Unit-II

OPERATIONS RESEARCH FOR BUSINESS DECISIONS

MMS-608

By – Ranu Singh

Department of Mathematics & Scientific Computing,
Madan Mohan Malaviya University of Technology,
Gorakhpur (UP), India-273010
Email: ranusinghmmmut@gmail.com

Syllabus (unit II)

(Transportation problem)

- Types of transportation problem
- Mathematical model
- Transportation algorithm

(Assignment problem)

- Allocation and assignment problems and models
- Processing of job through machines

ABOUT TRANSPORTATION PROBLEM

- Transportation problem talks about transporting items from a given set of supply point to a given set of destinations point.
- A transportation problem when expressed in terms of an LP model can also be solved by the simplex method.
- The structure of transportation problem involves a large number of shipping routes from several supply centres to several demand centres. Thus, objective is to determine shipping routes between supply centres and demand centres in order to satisfy the required quantity of goods or services at each destination centre, with available quantity of goods or services at each supply centre at the minimum transportation cost and / time.
- Transportation problem is of two types:

a) Balanced transportation problem: (total demand = total supply)

b) Unbalanced transportation problem: (total demand ≠ total supply)

MATHEMATICAL MODEL OF TRANSPORTATION PROBLEM

Example 1: A company has three production facilities *S*1, *S*2 and *S*3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses *D*1, *D*2, *D*3 and *D*4 with requirement of 5, 8, 7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply (availability)
*S*1	19	30	50	10	7
*S*2	70	30	40	60	9
*S*3	40	8	70	20	18
Demand (requirement)	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Solution: Let xij = number of units of the product to be transported from a production facility i ($i = 1, 2, 3$) to a warehouse j ($j = 1, 2, 3, 4$)

The transportation problem is stated as an LP model as follows:

$$\text{Minimize (total transportation cost) } Z = 19\,x11 + 30\,x12 + 50\,x13 + 10\,x14 + 70\,x21 + 30\,x22 + 40\,x23 + 60\,x24 + 40\,x31 + 8\,x32 + 70\,x33 + 20\,x34$$

Continued….

subject to the constraints

$$\left.\begin{aligned} x11 + x12 + x13 + x14 &= 7 \\ x21 + x22 + x23 + x24 &= 9 \\ x31 + x32 + x33 + x34 &= 18 \end{aligned}\right\} \quad \text{(Supply)}$$

$$\left.\begin{aligned} x11 + x21 + x31 &= 5 \\ x12 + x22 + x32 &= 8 \\ x13 + x23 + x33 &= 7 \\ x14 + x24 + x34 &= 14 \end{aligned}\right\} \quad \text{(Demand)}$$

and $xij \geq 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, and 4.

In the above LP model, there are $m \times n = 3 \times 4 = 12$ decision variables, xij and $m + n = 7$ constraints, where m are the number of rows and n are the number of columns in a general transportation table.

GENERAL MATHEMATICAL MODEL OF TRANSPORTATION PROBLEM

- Let there be *m* sources of supply, *S*1, *S*2, . . ., *Sm* having *ai* (*i* = 1, 2, . . ., *m*) units of supply (or capacity), respectively to be transported to *n* destinations, *D*1, *D*2, . . ., *Dn* with *bj* (*j* = 1, 2, . . ., *n*) units of demand (or requirement), respectively. Let *Cij* be the cost of shipping one unit of the commodity from source *i* to destination *j*. If *xij* represents number of units shipped from source *i* to destination *j*, the problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand conditions. Mathematically, the transportation problem, in general, may be stated as follows:

minimize (total cost) Z= $\sum_{i=1}^{m} \sum_{j=1}^{n} Cij . xij$

subject to the constraints

$$\sum_{j=1}^{n} xij = ai \, , i = 1, 2, \ldots . , m \; (supply\ constraints)$$

$$\sum_{i=1}^{m} xij = bj \, , j = 1, 2, \ldots . , n \; (demand\ constraints)$$

and xij$\geq$ 0 for all i and j.

Continued…

Remarks:

- When the total supply is equal to the total demand, the problem is called a balanced transportation problem, otherwise it is called an unbalanced transportation problem. The unbalanced transportation problem can be made balanced by adding a dummy supply centre (row) or a dummy demand centre (column) as the need arises.
- When the number of positive allocations (values of decision variables) at any stage of the feasible solution is less than the required number (rows + columns – 1), i.e. number of independent constraint equations, the solution is said to be degenerate, otherwise non-degenerate.
- Cells in the transportation table having positive allocation, i.e., $xij > 0$ are called occupied cells, otherwise are known as non-occupied (or empty) cells.

THE TRANSPORTATION ALGORITHM

The algorithm for solving a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and arrange the data in the matrix form The formulation of the transportation problem is similar to the LP problem formulation. In transportation problem, the objective function is the total transportation cost and the constraints are the amount of supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution In this chapter, following three different methods are discussed to obtain an initial solution:

- North-West Corner Method,
- Least Cost Method, and
- Vogel’s Approximation (or Penalty) Method.

The initial solution obtained by any of the three methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called rim conditions).

Continued…

(ii) The number of positive allocations must be equal to m + n – 1, where m is the number of rows

and n is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise, degenerate solution.

Step 3: Test the initial solution for optimality In this chapter, the Modified Distribution (MODI) method is discussed to test the optimality of the solution obtained in Step 2. If the current solution is optimal, then stop. Otherwise, determine a new improved solution.

Step 4: Updating the solution Repeat Step 3 until an optimal solution is reached.

METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

North-West Corner Method:

Step 1: Start with the cell at the upper left (north-west) corner of the transportation table (or matrix) and allocate commodity equal to the minimum of the rim values for the first row and first column, i.e. min (a1, b1).

Step 2: **(a)** If allocation made in Step 1 is equal to the supply available at first source (a1, in first row), then move vertically down to the cell (2, 1), i.e., second row and first column. Apply Step 1 again, for next allocation.

(b) If allocation made in Step 1 is equal to the demand of the first destination (b1 in first column), then move horizontally to the cell (1, 2), i.e., first row and second column. Apply Step 1 again for next allocation.

(c) If $a1 = b1$, allocate $x11 = a1$ or $b1$ and move diagonally to the cell (2, 2).

Step 3: Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table.

Question: Use North-West Corner Method (NWCM) to find an initial basic feasible solution to the transportation problem using data of **Example 1.**

Solution: The cell (*S*1, *D*1) is the north-west corner cell in the given transportation table. The rim values for row *S*1 and column *D*1 are compared. The smaller of the two, i.e. 5, is assigned as the first allocation.

	D1	D2	D3	D4	Supply
S1	19 ⑤	30	50	10	~~7~~ 2
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	~~5~~	8	7	14	34

	D2	D3	D4	Supply
S1	30 ②	50	10	~~2~~
S2	30	40	60	9
S3	8	70	20	18
Demand	~~8~~ 6	7	14	

	D2	D3	D4	Supply
S2	30 ⑥	40	60	~~9~~ 3
S3	8	70	20	18
Demand	~~6~~	7	14	

	D3	D4	Supply
S2	40 ③	60	~~3~~
S3	70	20	18
Demand	~~7~~ 4	14	

	D3	D4	Supply
S3	70 ④	20 ⑭	~~18~~ ~~14~~
Demand	~~4~~	~~14~~	

Continued…

Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal to $m + n - 1 = 3 + 4 - 1 = 6$. If yes, then solution is non-degenerate feasible solution, Otherwise degenerate solution.

The total transportation cost of the initial solution is obtained by multiplying the quantity xij in the occupied cells with the corresponding unit cost cij and adding all the values together. Thus, the total transportation cost of this solution is

$$\text{Total cost} = 5 \times 19 + 2 \times 30 + 6 \times 30 + 3 \times 40 + 4 \times 70 + 14 \times 20 = \text{Rs } 1{,}015$$

Least Cost Method (LCM)

The main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step 2: After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

Question: Use Least Cost Method (LCM) to find an initial basic feasible solution to the transportation problem using data of **Example 1.**

Solution: The cell with lowest unit cost (i.e., 8) is (*S*3, *D*2). The maximum units which can be allocated to this cell is 8. This meets the complete demand of *D*2 and leave l0 units with *S*3**.** In the reduced table without column *D*2, the next smallest unit transportation cost, is 10 in cell (*S*1, *D*4). The maximum which can be allocated to this cell is 7. This exhausts the capacity of *S*1 and leaves 7 units with *D*4 as unsatisfied demand. Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied.

	D1	~~D2~~	D3	D4	Supply
*S*1	19	~~30~~	50	10	7
*S*2	70	~~30~~	40	60	9
*S*3	40	~~8 ⑧~~	70	20	~~18~~ 10
Demand	5	~~8~~	7	14	34

	D1	D3	D4	Supply
~~*S*1~~	~~19~~	~~50~~	~~10 ⑦~~	~~7~~
*S*2	70	40	60	9
*S*3	40	70	20	10
Demand	5	7	~~14~~ 7	

	D1	D3	~~D4~~	Supply
*S*2	70	40	~~60~~	9
*S*3	40	70	~~20 ⑦~~	~~10~~ 3
Demand	5	7	~~7~~	

	D1	~~D3~~	Supply
*S*2	70	~~40 ⑦~~	~~9~~ 2
*S*3	40	~~70~~	3
Demand	5	~~7~~	

	D1	Supply
*S*2	70 ②	~~2~~
~~*S*3~~	~~40 ③~~	~~3~~
Demand	~~5~~ ~~2~~	

Continued…

The total transportation cost of the initial solution by LCM is calculated as given below:

Total cost = 7 × 10 + 2 × 70 + 7 × 40 + 3 × 40 + 8 × 8 + 7 × 20 = Rs 814

 # The total transportation cost obtained by LCM is less than the cost obtained by NWCM.

Vogel’s Approximation Method (VAM)

Vogel’s approximation (penalty or regret) is preferred over NWCR and LCM methods. In this method, an allocation is made on the basis of the opportunity (or penalty or extra) cost that would have been incurred if the allocation in certain cells with minimum unit transportation cost were missed. Hence, allocations are made in such a way that the penalty cost is minimized. An initial solution obtained by using this method is nearer to an optimal solution or is the optimal solution itself. The steps of VAM are as follows:

Step 1: Calculate the penalties for each row (column) by taking the difference between the smallest and next smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if decision-maker fails to allocate to the cell with the minimum unit transportation cost.

Step 2: Select the row or column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row or column and satisfies the rim conditions. If there is a tie in the values of penalties, it can be broken by selecting the cell where the maximum allocation can be made.

Step 3: Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

Step 4: Repeat Steps 1 to 3 until the available supply at various sources and demand at various destinations is satisfied.

Question: Use Vagel’s Approximation Method (VAM) to find an initial basic feasible solution to the transportation problem using data of **Example 1.**

Solution: The differences (penalty costs) for each row and column have been calculated as shown below table. In the first round, the maximum penalty, 22 occurs in column *D*2. Thus the cell (*S*3, *D*2) having the least transportation cost is chosen for allocation. The maximum possible allocation in this cell is 8 units and it satisfies demand in column *D*2. Adjust the supply of *S*3 from 18 to 10 (18 – 8 = 10). Again repeat this process.

	D1	D2	D3	D4	supply	Row Penalty
S1	19	30	50	10	7	19-10=9
S2	70	30	40	60	9	40-30=10
S3	40	8 ⑧	70	20	~~18~~ 10	20-8=12
Demand	5	~~8~~	7	14	34	
Column penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	D1	D3	D4	supply	Row Penalty
S1	19 ⑤	50	10	~~7~~ 2	19-10=9
S2	70	40	60	9	60-40=20
S3	40	70	20	10	40-20=20
Demand	~~5~~	7	14		
Column penalty	40-19=21	50-40=10	20-10=10		

Continued…

	D3	D4	supply	Row Penalty
S1	50	10	2	50-10=40
S2	40	60	9	60-40=20
~~S3~~	~~70~~	~~20~~ ⑩	~~10~~	~~70-20=50~~
Demand	7	~~14~~ 4		
Column penalty	50-40=10	20-10=10		

	D3	D4	supply	Row Penalty
~~S1~~	~~50~~	~~10~~ ②	~~2~~	~~50-10=40~~
S2	40	60	9	60-40=20
Demand	7	~~4~~ 2		
Column penalty	50-40=10	60-10=50		

	D3	~~D4~~	supply	Row Penalty
S2	40	~~60~~ ②	~~9~~ 7	60-40=20
Demand	7	~~2~~		
Column penalty	40	~~60~~		

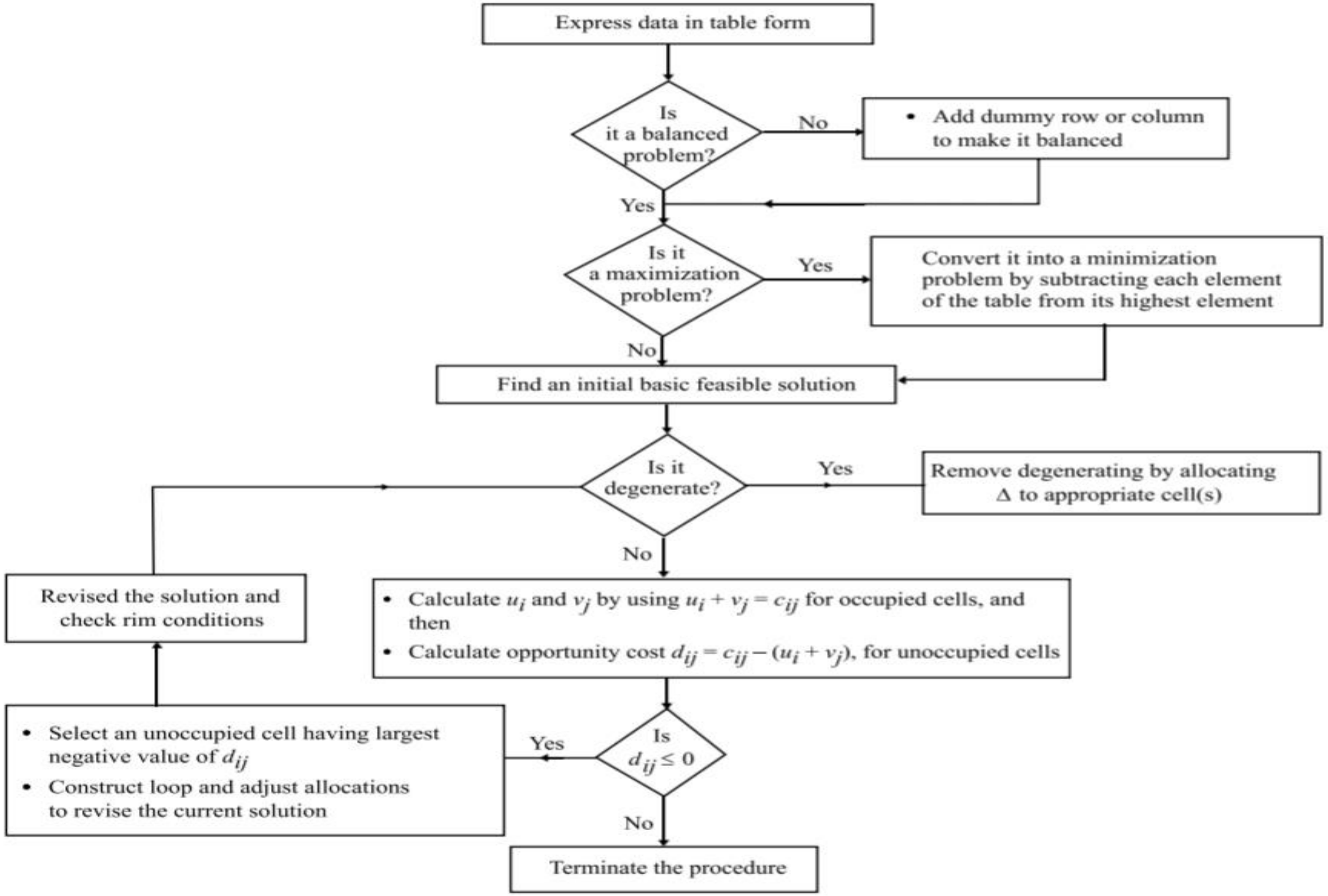
	D3	supply	Row Penalty
S2	40 ⑦	~~7~~	60-40=20
Demand	~~7~~		
Column penalty	40		

Continued…

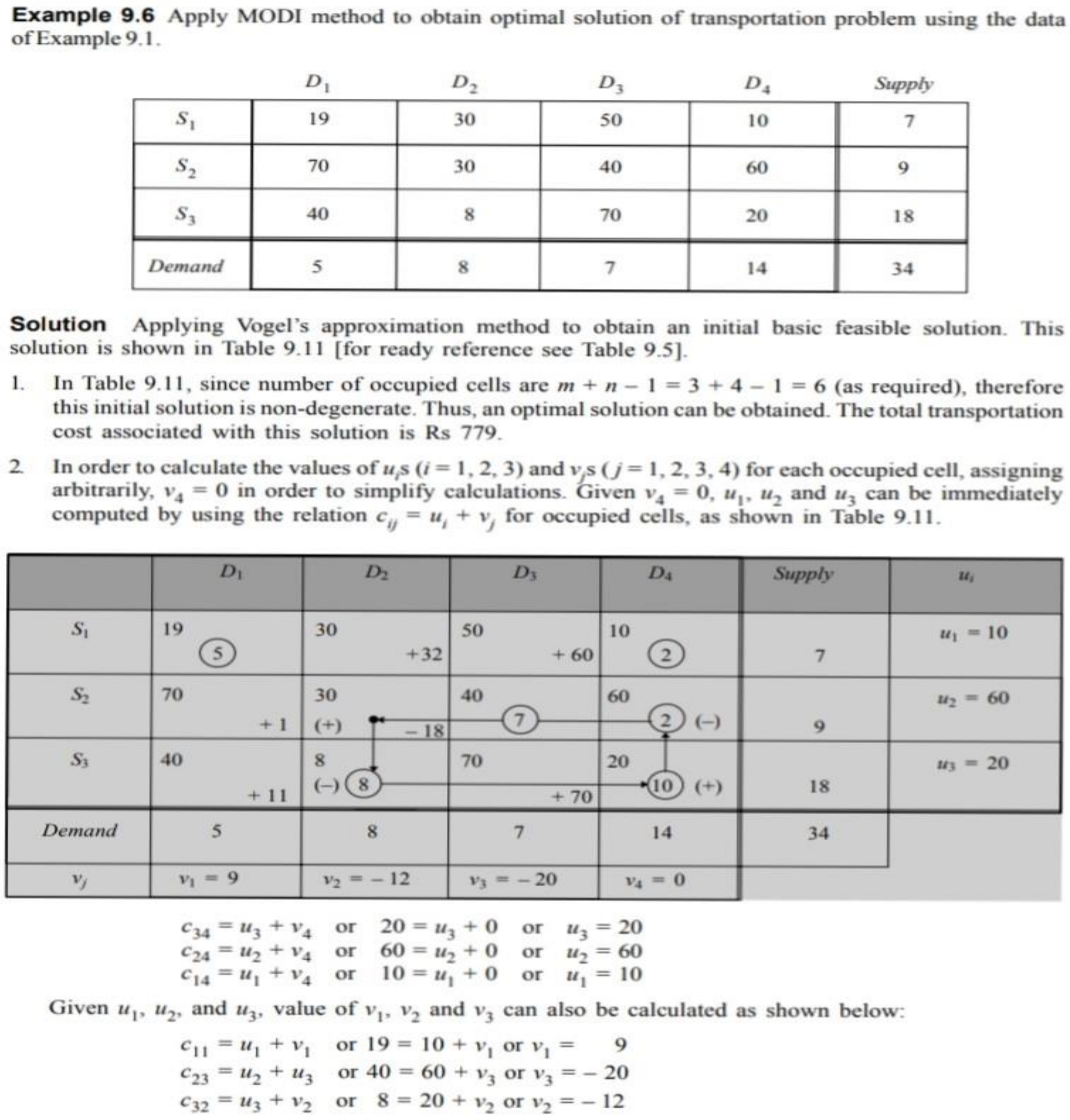
The total transportation cost associated with this method is:

Total cost = $5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20$ = Rs 779

Flow Chart of MODI Method(Test of Optimality)



Question



Example 9.6 Apply MODI method to obtain optimal solution of transportation problem using the data of Example 9.1.

	D_1	D_2	D_3	D_4	*Supply*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Solution Applying Vogel's approximation method to obtain an initial basic feasible solution. This solution is shown in Table 9.11 [for ready reference see Table 9.5].

1. In Table 9.11, since number of occupied cells are $m + n - 1 = 3 + 4 - 1 = 6$ (as required), therefore this initial solution is non-degenerate. Thus, an optimal solution can be obtained. The total transportation cost associated with this solution is Rs 779.
2. In order to calculate the values of u_is ($i = 1, 2, 3$) and v_js ($j = 1, 2, 3, 4$) for each occupied cell, assigning arbitrarily, $v_4 = 0$ in order to simplify calculations. Given $v_4 = 0$, u_1, u_2 and u_3 can be immediately computed by using the relation $c_{ij} = u_i + v_j$ for occupied cells, as shown in Table 9.11.

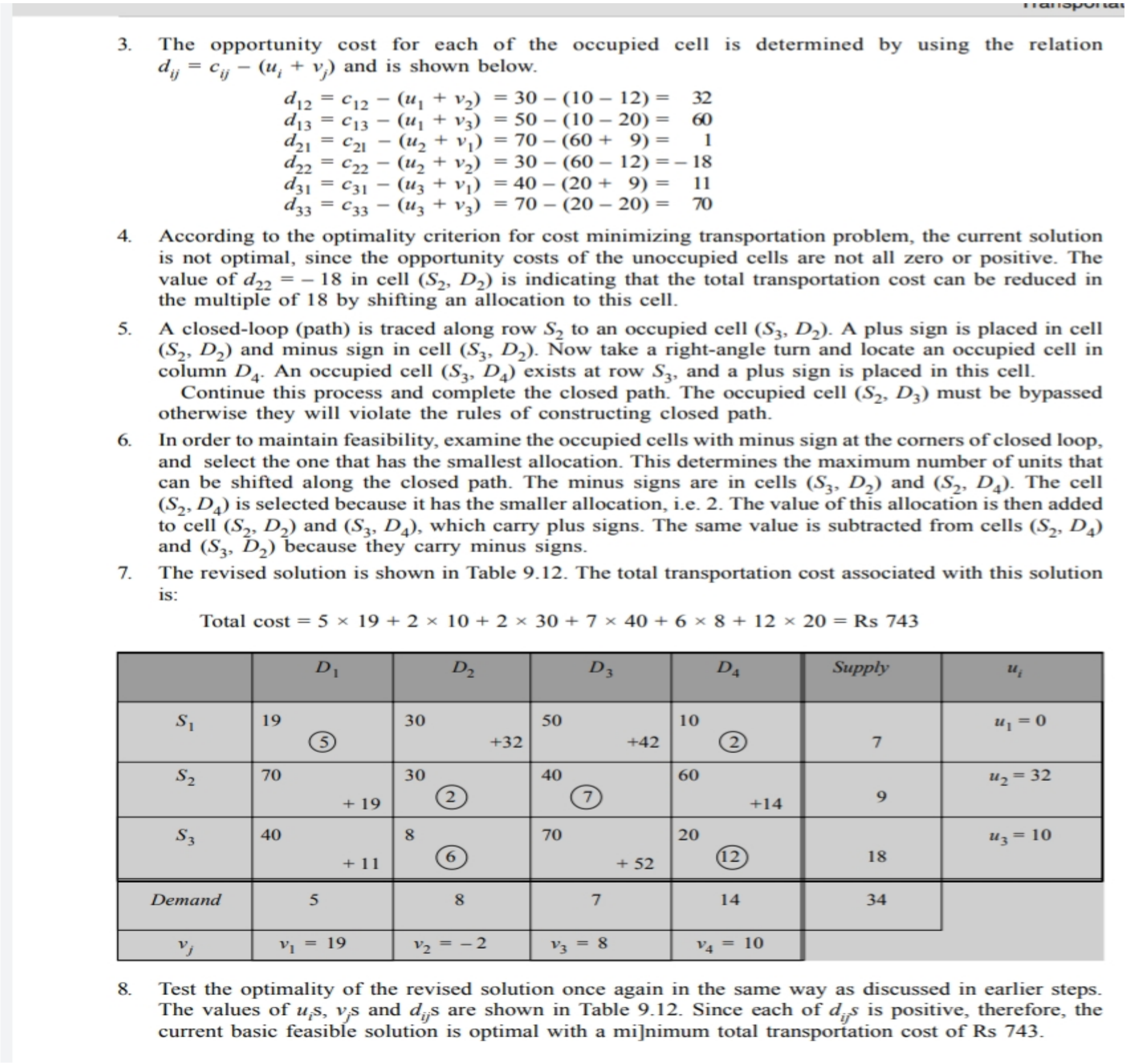
	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19 ⑤	30 +32	50 + 60	10 ②	7	$u_1 = 10$
S_2	70 + 1	30 (+) – 18	40 ⑦	60 ② (–)	9	$u_2 = 60$
S_3	40 + 11	8 (–) ⑧	70 + 70	20 ⑩ (+)	18	$u_3 = 20$
Demand	5	8	7	14	34	
v_j	$v_1 = 9$	$v_2 = -12$	$v_3 = -20$	$v_4 = 0$		

$$c_{34} = u_3 + v_4 \quad \text{or} \quad 20 = u_3 + 0 \quad \text{or} \quad u_3 = 20$$
$$c_{24} = u_2 + v_4 \quad \text{or} \quad 60 = u_2 + 0 \quad \text{or} \quad u_2 = 60$$
$$c_{14} = u_1 + v_4 \quad \text{or} \quad 10 = u_1 + 0 \quad \text{or} \quad u_1 = 10$$

Given u_1, u_2, and u_3, value of v_1, v_2 and v_3 can also be calculated as shown below:

$$c_{11} = u_1 + v_1 \quad \text{or } 19 = 10 + v_1 \text{ or } v_1 = \quad 9$$
$$c_{23} = u_2 + u_3 \quad \text{or } 40 = 60 + v_3 \text{ or } v_3 = -20$$
$$c_{32} = u_3 + v_2 \quad \text{or } \ 8 = 20 + v_2 \text{ or } v_2 = -12$$

Continued…



3. The opportunity cost for each of the occupied cell is determined by using the relation $d_{ij} = c_{ij} - (u_i + v_j)$ and is shown below.

$$\begin{aligned}
d_{12} &= c_{12} - (u_1 + v_2) = 30 - (10 - 12) = 32 \\
d_{13} &= c_{13} - (u_1 + v_3) = 50 - (10 - 20) = 60 \\
d_{21} &= c_{21} - (u_2 + v_1) = 70 - (60 + 9) = 1 \\
d_{22} &= c_{22} - (u_2 + v_2) = 30 - (60 - 12) = -18 \\
d_{31} &= c_{31} - (u_3 + v_1) = 40 - (20 + 9) = 11 \\
d_{33} &= c_{33} - (u_3 + v_3) = 70 - (20 - 20) = 70
\end{aligned}$$

4. According to the optimality criterion for cost minimizing transportation problem, the current solution is not optimal, since the opportunity costs of the unoccupied cells are not all zero or positive. The value of $d_{22} = -18$ in cell (S_2, D_2) is indicating that the total transportation cost can be reduced in the multiple of 18 by shifting an allocation to this cell.

5. A closed-loop (path) is traced along row S_2 to an occupied cell (S_3, D_2). A plus sign is placed in cell (S_2, D_2) and minus sign in cell (S_3, D_2). Now take a right-angle turn and locate an occupied cell in column D_4. An occupied cell (S_3, D_4) exists at row S_3, and a plus sign is placed in this cell.

   Continue this process and complete the closed path. The occupied cell (S_2, D_3) must be bypassed otherwise they will violate the rules of constructing closed path.

6. In order to maintain feasibility, examine the occupied cells with minus sign at the corners of closed loop, and select the one that has the smallest allocation. This determines the maximum number of units that can be shifted along the closed path. The minus signs are in cells (S_3, D_2) and (S_2, D_4). The cell (S_2, D_4) is selected because it has the smaller allocation, i.e. 2. The value of this allocation is then added to cell (S_2, D_2) and (S_3, D_4), which carry plus signs. The same value is subtracted from cells (S_2, D_4) and (S_3, D_2) because they carry minus signs.

7. The revised solution is shown in Table 9.12. The total transportation cost associated with this solution is:

   Total cost = 5 × 19 + 2 × 10 + 2 × 30 + 7 × 40 + 6 × 8 + 12 × 20 = Rs 743

	D_1	D_2	D_3	D_4	*Supply*	u_i
S_1	19 ⑤	30 +32	50 +42	10 ②	7	$u_1 = 0$
S_2	70 + 19	30 ②	40 ⑦	60 +14	9	$u_2 = 32$
S_3	40 + 11	8 ⑥	70 + 52	20 ⑫	18	$u_3 = 10$
Demand	5	8	7	14	34	
v_j	$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$		

8. Test the optimality of the revised solution once again in the same way as discussed in earlier steps. The values of u_is, v_js and d_{ij}s are shown in Table 9.12. Since each of d_{ij}s is positive, therefore, the current basic feasible solution is optimal with a mi]nimum total transportation cost of Rs 743.

Some problems for practice

1. **Solve the given transportation problem by NWCR, LCM and VAM method.**

	D1	D2	D3	D4	**supply**
S1	1	2	3	4	16
S2	4	3	2	0	8
S3	0	2	2	1	20
Demand	14	10	8	12	

2. **Solve the given problem by VAM method and test the optimality of the problem by modi method**

destination

source

	D1	D2	D3	Supply
O1	4	8	8	76
O2	16	24	16	82
O3	8	16	24	77
Demand	72	102	41	

Continued…

Hint for 2: Given transportation problem is unbalanced transportation problem because total demand is not equal to total supply. Total demand = 215

total supply = 235

	D1	D2	D3	D4(dummy)	Supply
O1	4	8	8	0	76
O2	16	24	16	0	82
O3	8	16	24	0	77
Demand	72	102	41	20	

Assignment Problem

An assignment problem is a particular case of a transportation problem where the resources (say facilities) are assignees and the destinations are activities (say jobs). Given *n* resources (or facilities) and *n* activities (or jobs), with effectiveness (in terms of cost, profit, time, etc.) of each resource for each activity. Then problem becomes to assign (or allocate) each resource to only one activity (job) and vice-versa so that the given measure of effectiveness is optimized. Some of the problems where the assignment technique may be useful are assignment of (i) workers to machines, (ii) salesmen to different sales areas, (iii) clerks to various checkout counters, (iv) classes to rooms, (v) vehicles to routes, (vi) contracts to bidders, etc.

Hungarian Method for Solving Assignment Problem:

The Hungarian method (minimization case) can be summarized in the following steps:

Step 1: Develop the cost matrix from the given problem If the number of rows are not equal to the number of columns, then add required number of dummy rows or columns. The cost element in dummy rows/columns are always zero.

Step 2: Find the opportunity cost matrix

(a) Identify the smallest element in each row of cost matrix and then subtract it from each element of that row, and

(b) In the reduced matrix obtained from 2(a), identify the smallest element in each column and then subtract it from each element of that column. Each row and column now have at least one zero element.

Continued…

Step 3: Make assignments in the opportunity cost matrix The procedure of making assignments is as follows:

(a) First round for making assignments

- Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making a square (□) around it. Then cross off (×) all other zeros in the corresponding column.
- Identify columns successively from left to right hand with exactly one zero element that has not been assigned. Make assignment to this single zero by making a square (□) around it and then cross off (×) all other zero elements in the corresponding row.

(b) Second round for making assignments

- If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection, then choose zero element arbitrarily for assignment.
- Repeat steps (a) and (b) successively until one of the following situations arise.

Step 4: Optimality criterion

(a) If all zero elements in the cost matrix are either marked with square (□) or are crossed off (×) and there is exactly one assignment in each row and column, then it is an optimal solution. The total cost associated with this solution is obtained by adding the original cost elements in the occupied cells.

Continued…

(b) If a zero element in a row or column was chosen arbitrarily for assignment in Step 4(a), there exists an alternative optimal solution.

(c) If there is no assignment in a row (or column), then this implies that the total number of assignments are less than the number of rows/columns in the square matrix. In such a situation proceed to Step 5.

Step 5: Revise the opportunity cost matrix Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from Step 3, by using the following procedure:

a) For each row in which no assignment was made, mark a tick (√)

b) Examine the marked rows. If any zero element is present in these rows, mark a tick (√) to the respective columns containing zeros.

c) Examine marked columns. If any assigned zero element is present in these columns, tick (√) the respective rows containing assigned zeros.

d) Repeat this process until no more rows or columns can be marked.

e) Draw a straight line through each marked column and each unmarked row.

If the number of lines drawn (or total assignments) is equal to the number of rows (or columns), the current solution is the optimal solution, otherwise go to Step 6.

Continued…

Step 6: Develop the new revised opportunity cost matrix

(a) Among the elements in the matrix not covered by any line, choose the smallest element. Call this value *k*.

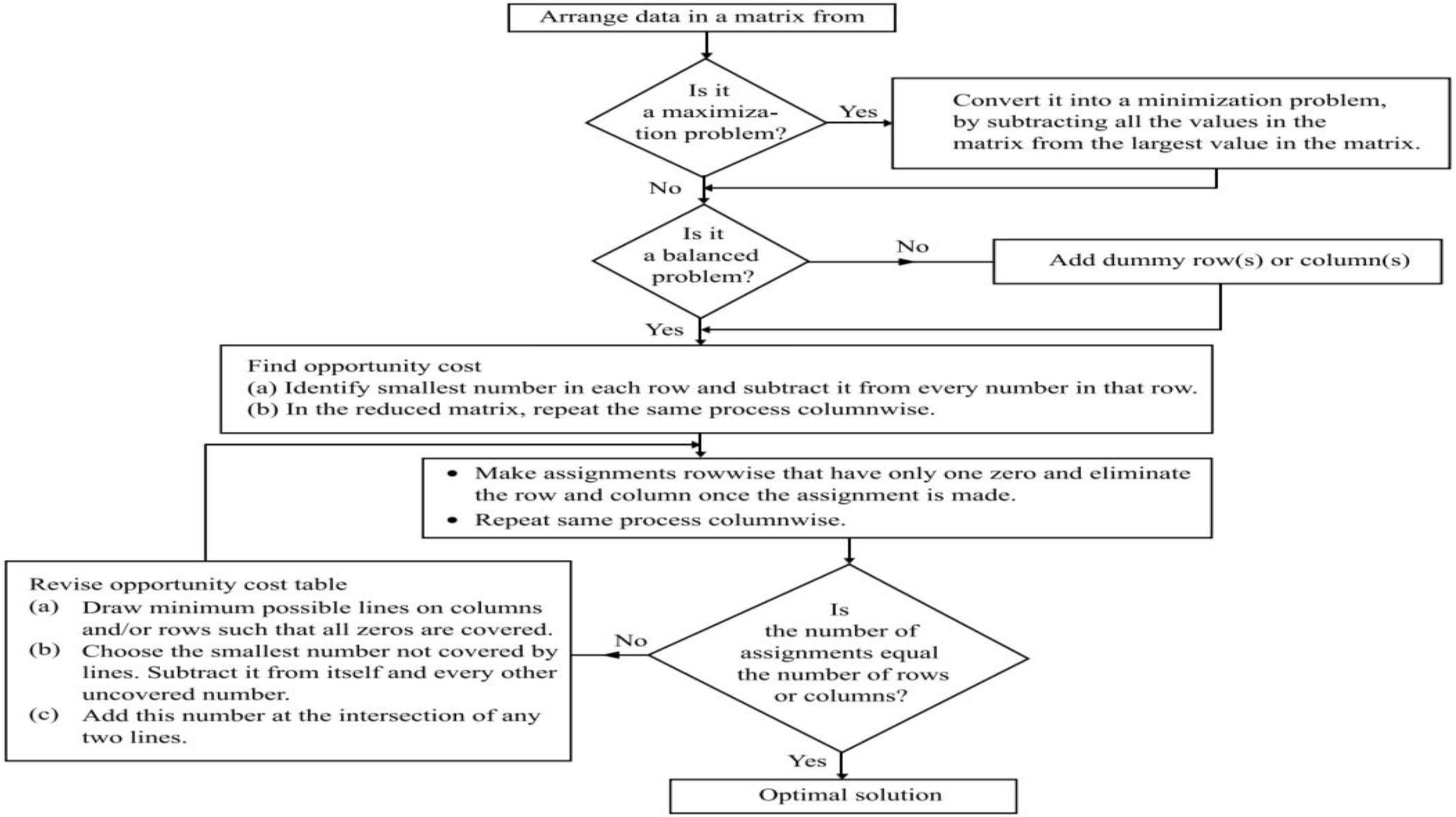
(b) Subtract *k* from every element in the matrix that is not covered by a line.

(c) Add *k* to every element in the matrix covered by the two lines, i.e. intersection of two lines.

(d) Elements in the matrix covered by one line remain unchanged.

Step 7: Repeat steps Repeat Steps 3 to 6 until an optimal solution is obtained.

Flow Chart of Hungarian Method



Arrange data in a matrix from

Is it a maximization problem?

Yes

Convert it into a minimization problem, by subtracting all the values in the matrix from the largest value in the matrix.

No

Is it a balanced problem?

No

Add dummy row(s) or column(s)

Yes

Find opportunity cost
(a) Identify smallest number in each row and subtract it from every number in that row.
(b) In the reduced matrix, repeat the same process columnwise.

- Make assignments rowwise that have only one zero and eliminate the row and column once the assignment is made.
- Repeat same process columnwise.

Is the number of assignments equal the number of rows or columns?

No

Revise opportunity cost table
(a) Draw minimum possible lines on columns and/or rows such that all zeros are covered.
(b) Choose the smallest number not covered by lines. Subtract it from itself and every other uncovered number.
(c) Add this number at the intersection of any two lines.

Yes

Optimal solution

Question

1. A computer centre has three expert programmers. The centre wants three application programmes to be developed. The head of the computer centre, after carefully studying the programmes to be developed, estimates the computer time in minutes required by the experts for the application programmes as follows:

programmers

$$\begin{array}{cc|ccc|}
 & & A & B & C \\ \hline
 & 1 & 120 & 100 & 80 \\
\textbf{programmes} & 2 & 80 & 90 & 110 \\
 & 3 & 110 & 140 & 120 \\ \hline
\end{array}$$

Assign the programmers to the programmes in such a way that the total computer time is minimum.

Solution: The minimum time element in rows 1, 2 and 3 is 80, 80 and 110, respectively. Subtract these elements from all elements in their respective row. The reduced time matrix is shown in below.

$$\begin{array}{|ccc|} \hline
40 & 20 & 0 \\
0 & 10 & 30 \\
0 & 30 & 10 \\ \hline
\end{array}$$

The minimum time element in columns *A*, *B* and *C* is 0, 10 and 0, respectively. Subtract these elements from all elements in their respective column in order to get the reduced time matrix.

Continued…

$$\begin{bmatrix} 40 & 10 & 0 \\ 0 & 0 & 30 \\ 0 & 20 & 10 \end{bmatrix}$$

Examine all the rows starting from the first, one-by-one, until a row containing single zero element is found. Make an assignment in these cells and cross off all zero elements in the assigned column respectively.

$$\begin{bmatrix} 40 & 10 & \boxed{0} \\ \cancel{0} & 0 & 30 \\ \boxed{0} & 20 & 10 \end{bmatrix} \qquad \begin{array}{c|ccc} & A & B & C \\ \hline 1 & 40 & 10 & \boxed{0} \\ 2 & \cancel{0} & \boxed{0} & 30 \\ 3 & \boxed{0} & 20 & 10 \end{array}$$

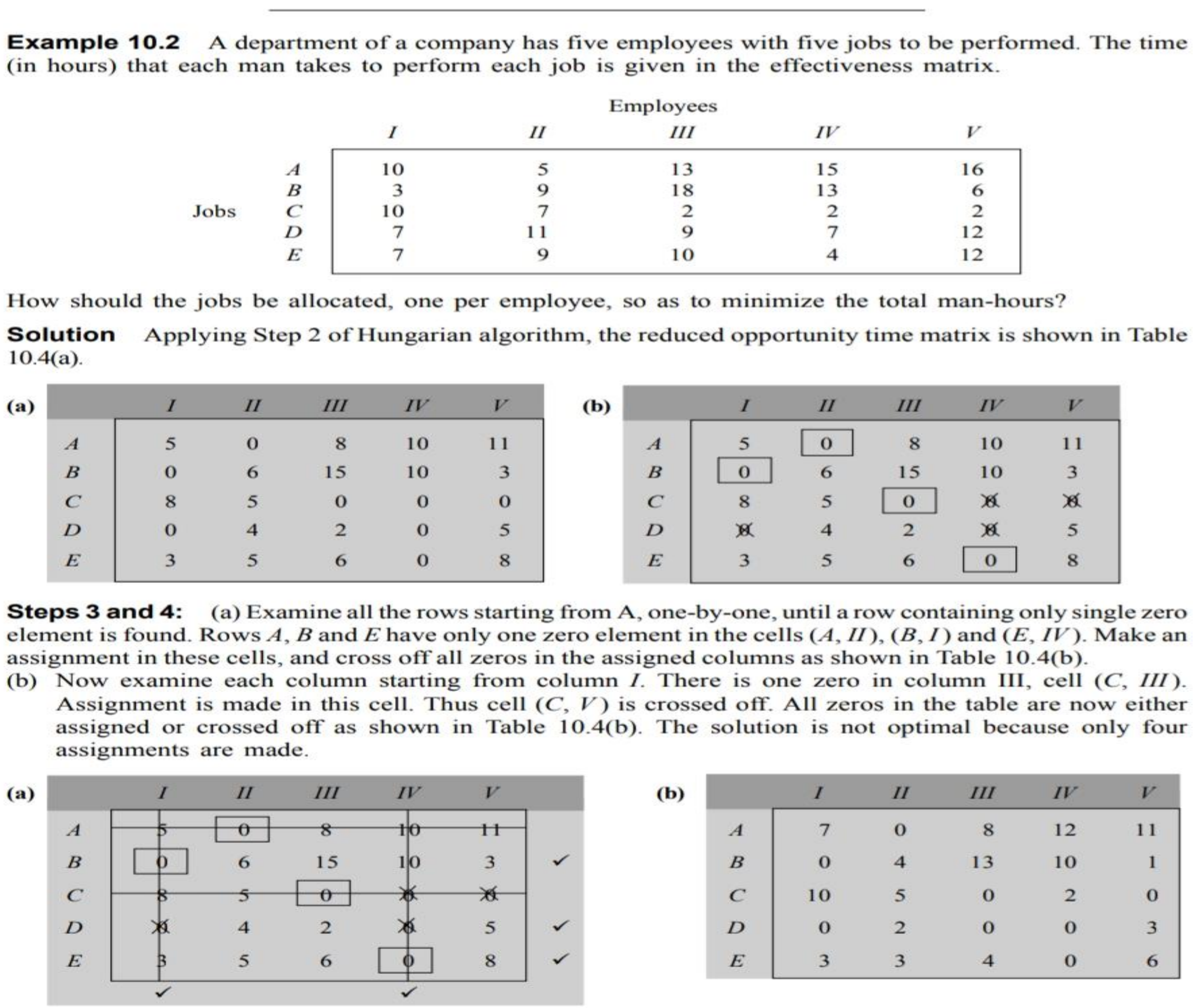
Since the number of assignments (= 3) equals the number of rows (= 3), the optimal solution is obtained.

The pattern of assignments among programmers and programmes with their respective time (in minutes) is given below:

1 → C time(minute)= 80
2 → B 90
3 → A + 110

280

Question



Example 10.2 A department of a company has five employees with five jobs to be performed. The time (in hours) that each man takes to perform each job is given in the effectiveness matrix.

		Employees				
		I	*II*	*III*	*IV*	*V*
Jobs	*A*	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

Solution Applying Step 2 of Hungarian algorithm, the reduced opportunity time matrix is shown in Table 10.4(a).

(a)

	I	*II*	*III*	*IV*	*V*
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

(b)

	I	*II*	*III*	*IV*	*V*
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
D	$\cancel{0}$	4	2	$\cancel{0}$	5
E	3	5	6	$\boxed{0}$	8

Steps 3 and 4: (a) Examine all the rows starting from A, one-by-one, until a row containing only single zero element is found. Rows *A*, *B* and *E* have only one zero element in the cells (*A*, *II*), (*B*, *I*) and (*E*, *IV*). Make an assignment in these cells, and cross off all zeros in the assigned columns as shown in Table 10.4(b).

(b) Now examine each column starting from column *I*. There is one zero in column III, cell (*C*, *III*). Assignment is made in this cell. Thus cell (*C*, *V*) is crossed off. All zeros in the table are now either assigned or crossed off as shown in Table 10.4(b). The solution is not optimal because only four assignments are made.

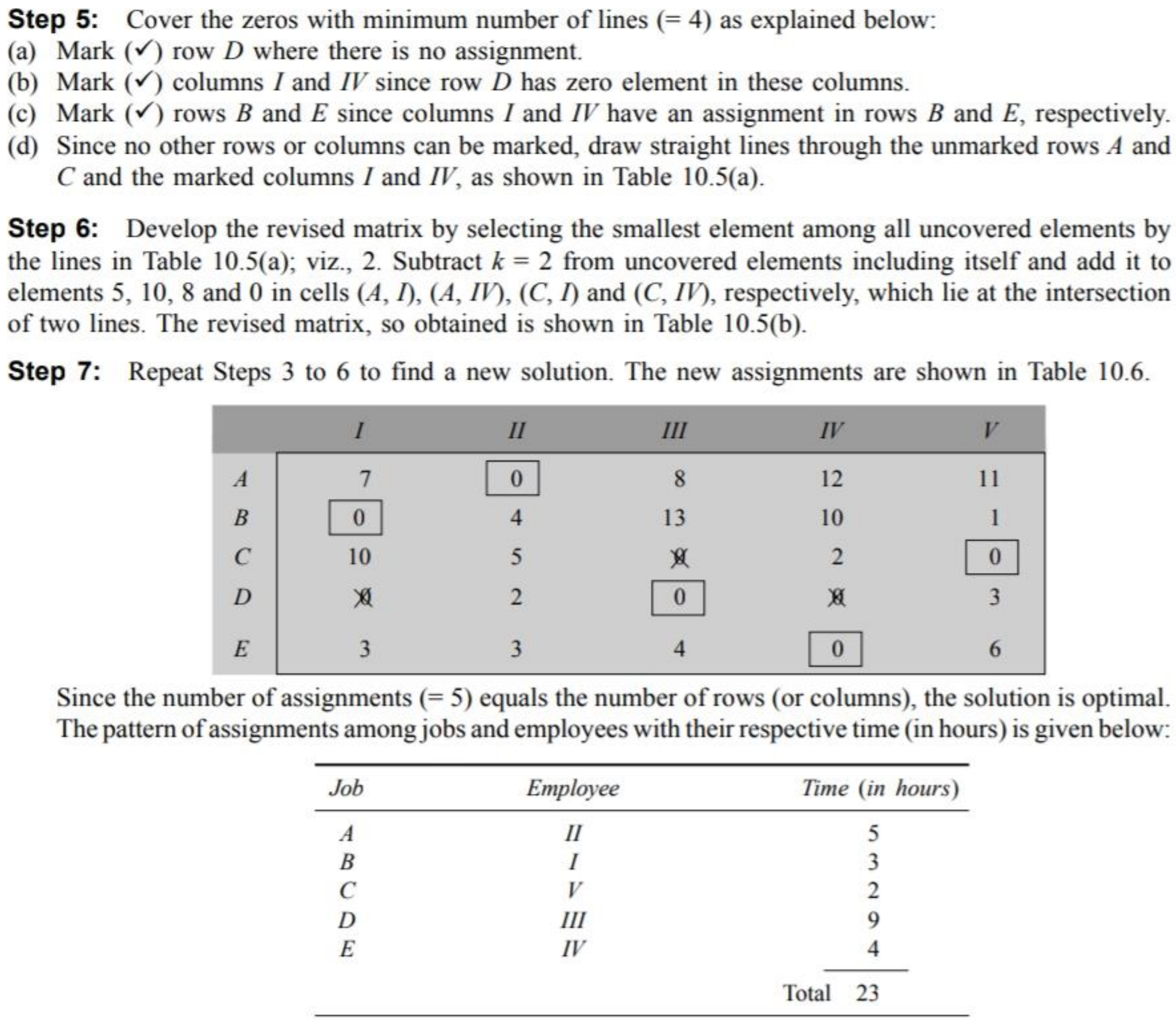
(a)

	I	*II*	*III*	*IV*	*V*	
A	5	$\boxed{0}$	8	10	11	
B	$\boxed{0}$	6	15	10	3	✓
C	8	5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
D	$\cancel{0}$	4	2	$\cancel{0}$	5	✓
E	3	5	6	$\boxed{0}$	8	✓
	✓			✓		

(b)

	I	*II*	*III*	*IV*	*V*
A	7	0	8	12	11
B	0	4	13	10	1
C	10	5	0	2	0
D	0	2	0	0	3
E	3	3	4	0	6

Continued…



Step 5: Cover the zeros with minimum number of lines (= 4) as explained below:

(a) Mark (✓) row *D* where there is no assignment.

(b) Mark (✓) columns *I* and *IV* since row *D* has zero element in these columns.

(c) Mark (✓) rows *B* and *E* since columns *I* and *IV* have an assignment in rows *B* and *E*, respectively.

(d) Since no other rows or columns can be marked, draw straight lines through the unmarked rows *A* and *C* and the marked columns *I* and *IV*, as shown in Table 10.5(a).

Step 6: Develop the revised matrix by selecting the smallest element among all uncovered elements by the lines in Table 10.5(a); viz., 2. Subtract $k = 2$ from uncovered elements including itself and add it to elements 5, 10, 8 and 0 in cells (*A*, *I*), (*A*, *IV*), (*C*, *I*) and (*C*, *IV*), respectively, which lie at the intersection of two lines. The revised matrix, so obtained is shown in Table 10.5(b).

Step 7: Repeat Steps 3 to 6 to find a new solution. The new assignments are shown in Table 10.6.

	I	*II*	*III*	*IV*	*V*
A	7	$\boxed{0}$	8	12	11
B	$\boxed{0}$	4	13	10	1
C	10	5	$\cancel{0}$	2	$\boxed{0}$
D	$\cancel{0}$	2	$\boxed{0}$	$\cancel{0}$	3
E	3	3	4	$\boxed{0}$	6

Since the number of assignments (= 5) equals the number of rows (or columns), the solution is optimal. The pattern of assignments among jobs and employees with their respective time (in hours) is given below:

Job	*Employee*	*Time (in hours)*
A	*II*	5
B	*I*	3
C	*V*	2
D	*III*	9
E	*IV*	4
		Total 23