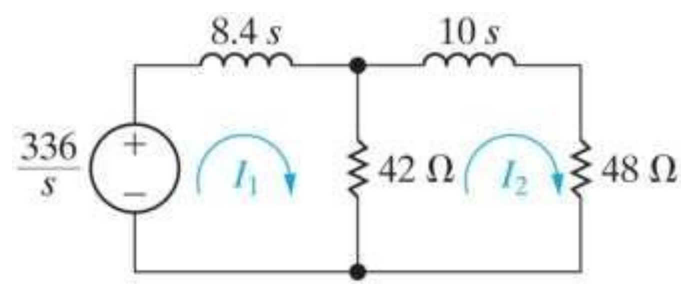
1. Redraw the circuit (nothing about the Laplace transform changes the types of elements or their interconnections).
2. Any voltages or currents with values given are Laplace-transformed using the functional and operational tables.
3. Any voltages or currents represented symbolically, using i(t) and v(t), are replaced with the symbols I(s) and V(s).
4. All component values are replaced with the corresponding complex impedance, Z(s).
5. Use DC circuit analysis techniques to write the s-domain equations and solve them.
6. Inverse-Laplace transform s-domain solutions to get time-domain solutions.

LaPlace Transform in Circuit Analysis

Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.





$$-\frac{336}{s} + (42 + 8.4s)I_1 - 42I_2 = 0$$

$$(10s + 90)I_2 - 42I_1 = 0 \qquad \Rightarrow \qquad I_1 = \frac{10s + 90}{42} I_2$$

Substituting, $$-\frac{336}{s} + \left[\frac{(42 + 8.4s)(10s + 90)}{42} - 42\right] I_2 = 0$$

$$\Rightarrow \qquad I_2(s) = \frac{336(42)}{s[(42 + 8.4s)(10s + 90) - 42^2]} = \frac{168}{s^3 + 14s^2 + 24s}$$

$$I_1(s) = \frac{10s + 90}{42}\left[\frac{168}{s^3 + 14s^2 + 24s}\right] = \frac{40s + 360}{s^3 + 14s^2 + 24s}$$

LaPlace Transform in Circuit Analysis

Recipe for Laplace transform circuit analysis:

1. Redraw the circuit (nothing about the Laplace transform changes the types of elements or their interconnections).
2. Any voltages or currents with values given are Laplace-transformed using the functional and operational tables.
3. Any voltages or currents represented symbolically, using i(t) and v(t), are replaced with the symbols I(s) and V(s).
4. All component values are replaced with the corresponding complex impedance, Z(s).
5. Use DC circuit analysis techniques to write the s-domain equations and solve them.
6. Inverse-Laplace transform s-domain solutions to get time-domain solutions.

LaPlace Transform in Circuit Analysis

Finding the inverse Laplace transform:

$$f(t) = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \qquad t > 0$$

This is a contour integral in the complex plane, where the complex number c must be chosen such that the path of integration is in the convergence area along a line parallel to the imaginary axis at distance c from it, where c must be larger than the real parts of all singular values of F(s)!

There must be a better way …

LaPlace Transform in Circuit Analysis

Inverse Laplace transform using partial fraction expansion:

- Every s-domain quantity, V(s) and I(s), will be in the form

$$\frac{N(s)}{D(s)}$$

where N(s) is the numerator polynomial in s, and has real coefficients, and D(s) is the denominator polynomial in s, and also has real coefficients, and

$$\mathrm{O}\{N(s)\} < \mathrm{O}\{D(s)\}$$

- Since D(s) has real coefficients, it can always be factored, where the factors can be in the following forms:
  - ✓Real and distinct
  - ✓Real and repeated
  - ✓Complex conjugates and distinct
  - ✓Complex conjugates and repeated

LaPlace Transform in Circuit Analysis

Inverse Laplace transform using partial fraction expansion:

- The roots of D(s) (the values of s that make D(s) = 0) are called **poles**.
- The roots of N(s) (the values of s that make N(s) = 0) are called **zeros**.

Back to the example:

$$I_1(s) = \frac{40s + 360}{s^3 + 14s^2 + 24s} = \frac{40(s + 9)}{s(s + 2)(s + 12)}$$

$$I_2(s) = \frac{168}{s^3 + 14s^2 + 24s} = \frac{168}{s(s + 2)(s + 12)}$$

Find the zeros of $I_1(s)$.

$$I_1(s) = \frac{40(s+9)}{s(s+2)(s+12)}$$

A. $s = -9$ rad/s

B. $s = -9$ rad/s

C. There aren't any zeros

Find the poles of $I_1(s)$.

$$I_1(s) = \frac{40(s+9)}{s(s+2)(s+12)}$$

A. $s = 2$ rad/s, $s = 12$ rad/s

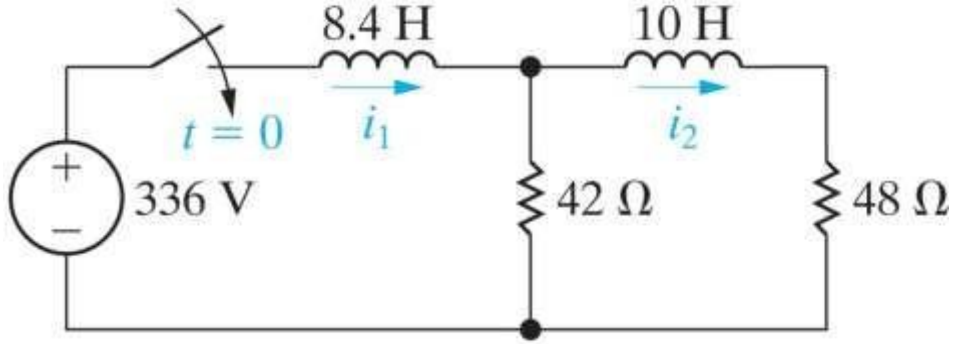
B. $s = -2$ rad/s, $s = -12$ rad/s

C. $s = 0$ rad/s, $s = -2$ rad/s, $s = -12$ rad/s

LaPlace Transform in Circuit Analysis

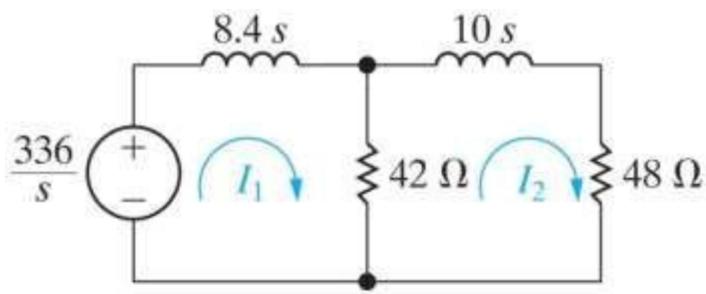
Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for t > 0.



$$I_1(s) = \frac{40s+360}{s(s+2)(s+12)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+12}$$



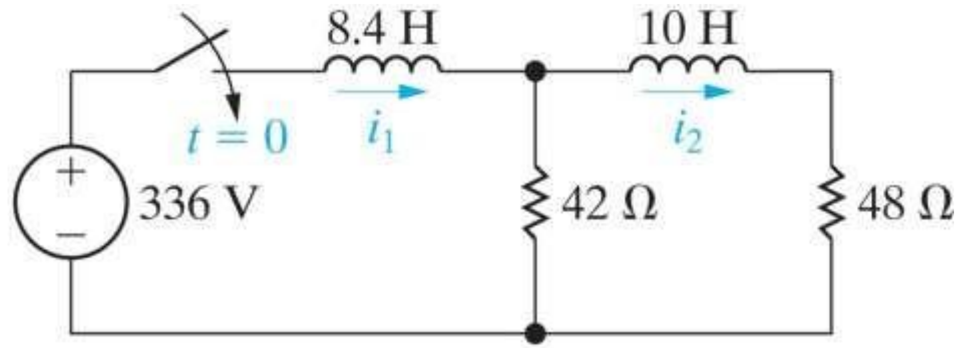
$$K_1 = \left.\frac{40s+360}{(s+2)(s+12)}\right|_{s=0} = 15; \qquad K_2 = \left.\frac{40s+360}{s(s+12)}\right|_{s=-2} = -14; \qquad K_3 = \left.\frac{40s+360}{s(s+2)}\right|_{s=-12} = -1$$

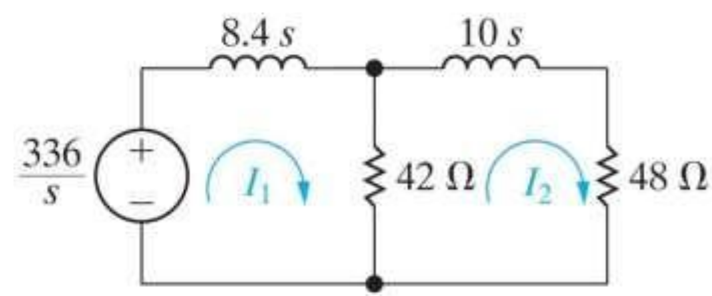
$$\therefore \qquad I_1(s) = \frac{15}{s} + \frac{-14}{s+2} + \frac{-1}{s+12}$$

LaPlace Transform in Circuit Analysis

Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.





$$i_1(t) = \mathcal{L}^{-1}\left\{\frac{15}{s} + \frac{-14}{s+2} + \frac{-1}{s+12}\right\}$$

$$= [15 - 14e^{-2t} - e^{-12t}]u(t) \text{ A}$$

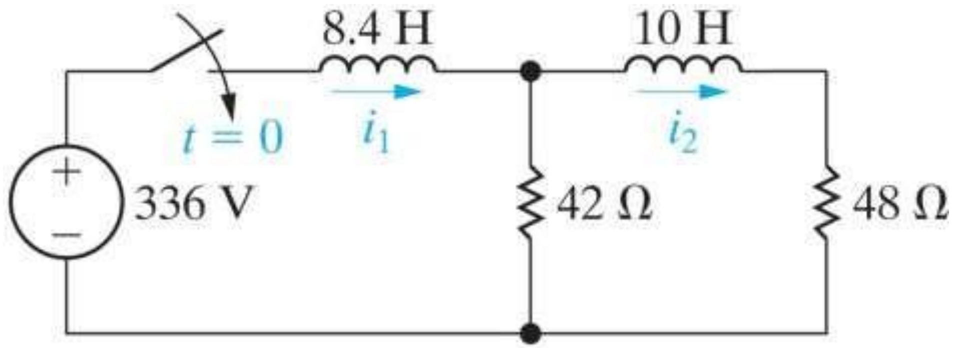
The forced response is $15u(t)$ A;

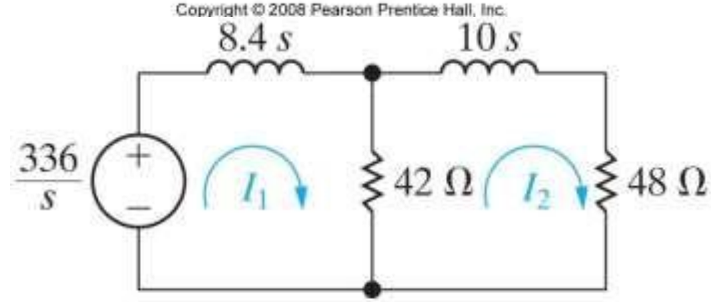
The natural response is $[-14e^{-2t} - e^{-12t}]u(t)$ A.

LaPlace Transform in Circuit Analysis

Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.





$$I_2(s) = \frac{168}{s(s+2)(s+12)}$$

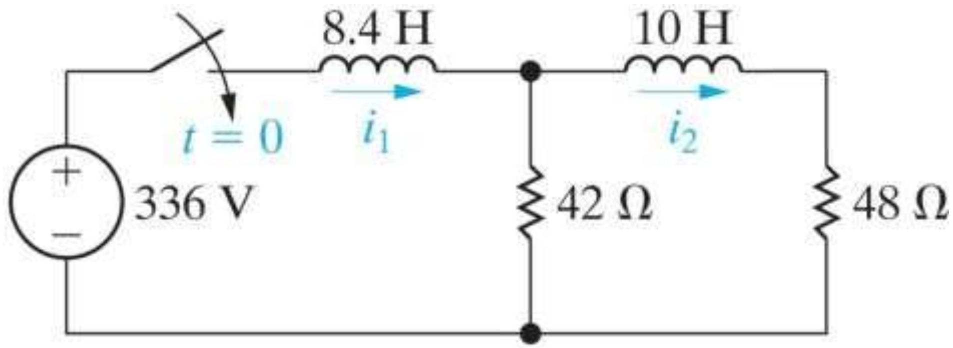
$$= \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+12}$$

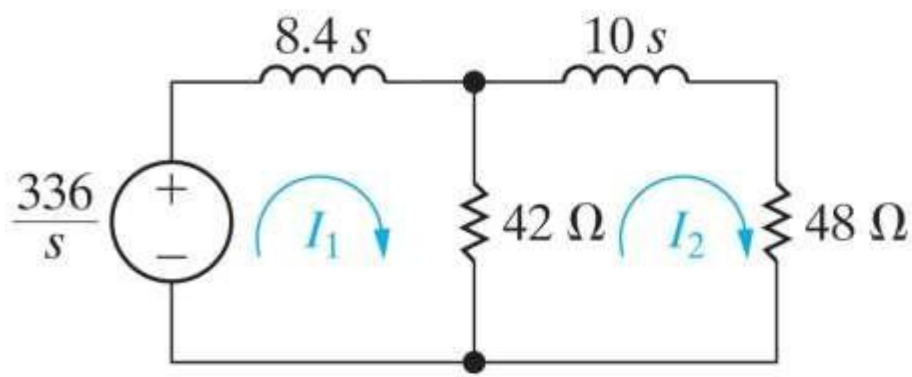
$$K_1 = \left.\frac{168}{(s+2)(s+12)}\right|_{s=0} = 7; \qquad K_2 = \left.\frac{168}{s(s+12)}\right|_{s=-2} = -8.4; \qquad K_3 = \left.\frac{168}{s(s+2)}\right|_{s=-12} = 1.4$$

$$\therefore \qquad I_2(s) = \frac{7}{s} + \frac{-8.4}{s+2} + \frac{1.4}{s+12}$$

LaPlace Transform in Circuit Analysis

Example:
There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.





$$i_2(t) = \mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{-8.4}{s+2} + \frac{1.4}{s+12}\right\}$$

$$= [7 - 8.4e^{-2t} + 1.4e^{-12t}]u(t)\text{ A}$$

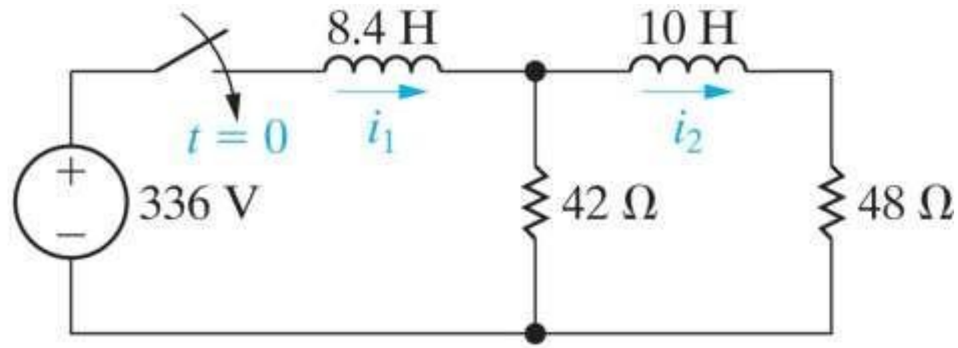
The forced response is $7u(t)$ A;

The natural response is $[-8.4e^{-2t} - 1.4e^{-12t}]u(t)$ A.

LaPlace Transform in Circuit Analysis

Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.



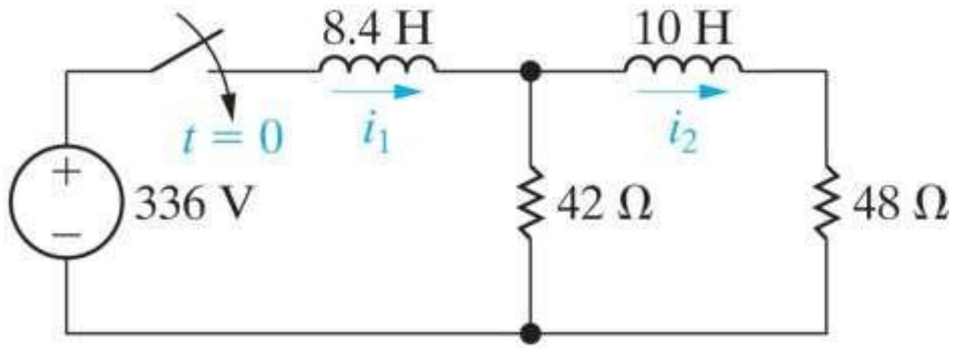
$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A$$

$$i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A$$

Check the answers at t = 0 and t = ∞ to make sure the circuit and the equations match!

LaPlace Transform in Circuit Analysis

Example:
There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.



$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A$$

$$i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A$$

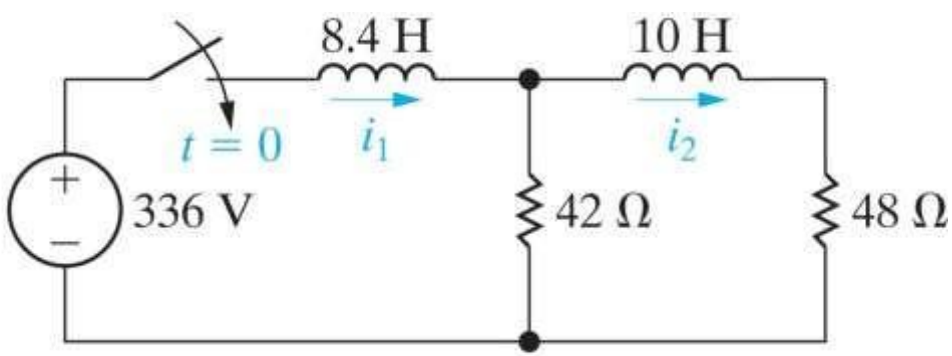
At $t = 0$, the circuit has no initial stored energy, so $i_1(0) = 0$ and $i_2(0) = 0$. Now check the equations:

$$i_1(0) = (15 - 14 - 1)(1) = 0$$

$$i_2(0) = (7 - 8.4 + 1.4)(1) = 0$$

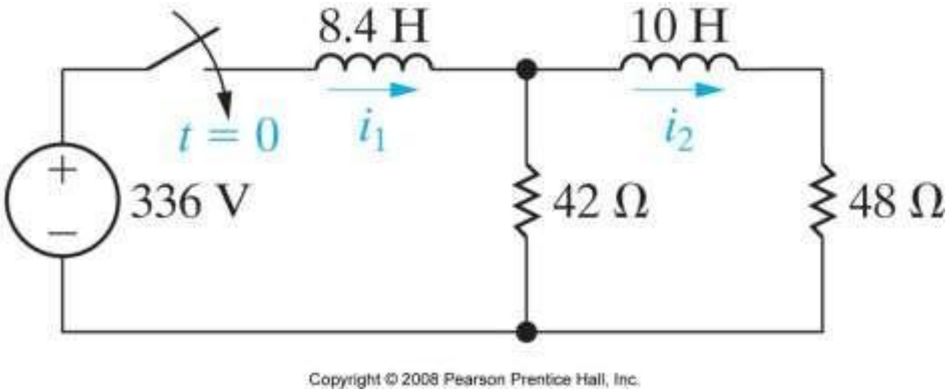
As $t \to \infty$, the inductors behave like

A. Inductors
B. Open circuits
C. Short circuits



LaPlace Transform in Circuit Analysis

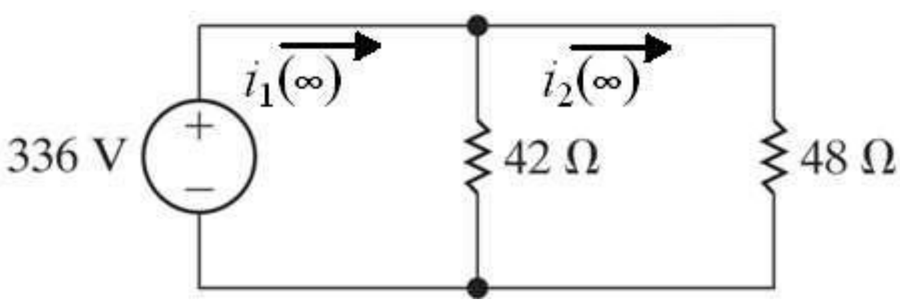
Example:
There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.



$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A \quad \Rightarrow \quad i_1(\infty) = 15 - 0 - 0 = 15\,\mathrm{A}$$

$$i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A \quad \Rightarrow \quad i_2(\infty) = 7 - 0 - 0 = 7\,\mathrm{A}$$

Draw the circuit for $t = \infty$ and check these solutions.



$$42 \,||\, 48 = 22.4\Omega$$

$$i_1(\infty) = \frac{336}{22.4} = 15\,\mathrm{A(check!)}$$

$$i_2(\infty) = \frac{22.4}{48}(15) = 7\,\mathrm{A(check!)}$$

LaPlace Transform in Circuit Analysis

We can also check the initial and final values in the s-domain, before we begin the process of inverse-Laplace transforming our s-domain solutions. To do this, use the **Initial Value Theorem (IVT)** and the **Final Value Theorem (FVT)**.

- The initial value theorem:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

This theorem is valid if and only if f(t) has no impulse functions.

- The final value theorem:

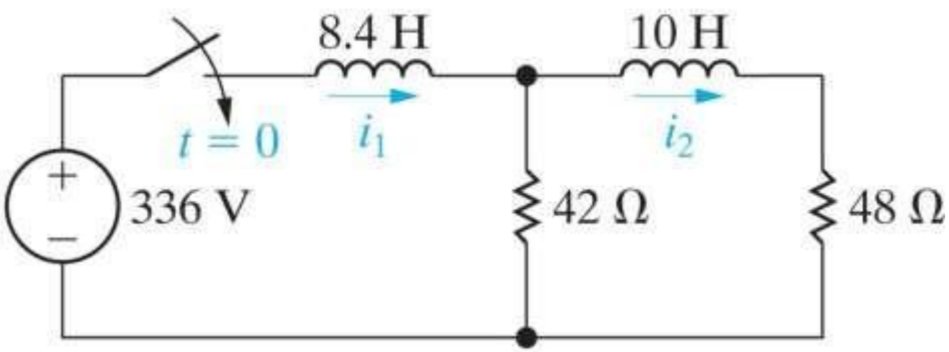
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

This theorem is valid if and only if all but one of the poles of F(s) are in the left-half of the complex plane, and the one that is not can only be at the origin.

LaPlace Transform in Circuit Analysis

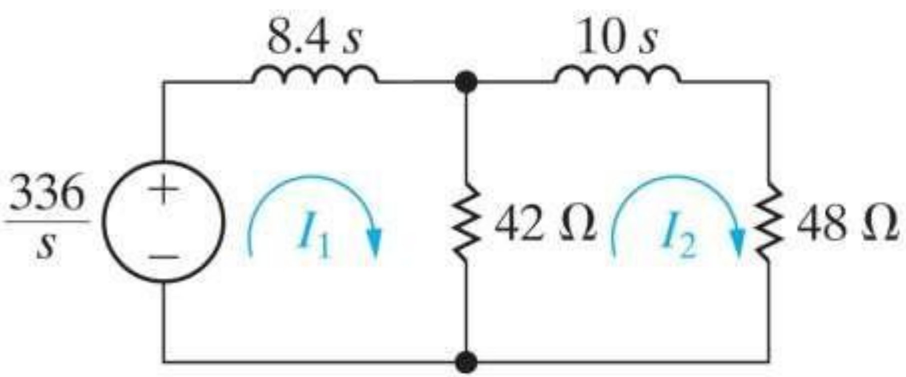
Example:

There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.



$$I_1(s) = \frac{40s + 360}{s^3 + 14s^2 + 24s}$$

$$I_2(s) = \frac{168}{s^3 + 14s^2 + 24s}$$

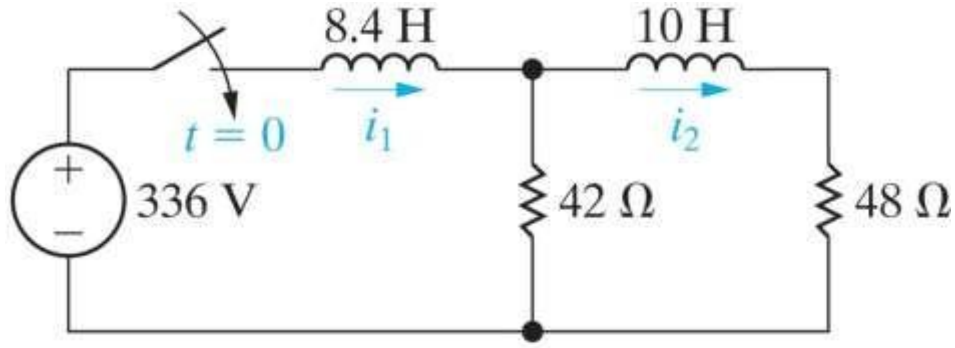


Check your answers using the IVT and the FVT.

LaPlace Transform in Circuit Analysis

IVT:

From the circuit, $i_1(0) = 0$ and $i_2(0) = 0$.



$$I_1(s) = \frac{40s + 360}{s^3 + 14s^2 + 24s}$$

$$\lim_{t \to 0} i_1(t) = \lim_{s \to \infty} sI_1(s)$$

$$= \lim_{s \to \infty} \frac{40s^2 + 360s}{s^3 + 14s^2 + 24s}$$

$$= \lim_{1/s \to 0} \frac{(40/s) + (360/s^2)}{1 + (14/s) + (24/s^2)}$$

$$= 0\,\mathrm{A(check!)}$$

$$I_2(s) = \frac{168}{s^3 + 14s^2 + 24s}$$

$$\lim_{t \to \infty} i_1(t) = \lim_{s \to \infty} sI_1(s)$$

$$= \lim_{s \to \infty} \frac{168s}{s^3 + 14s^2 + 24s}$$

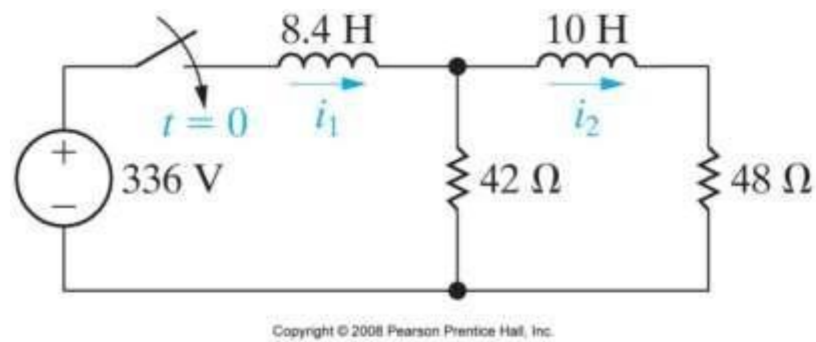
$$= \lim_{1/s \to 0} \frac{(168/s^2)}{1 + (14/s) + (24/s^2)}$$

$$= 0\,\mathrm{A(check!)}$$

LaPlace Transform in Circuit Analysis

FVT:

From the circuit, $i_1(\infty) = 15$ A and $i_2(\infty) = 7$ A.



$$I_1(s) = \frac{40s + 360}{s^3 + 14s^2 + 24s}$$

$$\lim_{t\to\infty} i_1(t) = \lim_{s\to 0} sI_1(s)$$

$$= \lim_{s\to 0} \frac{40s^2 + 360s}{s^3 + 14s^2 + 24s}$$

$$= \lim_{s\to 0} \frac{40s + 360}{s^2 + 14s + 24}$$

$$= \frac{360}{24} = 15\text{A(check!)}$$

$$I_2(s) = \frac{168}{s^3 + 14s^2 + 24s}$$

$$\lim_{t\to\infty} i_1(t) = \lim_{s\to 0} sI_1(s)$$

$$= \lim_{s\to 0} \frac{168s}{s^3 + 14s^2 + 24s}$$

$$= \lim_{s\to 0} \frac{168}{s^2 + 14s + 24}$$

$$= \frac{168}{24} = 7\text{A(check!)}$$

LaPlace Transform in Circuit Analysis

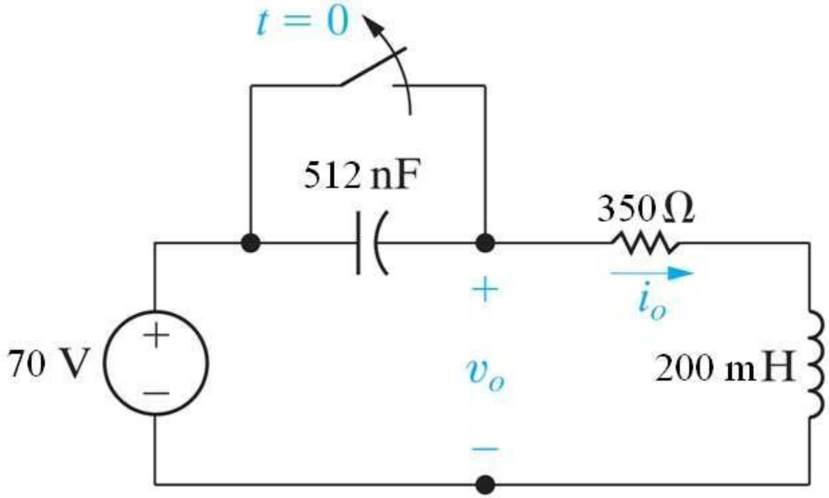
Recipe for Laplace transform circuit analysis:

1. Redraw the circuit (nothing about the Laplace transform changes the types of elements or their interconnections).
2. Any voltages or currents with values given are Laplace-transformed using the functional and operational tables.
3. Any voltages or currents represented symbolically, using i(t) and v(t), are replaced with the symbols I(s) and V(s).
4. All component values are replaced with the corresponding complex impedance, Z(s).
5. Use DC circuit analysis techniques to write the s-domain equations and solve them. Check your solutions with IVT and FVT.
6. Inverse-Laplace transform s-domain solutions to get time-domain solutions. Check your solutions at $t = 0$ and $t = \infty$.

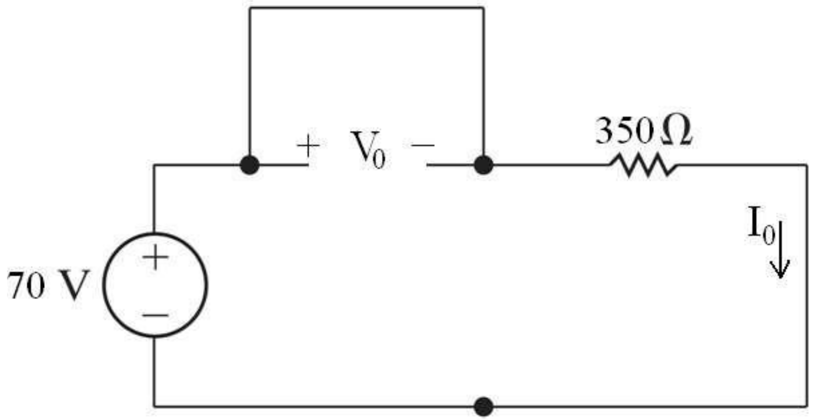
LaPlace Transform in Circuit Analysis

Example:

Find $v_0(t)$ for $t > 0$.



Begin by finding the initial conditions for this circuit.

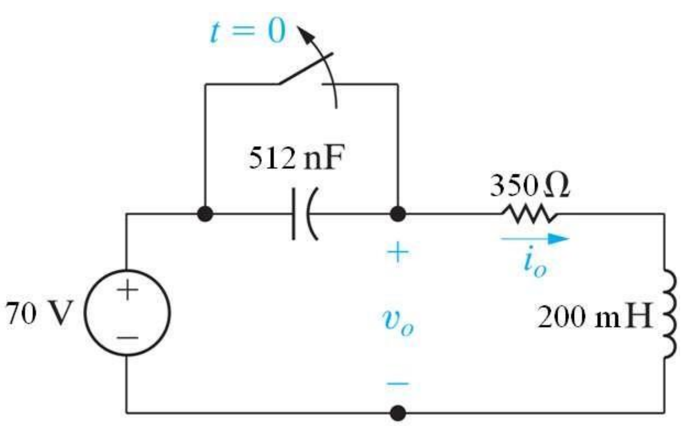


$$V_o = 0\,\mathrm{V}$$

$$\mathrm{I_o} = \frac{70}{350} = 0.2\,\mathrm{A}$$

Give the basic interconnections of this circuit, should we use a voltage source or a current source to represent the initial condition for the inductor?

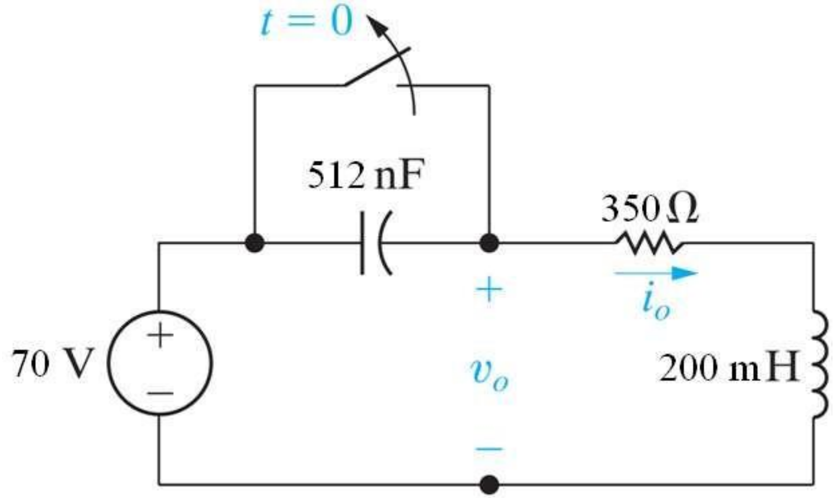
- A. Voltage source
- B. Current source
- C. Doesn't matter



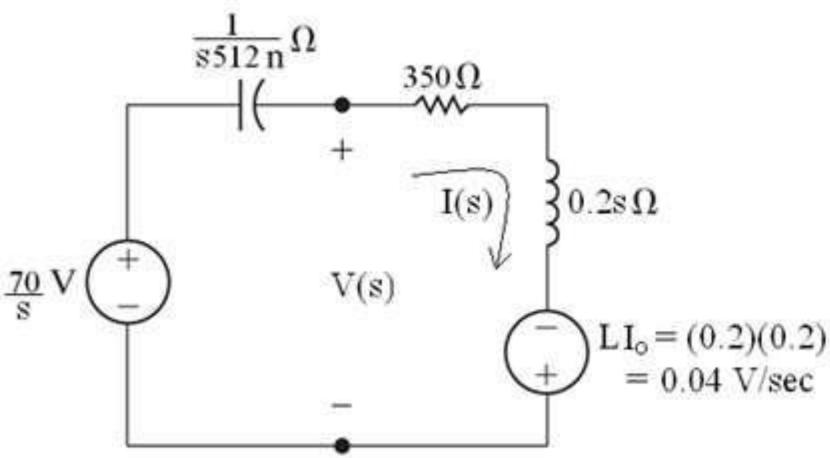
LaPlace Transform in Circuit Analysis

Example:

Find $v_0(t)$ for $t > 0$.



Laplace transform the circuit and solve for $V_0(s)$.



$$I(s) = \frac{70/s + 0.04}{1/s(512\text{n}) + 350 + 0.2s}$$

$$V(s) = (350 + 0.2s)I(s) - 0.04$$

$$= \frac{(350 + 0.2s)(70/s + 0.04)}{1/s(512\text{n}) + 350 + 0.2s} - 0.04$$

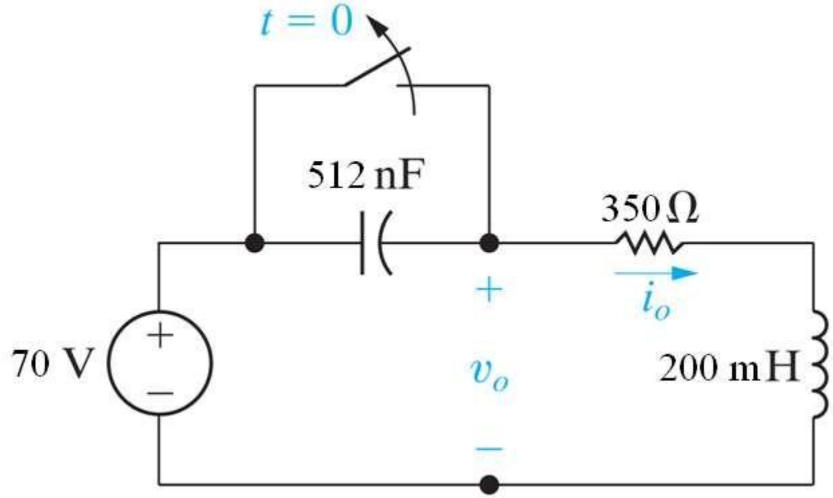
$$= \frac{70s - 268{,}125}{s^2 + 1750s + 9{,}765{,}625}$$

LaPlace Transform in Circuit Analysis

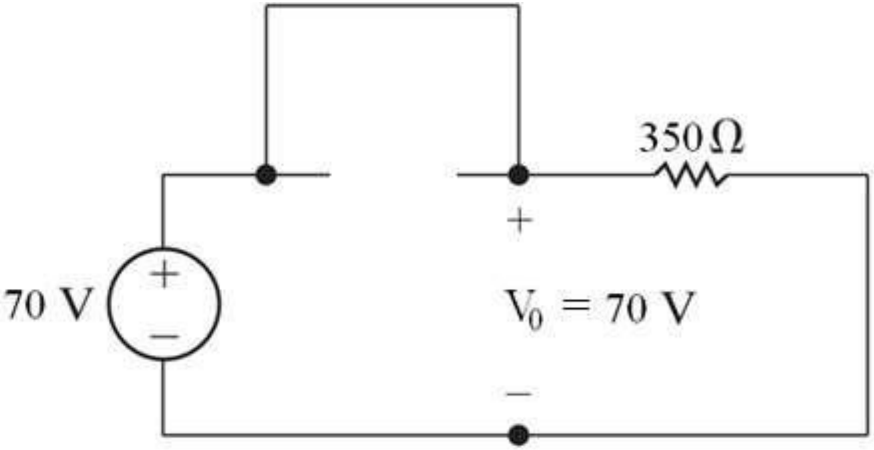
Example:

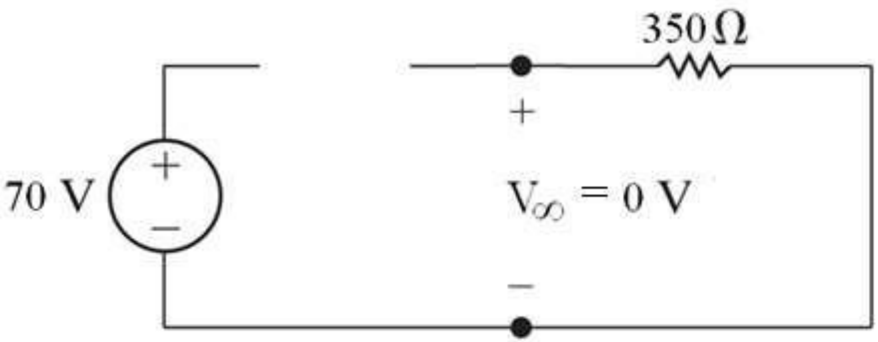
Find $v_0(t)$ for $t > 0$.

$$V_0(s) = \frac{70s - 268{,}125}{s^2 + 1750s + 9{,}765{,}625}$$

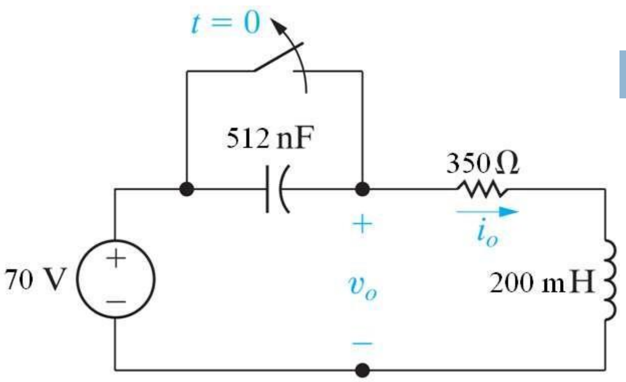


Use the IVT and FVT to check $V_0(s)$.





LaPlace Transform in Circuit Analysis



Example:

Find $v_0(t)$ for $t > 0$.

IVT

$$V_0(s) = \frac{70s - 268{,}125}{s^2 + 1750s + 9{,}765{,}625}$$

$$\lim_{t\to 0} v_o(t) = \lim_{s\to\infty} sV_o(s)$$

$$= \lim_{s\to\infty} \frac{70s^2 - 268{,}125s}{s^2 + 1750s + 9{,}765{,}625}$$

$$= \lim_{1/s\to 0} \frac{70 - 268{,}125/s}{1 + 1750/s + 9{,}765{,}625/s^2}$$

$$= \frac{70}{1} = 70\text{ V(check!)}$$

FVT

$$V_0(s) = \frac{70s - 268{,}125}{s^2 + 1750s + 9{,}765{,}625}$$

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} sV_o(s)$$

$$= \lim_{s\to 0} \frac{70s^2 - 268{,}125s}{s^2 + 1750s + 9{,}765{,}625}$$

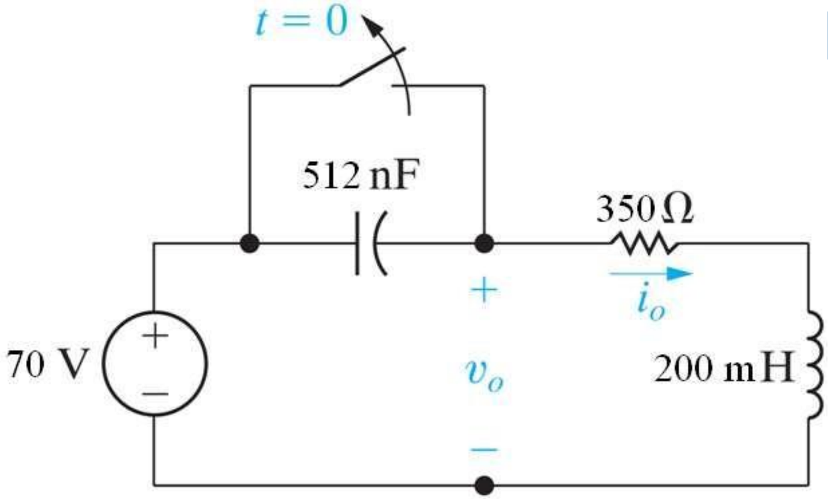
$$= \lim_{s\to 0} \frac{0}{9{,}765{,}625}$$

$$= 0\text{ V(check!)}$$

LaPlace Transform in Circuit Analysis

Example:

Find $v_0(t)$ for $t > 0$.



$$V_0(s) = \frac{70s - 268{,}125}{(s + 875 - j3000)(s + 875 + j3000)}$$

Partial fraction expansion:

$$V_0(s) = \frac{K_1}{(s + 875 - j3000)} + \frac{K_2}{(s + 875 + j3000)}$$

$$K_1 = \left.\frac{70s - 268{,}125}{(s + 875 + j3000)}\right|_{s=-875+j3000} = \frac{70(-875 + j3000) - 268{,}125}{[(-875 + j3000) + 875 + j3000]} = 65.1\angle 57.48°$$

$$K_2 = \left.\frac{70s - 268{,}125}{(s + 875 - j3000)}\right|_{s=-875-j3000} = \frac{70(-875 - j3000) - 268{,}125}{[(-875 - j3000) + 875 + - j3000]} = 65.1\angle - 57.48°$$

When two partial fraction denominators are complex conjugates, their numerators are

A. Equal
B. Unrelated
C. Complex conjugates

LaPlace Transform in Circuit Analysis

Aside – look at the inverse Laplace transform of partial fractions that are complex conjugates.

$$F(s)=\frac{10s}{s^2+2s+5}=\frac{K_1}{s+1-j2}+\frac{K_1^*}{s+1+j2}$$

$$K_1=\left.\frac{10s}{s+1+j2}\right|_{s=-1+j2}=\frac{10(-1+j2)}{-1+j2+1+j2}=5.59\angle 26.57°$$

$$\therefore \qquad F(s)=\frac{5.59\angle 26.57°}{s+1-j2}+\frac{5.59\angle -26.57°}{s+1+j2}$$

$$\Rightarrow \qquad f(t)=5.59e^{j26.57°}e^{-(1-j2)t}+5.59e^{-j26.57°}e^{-(1+j2)t}$$

$$=5.59e^{-t}e^{j(2t+26.57°)}+5.59e^{-t}e^{-j(2t+26.57°)}$$

$$=5.59e^{-t}[\cos(2t+26.57°)+j\sin(2t+26.57°)]$$

$$+5.59e^{-t}[\cos(2t+26.57°)-j\sin(2t+26.57°)]$$

$$=2(5.59)e^{-t}\cos(2t+26.57°)$$

LaPlace Transform in Circuit Analysis

The parts of the time-domain expression come from a single partial fraction term:

$$F(s) = \frac{5.59\angle 26.57^\circ}{s+1-j2} + \frac{5.59\angle -26.57^\circ}{s+1+j2}$$

$$f(t) = 2(5.59)e^{-t}\cos(2t + 26.57^\circ)$$

Important – you must use the numerator of the partial fraction whose denominator has the negative imaginary part!

LaPlace Transform in Circuit Analysis

The general Laplace transform (from the table below the "Functional Transforms" table)

$$F(s) = \frac{|K|\angle\theta}{s+a-jb} + \frac{|K|\angle-\theta}{s+a-jb}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = 2|K|e^{-at}\cos(bt+\theta)$$

$$V_0(s) = \frac{65.1\angle 57.48^\circ}{(s+875-j3000)} + \frac{65.1\angle -57.48^\circ}{(s+875+j3000)}$$

The partial fraction expansion for $V_0(s)$ is shown above. When we inverse-Laplace transform, which partial fraction term should we use?

A. The first term
B. The second term
C. It doesn't matter

$$V_0(s) = \frac{65.1\angle 57.48^\circ}{(s+875-j3000)} + \frac{65.1\angle -57.48^\circ}{(s+875+j3000)}$$

The time-domain function for $v_o(t)$ will include a cosine at what frequency?

A. 875 rad/s

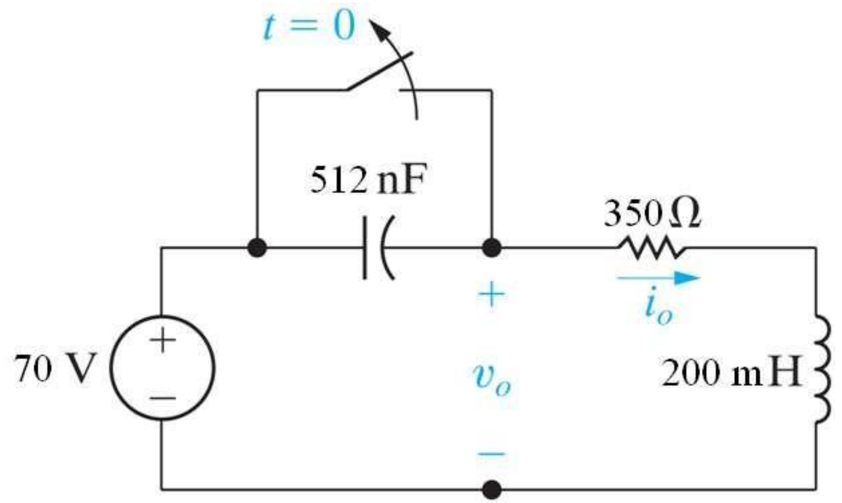
B. 130.2 rad/s

C. 3000 rad/s

LaPlace Transform in Circuit Analysis

Example:

Find $v_0(t)$ for $t > 0$.



$$V_0(s) = \frac{65.1\angle 57.48°}{(s+875-j3000)} + \frac{65.1\angle -57.48°}{(s+875+j3000)}$$

Inverse Laplace transform:

$$v_0(t) = 2(65.1)e^{-875t}\cos(3000t + 57.48°) = 130.2e^{-875t}\cos(3000t + 57.48°)\text{ V}$$

Check at $t = 0$ and $t \to \infty$:

$$v_0(0) = 130.2(1)\cos(57.48°) = 70\text{ V}$$

$$v_0(\infty) = 130.2(0)\cos(\ldots) = 0\text{ V}$$

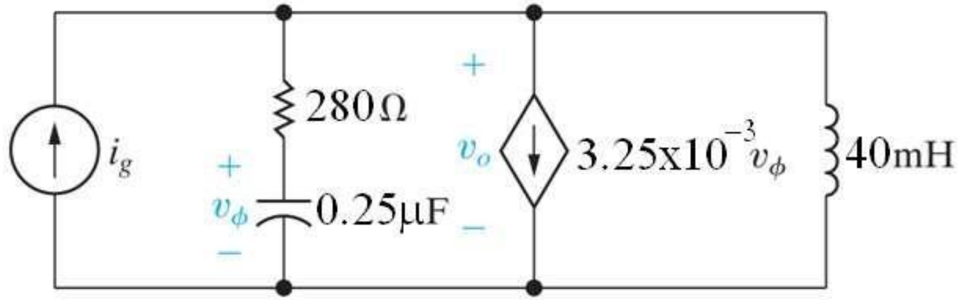
This example is a series RLC circuit. Its response form, repeated below, is characterized as:

$$v_0(t) = 130.2e^{-875t}\cos(3000t + 57.48°)\ \text{V}$$

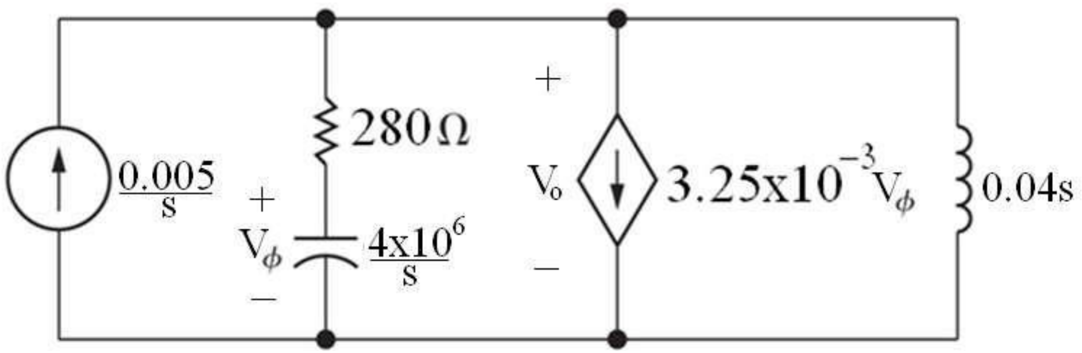
A. Underdamped
B. Overdamped
C. Critically damped

LaPlace Transform in Circuit Analysis

Example:
There is no initial energy stored in this circuit.
Find v_o if $i_g = 5u(t)$ mA.

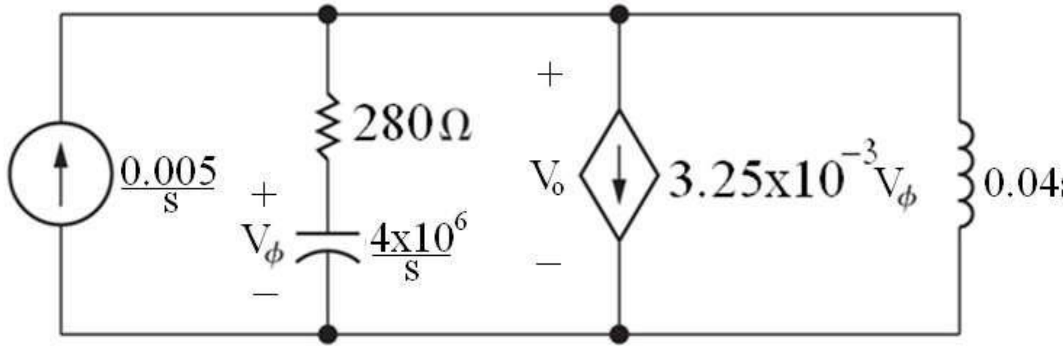


Laplace transform the circuit:



LaPlace Transform in Circuit Analysis

Example:
Find $V_o(s)$:



$$-\frac{0.005}{s}+\frac{V_o}{280+4\times10^6/s}+3.25\times10^{-3}V_\phi+\frac{V_o}{0.04s}=0 \qquad \text{KCL at topnode}$$

$$V_\phi=\frac{4\times10^6/s}{280+4\times10^6/s}V_o=\frac{\square_o 4\times10^6 V}{280s+4\times10^6} \qquad \text{voltage division}$$

$$\therefore \qquad V_o\left[\frac{s}{280s+4\times10^6}+\frac{13{,}000}{280s+4\times10^6}+\frac{25}{s}\right]=\frac{0.005}{s}$$

$$\Rightarrow \qquad V_o\left[\frac{s^2+13{,}000s+25\left(280s+4\times10^6\right)}{s\left(280s+4\times10^6\right)}\right]=\frac{0.005}{s}$$

$$\Rightarrow \qquad V_o=\frac{1.4s+20{,}000}{s^2+20{,}000s+10^8}$$

$$V_o = \frac{1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8}$$

This s-domain expression has_____ zeros and_____ poles.

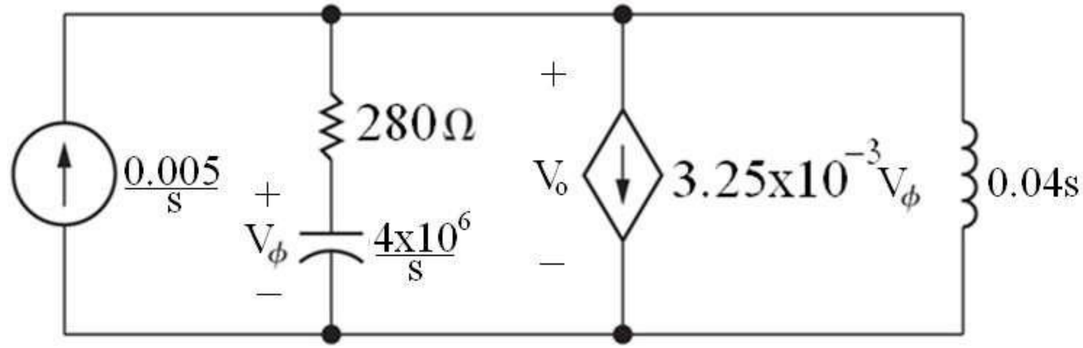
A. 0, 2

B. 1, 2

C. 2, 1

LaPlace Transform in Circuit Analysis

Example:
Check your s-domain answer:



IVT

$$V_0(s) = \frac{\square 1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8}$$

$$\lim_{t \to 0} v_0(t) = \lim_{s \to \infty} sF(s)$$

$$= \lim_{s \to \infty} \frac{1.4s^2 + 20{,}000s}{s^2 + 20{,}000s + 10^8}$$

$$= \lim_{1/s \to 0} \frac{1.4 + 20{,}000/s}{1 + 20{,}000/s + 10^8/s^2}$$

$$= 1.4\,\mathrm{V}$$

FVT

$$V_0(s) = \frac{1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8}$$

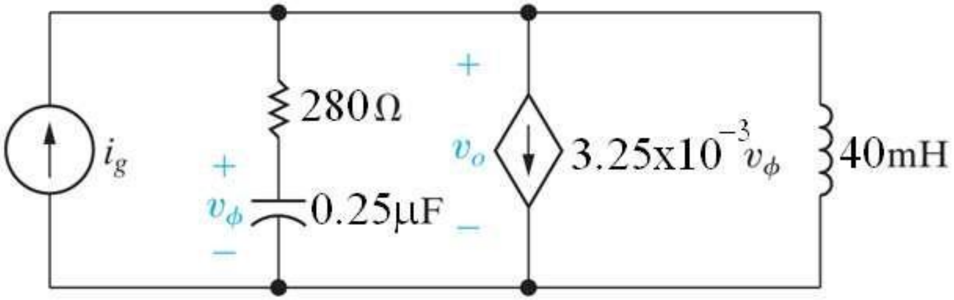
$$\lim_{t \to \infty} v_0(t) = \lim_{s \to 0} sF(s)$$

$$= \lim_{s \to 0} \frac{1.4s^2 + 20{,}000s}{s^2 + 20{,}000s + 10^8}$$

$$= \frac{0}{10^8} = 0\,\mathrm{V}$$

Warning – this one's tricky!

Just after t = 0, there is no initial stored energy in the circuit. Therefore, the capacitor behaves like a ______and the inductor behaves like a ____.



A. Open circuit/short circuit

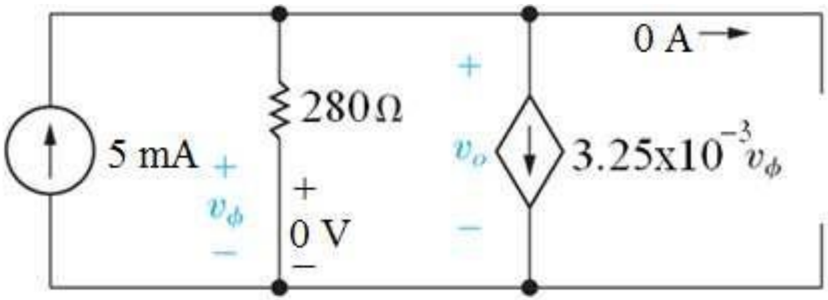
B. Open circuit/open circuit

C. Short circuit/short circuit

D. Short circuit/open circuit

LaPlace Transform in Circuit Analysis

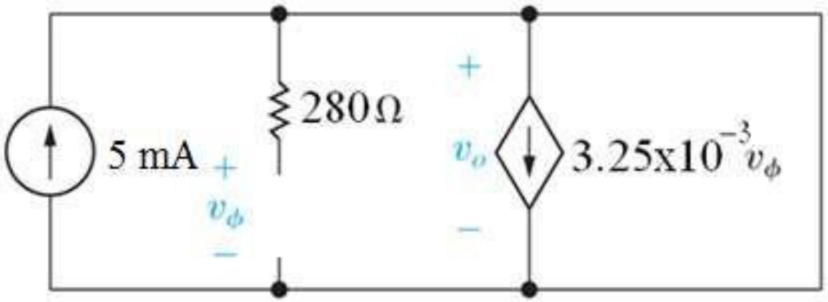
For t = 0



$$v_0(0) = (0.005)(280)$$

$$= 1.4\,\text{V (check!)}$$

For t $\rightarrow \infty$

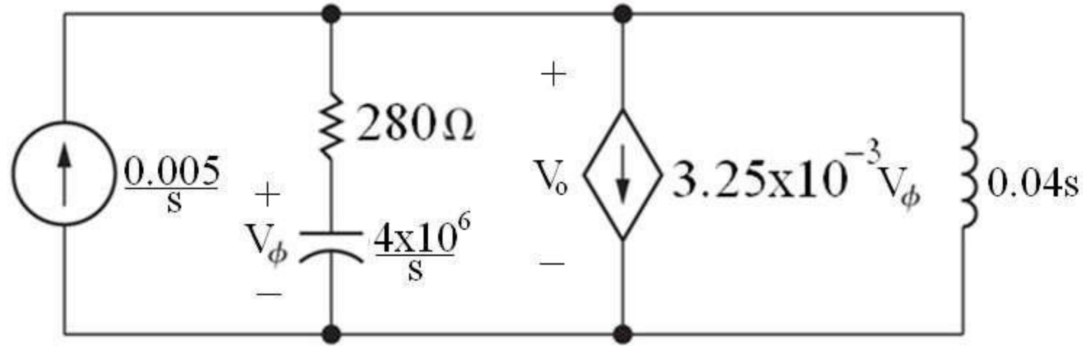


$$v_0(0) = 0\,\text{V}$$

(it is the voltage across a wire!)

LaPlace Transform in Circuit Analysis

Example:
Partial fraction expansion:



$$V_0(s) = \frac{1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8} = \frac{1.4s + 20{,}000}{(s + 10{,}000)^2}$$

$$= \frac{K_1}{(s + 10{,}000)^2} + \frac{K_2}{(s + 10{,}000)}$$

$$V_0(s) = \frac{K_1}{(s+10{,}000)^2} + \frac{K_2}{(s+10{,}000)}$$

In the partial fraction expansion given here, K_1 and K_2 are

A. Both real numbers

B. Complex conjugates

C. Need more information

LaPlace Transform in Circuit Analysis

Aside – find the partial fraction expansion when there are repeated real roots.

$$F(s) = \frac{4s^2 + 7s + 1}{s(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$$

$$K_1 = \left.\frac{4s^2 + 7s + 1}{(s+1)^2}\right|_{s=0} = \frac{1}{1} = 1$$

$$K_2 = \left.\frac{4s^2 + 7s + 1}{s}\right|_{s=-1} = \frac{4 - 7 + 1}{-1} = 2$$

$$K_3 = \left.\frac{4s^2 + 7s + 1}{s(s+1)}\right|_{s=-1} = \frac{4 - 7 + 1}{(-1)(0)} = \text{undefined!}$$

LaPlace Transform in Circuit Analysis

Aside – find the partial fraction expansion when there are repeated real roots. How do we find the coefficient of the term with just one copy of the repeated root?

$$(s+1)^2 F(s) = \frac{K_1 (s+1)^2}{s} + \frac{K_2 (s+1)^2}{(s+1)^2} + \frac{K_3 (s+1)^2}{s+1}$$

Eliminate these two terms

Keep this term!

$$\frac{d}{ds}\left[(s+1)^2 F(s)\right]\Big|_{s=-1} = \frac{d}{ds}\left[\frac{K_1 (s+1)^2}{s}\right]\Bigg|_{s=-1} + \frac{d}{ds}\left[\frac{K_2 (s+1)^2}{(s+1)^2}\right]\Bigg|_{s=-1} + \frac{d}{ds}\left[\frac{K_3 (s+1)^2}{s+1}\right]\Bigg|_{s=-1}$$

= 0 because the derivative still has (s+1) in the numerator

= 0 because the derivative of a constant is 0

= K_3 because the derivative of $K_3(s+1)$ is K_3

LaPlace Transform in Circuit Analysis

Aside – find the partial fraction expansion when there are repeated real roots.

$$F(s) = \frac{4s^2 + 7s + 1}{s(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{(s+1)^2} + \frac{K_3}{s+1}$$

$$K_1 = \left.\frac{4s^2 + 7s + 1}{(s+1)^2}\right|_{s=0} = \frac{4(0)^2 + 7(0) + 1}{(0+1)} = 1$$

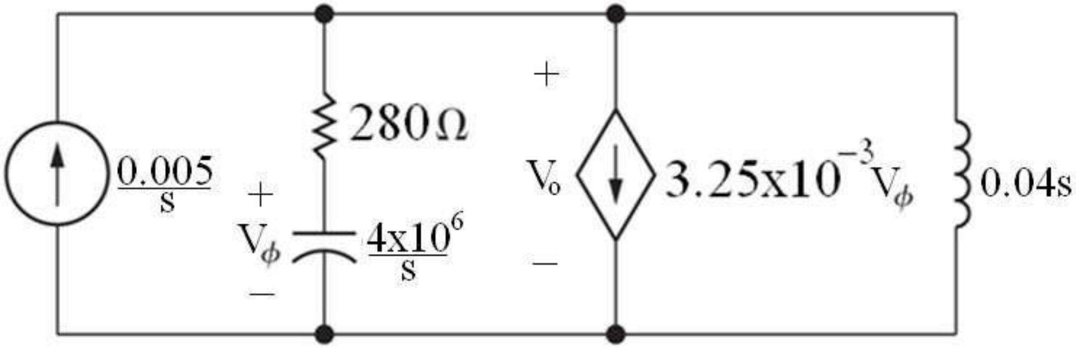
$$K_2 = \left.\frac{4s^2 + 7s + 1}{s}\right|_{s=-1} = \frac{4(-1)^2 + 7(-1) + 1}{(-1)} = 2$$

$$K_3 = \left.\frac{d}{ds}\left[\frac{4s^2 + 7s + 1}{s}\right]\right|_{s=-1} = \left.\left[\frac{8s + 7}{s} - \frac{4s^2 + 7s + 1}{s^2}\right]\right|_{s=-1}$$

$$= \frac{8(-1) + 7}{(-1)} - \frac{4(-1)^2 + 7(-1) + 1}{(-1)^2} = 3$$

LaPlace Transform in Circuit Analysis

Back to the example; find the partial fraction expansion:



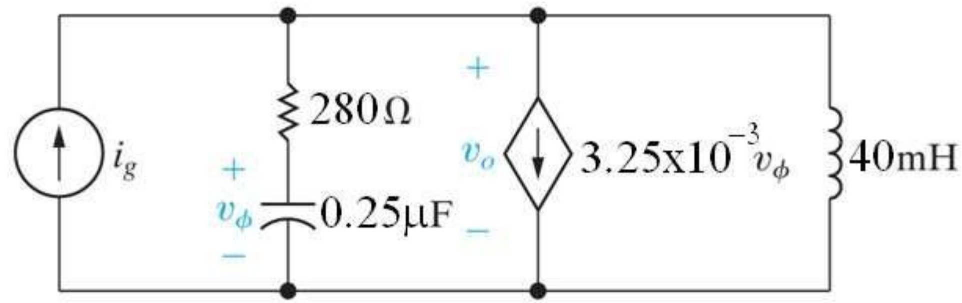
$$V_0(s) = \frac{1.4s + 20{,}000}{(s+10{,}000)^2} = \frac{K_1}{(s+10{,}000)^2} + \frac{K_2}{(s+10{,}000)}$$

$$K_1 = 1.4s + 20{,}000\Big|_{s=-10{,}000} = 6000$$

$$K_2 = \frac{d}{ds}\left[1.4s + 20{,}000\right]\Big|_{s=-10{,}000} = 1.4$$

LaPlace Transform in Circuit Analysis

Example:
Find $v_0(t)$ for $t > 0$.



Inverse Laplace transform the result in the s-domain to get the time-domain result:

$$V_0(s) = \frac{6000}{(s+10{,}000)^2} + \frac{1.4}{(s+10{,}000)}$$

$$v_0\,(t) = \left[6000te^{-10{,}000t} + 1.4e^{-10{,}000t}\right]u(t)\,\text{V (see the Laplace tables)}$$

$$v_0(0) = 1.4\,\text{V (check!)}$$

$$v_0(\infty) = 0\,\text{V (check!)}$$

$$v_o(t) = [6000te^{-10,000t} + 1.4e^{-10,000t}]u(t) \text{ V}$$

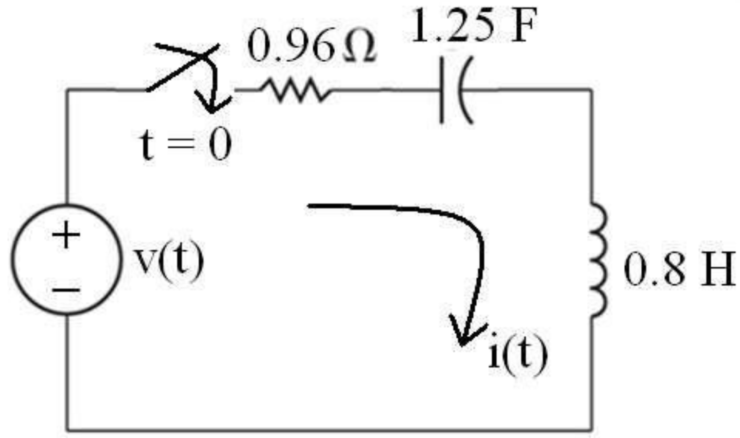
We have seen this response form in our analysis of second-order RLC circuits; it is called:

A. Overdamped
B. Underdamped
C. Critically damped

LaPlace Transform in Circuit Analysis

Example:

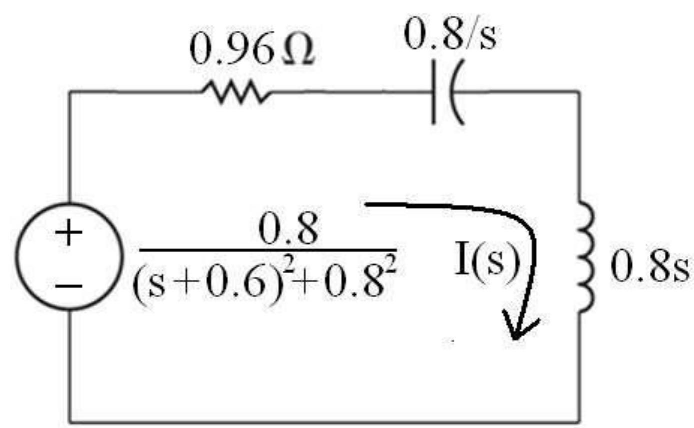
There is no initial energy stored in this circuit. Find i(t) if $v(t) = e^{-0.6t}\sin 0.8t$ V.



Laplace transform the circuit:

$$\mathcal{L}\left[e^{-0.6t}\sin 0.8t\right] = \frac{0.8}{(s+0.6)^2+0.8^2}$$

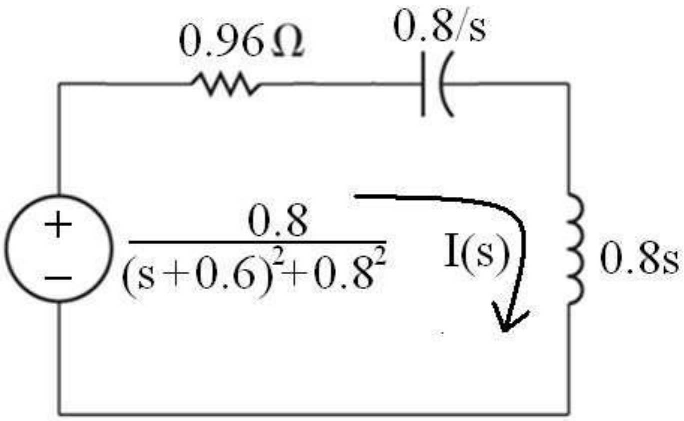
$$= \frac{0.8}{s^2+1.2s+1}$$



LaPlace Transform in Circuit Analysis

Example:

Find I(s):



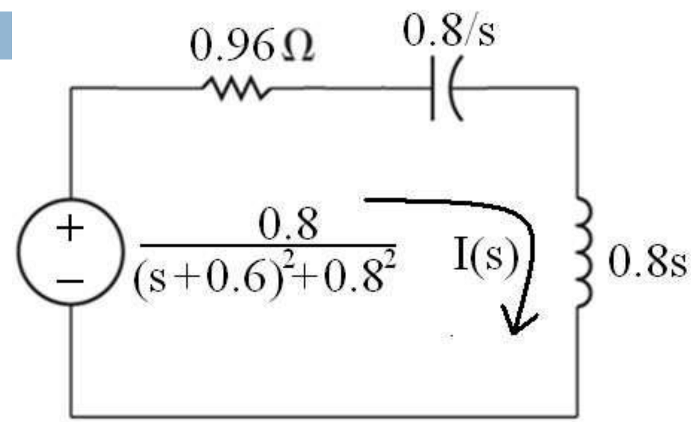
$$\left(0.96+\frac{0.8}{s}+0.8s\right)I(s)=\frac{0.8}{s^2+1.2s+1}$$

$$\therefore\left(\frac{0.8s^2+0.96s+0.8}{s}\right)I(s)=\frac{0.8}{s^2+1.2s+1}$$

$$\Rightarrow \qquad I(s)=\frac{s}{\left(s^2+1.2s+1\right)}$$

LaPlace Transform in Circuit Analysis

Example:
Check your s-domain answer:



IVT

$$I(s) = \frac{s}{(s^2 + 1.2s + 1)^2}$$

$$\lim_{t \to 0} i(t) = \lim_{s \to \infty} sI(s)$$

$$= \lim_{s \to \infty} \frac{s^2}{(s^2 + 1.2s + 1)^2}$$

$$= \lim_{1/s \to 0} \frac{1/s^2}{(1 + 1.2/s + 1/s^2)^2} = 0$$

FVT

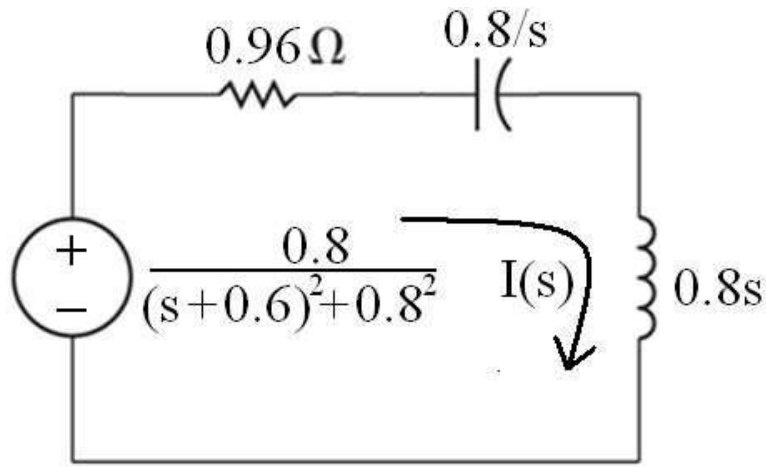
$$I(s) = \frac{s}{(s^2 + 1.2s + 1)^2}$$

$$\lim_{t \to \infty} i(t) = \lim_{s \to 0} sI(s)$$

$$= \lim_{s \to 0} \frac{s^2}{(s^2 + 1.2s + 1)^2} = 0$$

LaPlace Transform in Circuit Analysis

Example:
Partial fraction expansion:

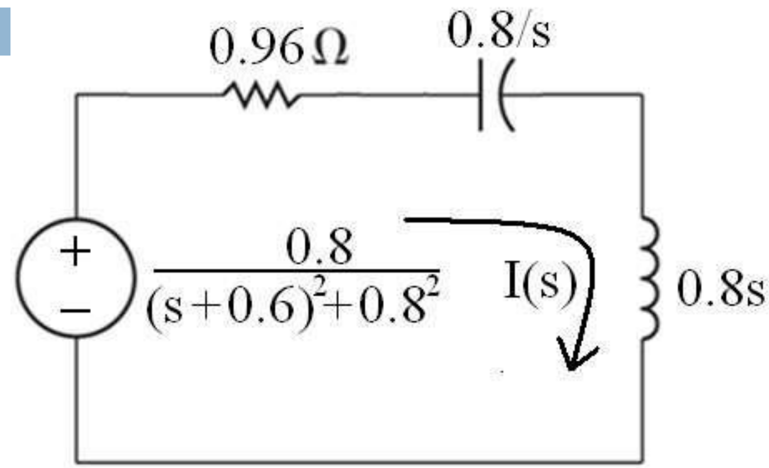


$$I(s) = \frac{s}{(s^2+1.2s+1)^2} = \frac{K_1}{(s+0.6-j0.8)^2} + \frac{K_2}{(s+0.6-j0.8)}$$

$$+\frac{K_1^*}{(s+0.6+j0.8)^2} + \frac{K_2^*}{(s+0.6+j0.8)}$$

LaPlace Transform in Circuit Analysis

Partial fraction expansion, continued:



$$I(s) = \frac{K_1}{(s+0.6-j0.8)^2} + \frac{K_2}{(s+0.6-j0.8)} + \ldots$$

$$K_1 = \left.\frac{s}{(s+0.6+j0.8)^2}\right|_{s=-0.6+j0.8} = \frac{-0.6+j0.8}{(-0.6+j0.8+0.6+j0.8)^2} = 0.39\angle -53.13^\circ$$

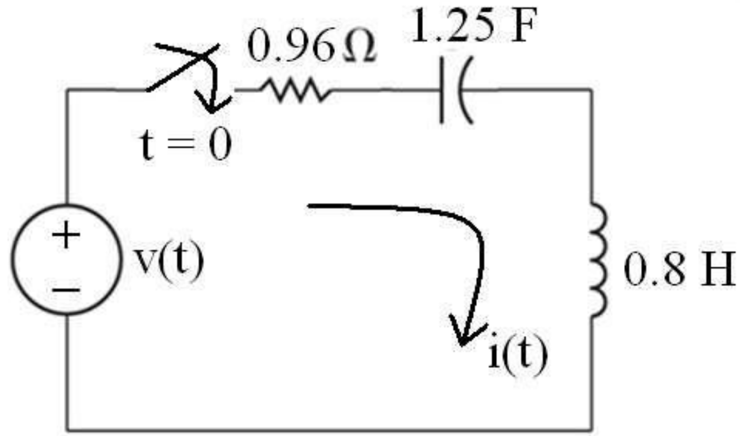
$$K_2 = \frac{d}{ds}\left[\frac{s}{(s+0.6+j0.8)^2}\right] = \left[\frac{1}{(s+0.6+j0.8)^2} - \left[\frac{2s}{(s+0.6+j0.8)^3}\right]\right]_{s=-0.6+j0.8}$$

$$= \frac{1}{[2(j0.8)]^2} - \frac{2(-0.6+j0.8)}{[2(j0.8)]^3} = 0.29\angle -90^\circ$$

LaPlace Transform in Circuit Analysis

Example:

There is no initial energy stored in this circuit. Find i(t) if v(t) = $e^{-0.6t}\sin 0.8t$ V.



Inverse Laplace transform the result in the s-domain to get the time-domain result:

$$I(s) = \frac{0.39\angle -53.13^\circ}{(s+0.6-j0.8)^2} + \frac{\square 0.29\angle 90^\circ}{(s+0.6-j0.8)} + \ldots$$

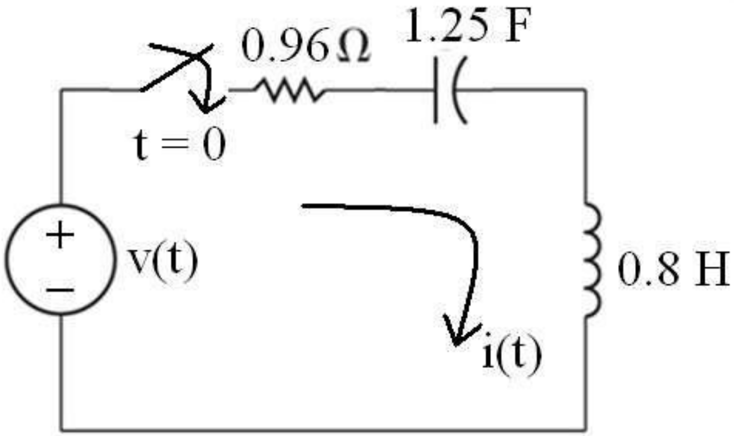
$$i(t) = 2(0.39)te^{-0.6t}\cos(0.8t - 53.13^\circ) + 2(0.29)e^{-0.6t}\cos(0.8t + 90^\circ)$$

$$= \left[0.78te^{-0.6t}\cos(0.8t - 53.13^\circ) + 0.58e^{-0.6t}\cos(0.8t + 90^\circ)\right]u(t)\ \text{A}$$

Which term of the solution represents the forced response?

Example:

There is no initial energy stored in this circuit. Find i(t) if v(t) = $e^{-0.6t}\sin 0.8t$ V.



$$i(t) = [0.78te^{-0.6t}\cos(0.8t - 53.13°) + 0.58e^{-0.6t}\cos(0.8t + 90°)]u(t)\ \text{A}$$

A. First term

B. Second term

C. Neither

LaPlace Transform in Circuit Analysis

Recipe for Laplace transform circuit analysis:

1. Redraw the circuit – note that you need to find the initial conditions and decide how to represent them in the circuit.
2. Any voltages or currents with values given are Laplace-transformed using the functional and operational tables.
3. Any voltages or currents represented symbolically, using i(t) and v(t), are replaced with the symbols I(s) and V(s).
4. All component values are replaced with the corresponding complex impedance, Z(s), and the appropriate source representing initial conditions.
5. Use DC circuit analysis techniques to write the s-domain equations and solve them. Check your solutions with IVT and FVT.
6. Inverse-Laplace transform s-domain solutions (using the partial fraction expansion technique and the Laplace tables) to get time-domain solutions. Check your solutions at $t = 0$ and $t = \infty$.

LaPlace Transform in Circuit Analysis

Aside – How do you inverse Laplace transform F(s) if it is an improper rational function? (Note – this won't happen in linear circuits, but can happen in other systems modeled with differential equations!)

Example:

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+7}{(s+1)(s+2)}\right\} \qquad \text{(Note: } O\{D(s)\} > O\{N(s)\} \text{ does not hold!)}$$

See next slide!

LaPlace Transform in Circuit Analysis

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+7}{(s+1)(s+2)}\right\}$$

(Note: $O\{D(s)\} > O\{N(s)\}$ does not hold!)

$$\begin{array}{r} 1 \\ s^2+3s+2 \overline{\big) s^2+6s+7} \\ \underline{-\;s^2+3s+2} \\ 3s+5 \end{array}$$

$$\Rightarrow \frac{s^2+6s+7}{(s+1)(s+2)} = 1 + \frac{3s+5}{(s+1)(s+2)} = 1 + \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

$$K_1 = \left.\frac{3s+5}{(s+2)}\right|_{s=-1} = 2; \qquad K_2 = \left.\frac{3s+5}{(s+1)}\right|_{s=-2} = 1$$

$$\mathcal{L}^{-1}\left\{1+\frac{2}{(s+1)}+\frac{1}{(s+2)}\right\} = \delta(t) + \left[2e^{-t}+e^{-2t}\right]u(t)$$