Lec-1

"WKB Approximation Method"
OR
"Semi Classical Approximation Method"

WKB stands for : Name of three scientists
W - Weintzet
K - Kramers
B - Brillouin

* Problem Format asked in UGC-NTA-CSIR NET/JRF:-
Two Types of problem →

I $V(x) \Rightarrow$ given
$E_n = ?$
(For large value of n)
where E_n is bound state Energy for nth state.

II $V(x) \Rightarrow$ given
Transmission cofficient through potential Barrier $\Rightarrow T = ?$

⇒ Around 95% problems asked in CSIR can be done with in 30 sec just by a TRICK.

Approach solving Problems :
- Try first by trick
- If trick is not sufficient then do it by generalized method.

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Que-I — — $V(x)$ — — given
$E_n = ?$ asked.

options: (i) $E_n =$ (ii) $E_n =$ (iii) E_n (iv) E_n
also given in options –
$n = 0, 1, 2, 3 \ldots$
$n = 1, 2, 3, 4 \ldots$

In options n may start from 0 or 1. Notice it and move accordingly.

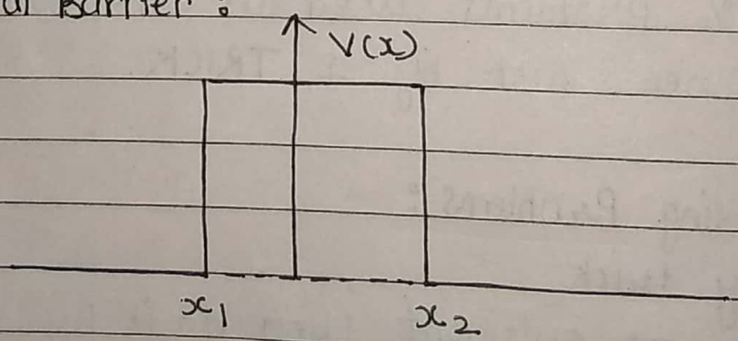
Type-II :- For 1D :-
Que → $V(x) \Rightarrow$ given.

$$T = e^{-2r}$$

$$T = e^{-2 \cdot \frac{1}{\hbar} \int_{x_1}^{x_2} p(x) \cdot dx}$$

i.e $$r = \frac{1}{\hbar} \int_{x_1}^{x_2} \sqrt{2m\{V(x) - E\}} \cdot dx$$

potential Barrier :



$x_1, x_2 \rightarrow$ Range of potential barrier.

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3D :

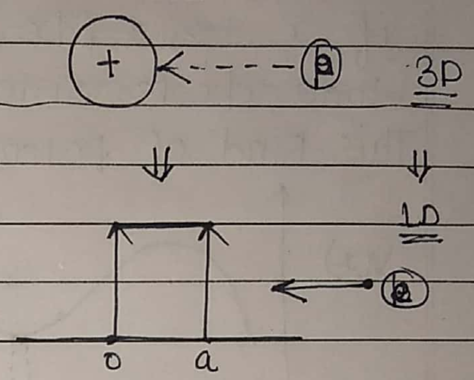
<u>Note :</u> WKB method is applicable only for 1D case

eg –

$V(r) \Rightarrow$ given

then treat it, visualising it in one Dimensional.

3D problem



$$r = \frac{1}{\hbar}\int_{r_1}^{r_2=0} \sqrt{2m\{V(r)-E\}}\,.\,\underbrace{dr}_{\text{one-D}}$$

—— x ——

What is WKB Method . ?

Why we call it " semi-classical Approximation method.

The WKB Approximation method is a technique for obtaining approximation solutions to time independent Schrodinger equations in 1-Dimension and specifically for slowly varying potentials.

$\Rightarrow$ $V(r) \rightarrow$ time independent $V \neq V(t)$

$\Rightarrow$ $V(r) \rightarrow$ Slowly Varying

$\Rightarrow$ WKB approximation method is applicable for 1D cases

$\Rightarrow$ WKB method is applicable for classically allowed region and classically forbidden region but not on turning points.

$\Rightarrow$ WKB method gives better results for large values of n. However you can find energy values for lo

$\Rightarrow$ WKB method is particularly used to find bound state energy values E_n & transmission cofficient through potential barriers.

★ $\Rightarrow$ If there is 1D problem then treat it as 3D beacuse WKB is applicable only for 1D.

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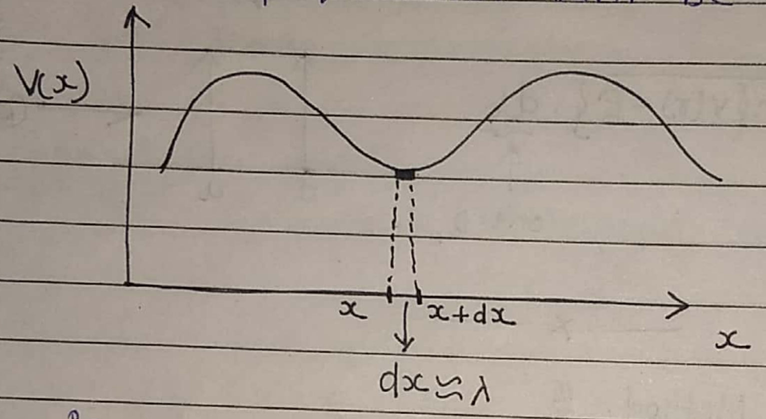
* Slowly varying potentials :

Slowly varying potentials are potentials which remains almost constant over a region of de-broglie wavelength.

if $x \longrightarrow x+dx$ then $V(x) \simeq$ constant
Where dx is order of debroglie wavelength.
This kind of potential will be slowly varying.



- if $\lambda \rightarrow$ very small $\Rightarrow dx$ is small.
  In classical regime ($\lambda \simeq 10^{-34}$ m), all potentials are very slowly varying (approax constant)

In classical regime; order of λ : $\Rightarrow$

100gm $\Rightarrow$ Ball $\Rightarrow$ 150km/h speed $\Rightarrow$ through $\Rightarrow \lambda = 10^{-34}$ ✓

$$\lambda = \frac{h}{p} = \frac{6.62\times10^{-34}}{mv} = \frac{6.62\times10^{-34}\times 3600}{100\times150\times1000\times10^{-3}}$$

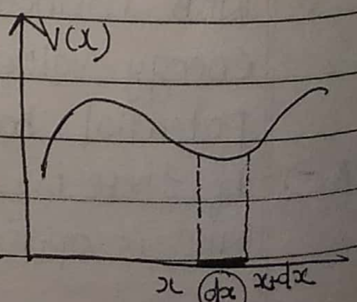
$$\lambda \simeq 10^{-34}\ m \Rightarrow \text{order}$$

Wavelength small $\Rightarrow$ Small change

But In semiclassical regime :

$\lambda \simeq 10^{-20}$ or 10^{-25} m.

In semiclassical regime all potentials are slowly varying ∵ dx is very small



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This is why WKB method is also called semi classical approximation method.
WKB method applicable for slowly varying potentials and slowly varying potential are present in semi classical regime thats why WKB is also called as semiclassical approximation method.

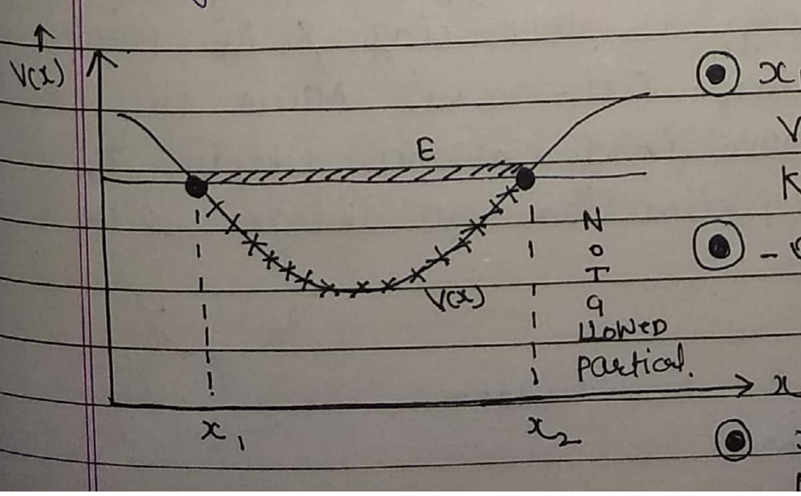
- classically allowed region (yes)
- classically forbidden region (yes)
- turning points. (No)

classically allowed region ⇒
Region in which kinetic energy of particle is positive
KE ⇒ +ve

classically forbidden region ⇒
The region in which KE is -ve negative.
KE ⇒ -ve which is not possible classically thats why this is classically forbidden.

Turning points ⇒ K.E ⇒ 0
At turning points particle's velocity is zero & it changes its direction (of velocity).

KE=0



- $x_1 < x < x_2$
  $V(x) < E$
  $KE \Rightarrow E - V(x) \Rightarrow$ +ve
- $-\infty < x < x_1$ & $x_2 < x < \infty$
  $V(x) > E$
  $T = KE = E - V(x) \Rightarrow$ -ve
- x_1 & x_2 = Turning point
  $E = V(x) \Rightarrow KE = $ zero.

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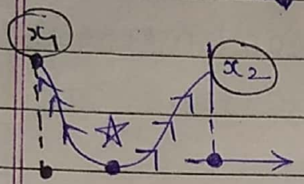
Movement of particle in classical region ⇒

Total energ E = constant

$E = T + V$

$\Downarrow \qquad \uparrow + \downarrow$ ⇒ oscillary motion

constant. $\quad$ OR $\quad \downarrow + \uparrow$ between x_1 & x_2



(Minima of PE ⇒ KE is maximum

Explanation ⇒

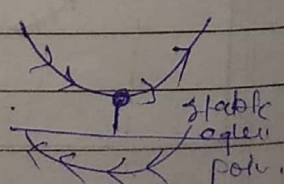
* As moving towards x_2 ⇒ potential energy is Increasing ⇒ kinetic energy is decreasing ⇒ At point x_2 = stops also potential energ is maximum

Every particle or physical entitity wants to minimise its potential energy.

particle can't move beyond x_2 beacuse there is forbidden region (can't move there). it means it will come back and that point is turning point. (where it changes its direction

As it moves back potential energy is going to decrease & at minima (☆) potential energy is minimum & kinetic energy is maximum. it KE is maximum it will suit in upward direction & in this way potential energy is Iesing towards x_1 ⇒ kinetic energy is decreasing & At point x_1 KE becomes zero. $(E = V(x))$ Again to ↓es PE particle will move in downward direction & Again at point x_1 direction will change which is turning point.

In this way this is oscillalory motion. In small oscillation stable eequillibrium.



* WKB Applicable for calasically allowed region & for classically forbidden region. but not applicable at turning points.
* Why WKB is not applicable at turning points = ?

At turning point KE ie $T = 0$ ∵ b/c $E = V$

$$\Rightarrow T = \frac{p^2}{2m} = 0$$

$$\Rightarrow \boxed{p = 0}$$

Debroglie wavelength $\Rightarrow \lambda = \frac{h}{p}$ $\quad p \to 0$

$$\Rightarrow \lambda \to \infty \text{ (very large)}$$

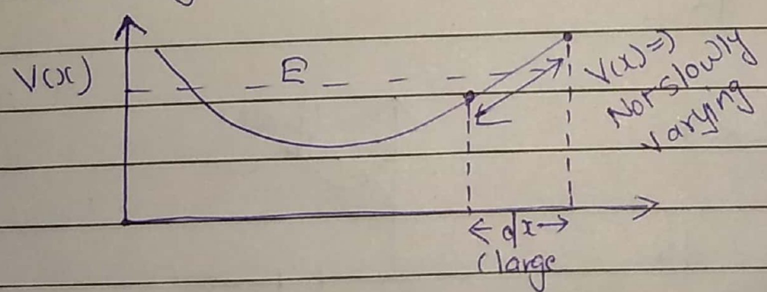
Slowly varying potential ⇒ For which WKB is applicable

$\Rightarrow x_2 \longrightarrow x_2 + dx \Rightarrow V(x_2) \simeq$ constant

$dx \leftarrow$ order of λ

But at turning point λ is very large ⇒ dx = large

For large dx, potential is not slowly varying in immidiate neighbourhood of turning points and that's why WKB is not applicable at turning points.



* Validity Condition of WKB method :-

$$\left| \frac{d\bar{\lambda}}{dx} \right| << 1 \qquad \text{Here } \bar{\lambda} = \frac{\lambda}{2\pi}$$

$$\because k = \frac{2\pi}{\lambda} \Rightarrow \frac{\lambda}{2\pi} = \frac{1}{k}$$

$$\Rightarrow \bar{\lambda} = \frac{1}{k} \Rightarrow d\bar{\lambda} = -\frac{1}{k^2} \cdot dk$$

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Validity :

$$\lambda = \frac{1}{k}$$

$$d\lambda = \frac{-1}{k^2} \cdot dk$$

$$\frac{d\lambda}{dx} = \frac{-1}{k^2} \cdot \frac{dk}{dx}$$

$$\left|\frac{d\lambda}{dx}\right| = \left|\frac{1}{k^2} \cdot \frac{dk}{dx}\right|$$

$$\Rightarrow \left|\frac{1}{k^2} \frac{dk}{dx}\right| << 1$$

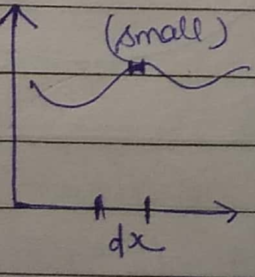
Validity condition for WKB method in two form -

- $\left|\frac{d\lambda}{dx}\right| << 1$

- $\left|\frac{1}{k^2} \cdot \frac{dk}{dx}\right| << 1$

* $\frac{d\lambda}{dx} << 1$

$\Rightarrow d\lambda << dx$



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Wave Function in classically allowed region ⇒

$$\psi(x) = \frac{C_1}{\sqrt{P(x)}} \cdot e^{+i\frac{1}{\hbar}\int P(x).dx} + \frac{C_2}{\sqrt{P(x)}} \cdot e^{-\frac{i}{\hbar}\int P(x).dx}$$

OR

$$\psi(x) = \frac{C_1}{\sqrt{P(x)}} \cdot e^{ikx} + \frac{C_2}{\sqrt{P(x)}} \cdot e^{-ikx}$$

Wavefunction for classically forbidden region ⇒

$$\psi(x) = \frac{C_1}{\sqrt{P(x)}} \cdot e^{\frac{1}{\hbar}\int P(x).dx} + \frac{C_2}{\sqrt{P(x)}} \cdot e^{-\frac{1}{\hbar}\int P(x).dx}$$

OR

$$\psi(x) = \frac{C_1}{\sqrt{P(x)}} \cdot e^{kx} + \frac{C_2}{\sqrt{P(x)}} \cdot e^{-kx}$$

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* Derivation For wavefunction ⇒
(In classically Allowed region) $E > V(x)$

Case 1 :- When potential is constant.
$V(x) = \text{constant} = V_0$

Note :
We have to solve / find wave function for "slowly varying potential but we are taking first for be "Constant potential" beacuse it will concrete our Understanding.

⇒ We have to solve schrodinger wave equation to find wave function for constant potential. ⇒

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V_0\, \psi(x) = E \cdot \psi(x)$$

$\because V(x) = V_0$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(E - V_0) \cdot \psi(x)$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\, \psi(x)$$

where $P = \sqrt{2m(E-V_0)}$ also $\frac{p^2}{\hbar^2} = k^2$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} = -k^2\psi(x)$$ ⇒ 2nd order equation
⇒ Complimentary function soln.

Solutions ⇒ $m^2 = -k^2$
$m = \pm ik$

$\psi(x) = A\, e^{\pm ikx}$ → two solutions
General solution will be Linear combination of these two

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$\Rightarrow$ $\psi(x) = C_1 e^{ikx} + C_2 \cdot e^{-ikx}$ (when V = Constant)

Case II :→ When potential is slowly varying.
Finding wave function in classically allowed region

Since potential is slowly varying, so we can assume wavefunction as following →

$$\psi(x) = A(x) \cdot e^{i\phi(x)} \quad \text{—(1)}$$ (~ approximate solution)

where;
A(x) & $\phi(x)$ are slowly varying function of (x).

A → Amplitude ~
Phase k → ~

* if $\boxed{\psi(x)}$ is the solution ⇒ it will satisfy schrodinger wave equation
SE!

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2\psi}{dx^2} + V(x) \cdot \psi(x) = E \cdot \psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left[E - V(x)\right] \cdot \psi(x)$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2}\psi(x)$$

where $p(x) = \sqrt{2m(E - V(x))}$

Since eqn (1) is solution of SE, so it will satisfy it.
Finding $\frac{d^2\psi}{dx^2}$ ⇒

$$\psi(x) = A(x)\, e^{i\phi(x)}$$

$$\frac{d\psi}{dx} = A' e^{i\phi} + A \cdot e^{i\phi}(i\phi')$$

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$$\frac{d\psi}{dx} = A' e^{i\phi} + A \cdot e^{i\phi} (i\phi)$$

$$\frac{d^2\psi}{dx^2} = A'' \cdot e^{i\phi} + A' \cdot e^{i\phi} (i\phi') + A' e^{i\phi} (i\phi') + A \cdot e^{i\phi} (i\phi')^2 + A \cdot e^{i\phi} (i\phi'')$$

$$\frac{d^2\psi}{dx^2} = \left\{ A'' \cdot e^{i\phi} - A(\phi')^2 \cdot e^{i\phi} \right\} + i \left\{ 2A'\phi' \cdot e^{i\phi} + A\phi'' e^{i\phi} \right\}$$

equation (2) is $\Rightarrow$

$$\frac{d^2\psi}{dx^2} = -\frac{p^2}{\hbar^2} \cdot \psi(x)$$

$$\left\{ A'' \cdot e^{i\phi} - A(\phi') \cdot e^{i\phi} \right\} + i \left\{ 2A'\phi' e^{i\phi} + A\phi'' e^{i\phi} \right\} = -\frac{p^2}{\hbar^2} \cdot A e^{i\phi}$$

Comparing real part & imaginary part both side $\Rightarrow$

$$A'' \cdot e^{i\phi} - A(\phi')^2 \cdot e^{i\phi} = -\frac{p^2}{\hbar^2} A e^{i\phi}$$

$$A'' - A(\phi')^2 = -\frac{p^2}{\hbar^2} A \quad \text{——— (3)}$$

Also $\rightarrow$

$$2A'\phi' e^{i\phi} + A\phi'' e^{i\phi} = 0$$

$$2A'\phi' + A\phi'' = 0 \quad \text{——— (4)}$$

or $(A^2\phi')' = 0$ ——— (5) $\quad \left[\because (A^2\phi')' = 2AA'\phi' + A^2\phi'' = 0 \right.$
$\left. \cancel{A}(2A'\phi' + A\phi'') = 0 \right]$

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From eq^n – 5

$$(A^2\phi')' = 0$$

$$A^2\phi' = \text{constant}$$ (क्योंकि constant का Derivative, zero होता है)

Let ; $A^2\phi' = c^2$

$$\Rightarrow A^2 = \frac{c^2}{\phi'}$$

$$\Rightarrow \boxed{A = \frac{c}{\sqrt{\phi'}}} \quad \text{—— (6)}$$

From eq^n — 3

$$A'' - A(\phi')^2 = \frac{-p^2 A}{\hbar^2}$$

$$A'' = A\left\{(\phi)^2 - \frac{p^2}{\hbar^2}\right\} \quad \text{—— (7)}$$

Assumption:

Since this is difficult to solve so we can assume that $A''(x)$ is negligible or zero

($A(x) \rightarrow$ is slowly varying potential.

$A'' \rightarrow$ double derivative $\Rightarrow$ negligible

And this Assumption is WKB Approximation and it is appropriate for slowly varying potential beacuse $A(x)$ is slowly varying.

$\boxed{A''=0} \rightarrow$ WKB approximation.

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From eq^n – (7) $A'' = A\left\{(i\phi')^2 - \frac{p^2}{\hbar^2}\right\}$

since $A'' = 0$

$\Rightarrow$

$$(\phi')^2 = \frac{p^2}{\hbar^2}$$

$$\phi' = \pm\frac{p}{\hbar} \quad \text{—— (8)}$$

$$\phi'(x) = +\frac{p(x)}{\hbar}$$

$$\frac{d\phi}{dx} = \pm\frac{p(x)}{\hbar}$$

$$\int d\phi = \pm\frac{1}{\hbar}\int p(x)\cdot dx$$

$$\Rightarrow \boxed{\phi(x) = \pm\frac{1}{\hbar}\int p(x)\cdot dx} \quad \text{—— (9)}$$

By eq^n (6) (7) & (8) $\rightarrow$

$$\psi(x) = A(x)\cdot e^{i\phi(x)}$$

$$\psi(x) = \frac{C_1}{\sqrt{p(x)}}\cdot e^{\pm i/\hbar\int p(x)\cdot dx}$$

(+) $\rightarrow \frac{C}{\sqrt{\frac{p(x)}{\hbar}}}$

all constant term in (C)

$\rightarrow$ This is solution of schroding eqn for slowly varying potential.

$\rightarrow$ General solution will be Linear Combination of these two ψ

$$\psi(x) = \frac{C_1}{\sqrt{p(x)}}\cdot e^{+\frac{i}{\hbar}\int p(x)\,dx} + \frac{C_2}{\sqrt{p(x)}}\cdot e^{-\frac{i}{\hbar}\int p(x)\cdot dx}$$

This is for classically allowed region.

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* Derivation For Wavefonction ⇒
(In classically forbidden region) : $E < V(x)$; $T = -ve$

Case 1 : When potential is constant
$V(x) = \text{constant} = V_0$

Schrodinger equation ⇒

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \underset{V_0}{V(x)}\cdot\psi(x) = E\cdot\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V_0)\psi(x)$$

$\because V_0 > E$

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\psi(x)$$

$$\frac{d^2\psi}{dx^2} = K^2\psi(x)$$

$$\left\{ K^2 = \frac{P^2}{\hbar^2} \qquad P = \sqrt{2m(E - V_0)} \right.$$

$m^2 = K^2$

$m = \pm K$

$\psi(x) = A e^{\pm kx}$

Generalised solution ⇒ $\psi(x) = C_1 e^{+kx} + C_2 e^{-kx}$

Case II :- When potential is slowly varying potential
(In classically forbidden region) $E < V(x)$
Wavefunction is ⇒

$$\psi(x) = \frac{C_1}{\sqrt{P(x)}} \cdot e^{+\frac{1}{\hbar}\int P(x)\cdot dx} + \frac{C_2}{\sqrt{P(x)}} \cdot e^{-\frac{1}{\hbar}\int P(x)\cdot dx}$$

diffy only

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Quantization Condition to obtain bound state energy Eigen values :— $E_n = ?$

- E_n or bound state energy eigen values are found out by a Quantization condition →

$$\frac{1}{\hbar}\int_{x_1}^{x_2} P(x)\,dx = \begin{cases} n\pi - (\beta_1+\beta_2) & n = 1,2,3\ldots \\ (n+1)\pi - (\beta_1+\beta_2) & n = 0,1,2,3\ldots \end{cases}$$

or

$$\boxed{\frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(E-V(x))}\,dx = \begin{cases} n\pi - (\beta_1+\beta_2) & n = 1,2,3\ldots \\ (n+1)\pi - (\beta_1+\beta_2) & n = 0,1,2,3\ldots \end{cases}}$$

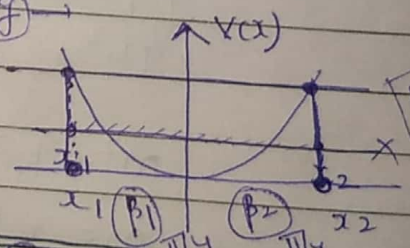
This is the Quantization condition.

where →

- $V(x)$ → potential (given in problem)
- Here x_1 & x_2 are turning points & these are found out by —

$$T = 0$$
$$E - V(x) = 0$$
$$E = V(x)$$

eg →



$E = \frac{1}{2} m w^2 x^2$

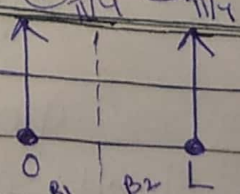
x_1, x_2

- β → phase factor

$$\beta = \begin{cases} 0 & \text{when there is Rigid Wall } V(x) = \infty \\ \frac{\pi}{4} & \text{when there is Smooth wall } V(x) = \text{finite} \end{cases}$$

$V(x) = \begin{cases} x & x > 0 \\ \infty & x \leq 0 \end{cases}$

eg



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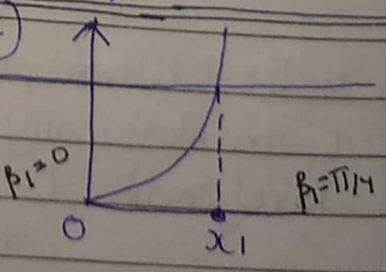
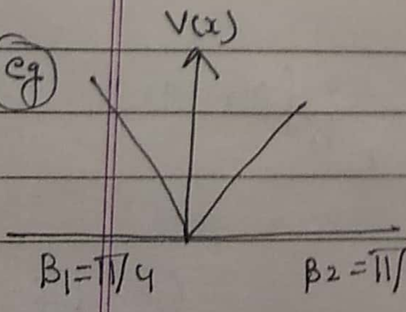
eg



x_1

$E = \frac{1}{2} m w^2 x_1^2 \quad x_1$ ✓

eg



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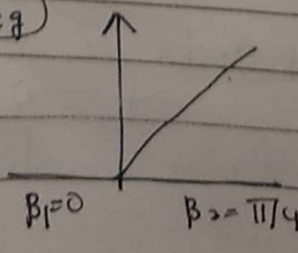


Note:

For constant potential, odd potentials and zero potential apply Quantization condition to find E_n

eg –

$$V(z) = \begin{cases} mgz & z > 0 \\ \infty & z \leq 0 \end{cases}$$

⇐ odd potential
⇐ Quant. Condition ✗

eg –

$$V(z) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

⇐ zero potential
⇐ Quant. condition ✗

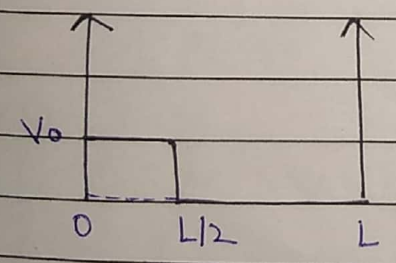
eg – Hydrogen atom

$V(r) = -\frac{Ze}{r}$ ⇒ constant potential
⇒ Quant. Condition ✗

eg –

$V(r) = kr$ ⇐ odd potential
Quant. cond. ~~trick~~

eg



potential ⇒

$$V(x) = \begin{cases} V_0 & 0 < x < L/2 \\ 0 & L/2 < x < L \\ \infty & \text{otherwise} \end{cases}$$

Imp. point :– whenever there is perturbation in the potential (above eg). then take as "normal potential" Don't think about perturbation.

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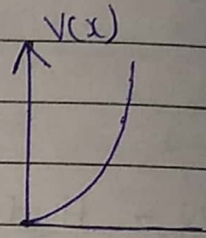
TRICK FOR EVEN POTENTIALS ONLY :

eg: 1 If $V(x) = \alpha(|x|)^m$ $\Rightarrow$ even potential $(|x|)$

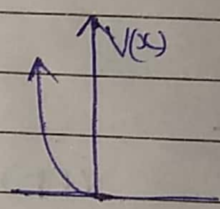
eg: 1 $V(x) = \alpha|x|$

eg: 2 $V(x) = \frac{1}{2}m\omega^2x^2$

eg: 3 $V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2 & x > 0 \\ \infty & x < 0 \end{cases}$



eg: 4 $V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2 & x < 0 \\ \infty & x > 0 \end{cases}$



eg: 5 $V(x) = kx^2$

eg: 6 $V(x) = kx^4$

eg: 7 $V(x) = kx^6$

In all these Question having given potential trick will be applied.

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TRICK :

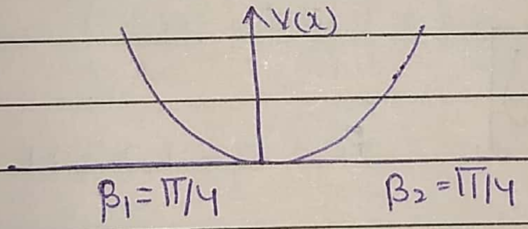
if $V(x) = \alpha(|x|)^m$

$$E_n = \alpha\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\cdot\frac{\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\right]^{\frac{2m}{m+2}}$$

* This is For $n = 1, 2, 3, 4 - -$
* If you want your answer For $n = 0, 1, 2, 3, 4 - - - -$ then replace n by $n+1$ $[n \longrightarrow (n+1)]$

* This formula is for smooth walls on both side.

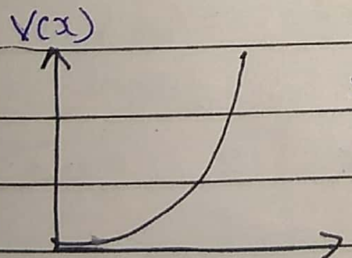
eg



$\Rightarrow$ Harmonic oscillator

$V(x) = \frac{1}{2}m\omega^2x^2$

*



$V(x) = \frac{1}{2}m\omega^2x^2$

To deal with this problem, notice the walls. if there is one rigid wall the add $(1/4)$ in $\left(n-\frac{1}{2}\right)$ term

In above Formula two walls are smoth $\left(-\frac{1}{4}-\frac{1}{4}\right)$
$\Rightarrow$ for one wall smooth only one time minum $1/4$ so for this add $+1/4$

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Ques:- 1 given that $V(x) = \alpha x^2$

Using trick →

$$E_n = \alpha\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\;\frac{\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{m}+1\right)}\right]^{\frac{2m}{m+2}}$$

$n = 1, 2, 3, \ldots$

Here. $\alpha = \alpha$

$m = 2$ (power)

$$E_n = \alpha\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\;\frac{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+1\right)}\right]^{\frac{2\times2}{2+2}}$$

$$E_n = \alpha\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\cancel{\pi}}{2m\alpha}}\times\frac{1}{\frac{1}{2}\cancel{\sqrt{\pi}}}\right]^{1}$$

$n = 1, 2, 3 \ldots$

$\sqrt{2}\times\sqrt{2}$

$$E_n = \alpha\left[\frac{\sqrt{2}\,\hbar}{\sqrt{m\alpha}}\left(n-\frac{1}{2}\right)\right]$$

For $n = 1, 2, 3, 4$ Ans

For $n = 0, 1, 2, 3 \ldots$

put $n = n+1$

$\Rightarrow \quad n-\frac{1}{2} \Rightarrow n+1-\frac{1}{2} = \left(n+\frac{1}{2}\right)$ ✓

$$E_n = \alpha\left[\sqrt{2}\,\hbar\left(n+\frac{1}{2}\right)\right]$$

Ans

$\Gamma\frac{3}{2} = \frac{1}{2}\times\sqrt{\pi}$ / $\Gamma 2 = 1!$

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Que-(2) $V(x) = \frac{1}{2}m\omega^2x^2$ $E_n = ?$ $n = 1, 2, 3$

Soln :- $\alpha = \frac{1}{2}m\omega^2$, $m = 2$

$\frac{2\times 2}{2+2} = \frac{4}{4}$

$$\Rightarrow E_n = \frac{1}{2}m\omega^2\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\frac{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+1\right)}\right]^{\frac{2\times2}{2+2}}$$

$$E_n = \frac{1}{2}m\omega^2\left[\left(n-\frac{1}{2}\right)\hbar\frac{\cancel{\sqrt{\pi}}}{\sqrt{2m\alpha}}\times\frac{\cancel{2}}{\cancel{\sqrt{\pi}}}\right]^1$$

$\cancel{\sqrt{2}\times\sqrt{2}}$

$$E_n = \frac{1m\omega^2}{\sqrt{2}\times\cancel{\sqrt{2}}}\left[\cancel{\sqrt{2}}\left(n-\frac{1}{2}\right)\hbar\right] \Rightarrow \frac{\hbar m\omega^2}{\sqrt{2}(\sqrt{m\alpha})}\left(n-\frac{1}{2}\right) \Rightarrow \text{Ans}$$

$\alpha = \frac{1}{2}m\omega^2$

For $n = 0, 1, 2, 3 \ldots$ $n \Rightarrow (n+1)$

$$E_n = \left(\frac{1}{2}m\omega^2\right)\left[\left(n+\frac{1}{2}\right)\hbar\times\frac{2}{\sqrt{2m\times\frac{1}{2}m\omega^2}}\right]$$

Que:-(3) $V(x) = k|x|$ $m = 1$

Here; $\alpha = k$, $m = 1$

$\frac{2\times1}{2+1}$

Soln- $$E_n = k\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2mk}}\frac{\Gamma(1+3/2)}{\Gamma(1+1)}\right]$$

$n = 1, 2, 3, 4.$

$$E_n = k\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2mk}}\times\frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{1}\right]^{2/3}$$

$$E_n = k\left[\left(n-\frac{1}{2}\right)\hbar\times\frac{3}{4}\frac{\pi}{\sqrt{2mk}}\right]^{2/3}.$$

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Dependency of E_n on 'n'
(For large values of n)

$V(x) = \alpha |x|^m$

$$E_n \propto (n)^{\frac{2m}{(m+2)}}$$ power

For large values of n, we may write

$$\left(n - \frac{1}{2}\right) \longrightarrow n$$

$\Rightarrow$

$$E_n = \alpha \left[n\hbar \sqrt{\frac{\pi}{2m\alpha}} \frac{\Gamma\left(\frac{1}{m} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{m} + 1\right)} \right]^{\frac{2m}{m+2}}$$

Dependency of E_n on α $\Rightarrow$

$$E_n = \alpha^1 \times \frac{1}{\alpha^{\frac{1}{2} \times \frac{2m}{m+2}}} \Rightarrow \alpha^{1 - \frac{m}{m+2}} \Rightarrow \alpha^{\frac{\cancel{m} + 2 - \cancel{m}}{m+2}}$$

$$\Rightarrow E_n \propto (\alpha)^{\frac{2}{m+2}}$$

?

Que→ 4 $V(x) = \alpha x^4$

$$E_n = \alpha \left[\left(n - \frac{1}{2}\right) \hbar \sqrt{\frac{\pi}{2m\alpha}} \frac{\Gamma\left(\frac{1}{4} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{4} + 1\right)} \right]^{\frac{2 \times 4}{2+4}}$$

$$E_n = \alpha \left[\left(n - \frac{1}{2}\right) \hbar \times \frac{\sqrt{\pi}}{\sqrt{2m\alpha}} \times \frac{\Gamma 7/4}{\Gamma 5/4} \right]^{4/3} \Rightarrow \text{Ans}$$

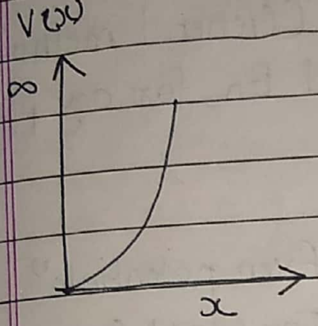
$\Gamma 7/4 = ?$

$\Gamma 5/4 = ?$ We can't find these values.

Que (5)

Half Harmonic oscillator:

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0 \\ \infty & x \le 0 \end{cases}$$



1 smooth wall

1 Rigid wall

$\Rightarrow \left(n - \frac{1}{2}\right) + \frac{1}{4} = n - \frac{1}{4}$

$\alpha = \frac{1}{2} m\omega^2 , \ m = 2$

$$E_n = \frac{1}{2} m\omega^2 \left[\left(n - \frac{1}{4}\right) \hbar \sqrt{\frac{\pi}{2m\alpha}} \frac{\Gamma\left(\frac{1}{2} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{2} + 1\right)} \right]^{\frac{2\times 2}{2+2}}$$

$n = 1, 2, 3, 4$

$$E_n = \frac{1}{2} m\omega^2 \left[\left(n - \frac{1}{4}\right) \hbar \frac{\cancel{\sqrt{\pi}}}{\sqrt{2m \times \frac{1}{\cancel{2}} m\omega^2}} \times \frac{1}{\Gamma\frac{3}{2}} \right]^1$$

$\Downarrow$

$\frac{1}{2} \times \cancel{\sqrt{\pi}}$

$$E_n = \frac{1}{2} m\omega^2 \left[\left(n - \frac{1}{4}\right) \hbar \times \frac{\sqrt{2} \times \sqrt{\cancel{2}}}{\cancel{\sqrt{2}} \times m\omega} \right]$$

$$E_n = \frac{1 \cancel{m}\omega^{\cancel{2}}}{\sqrt{2} \times \cancel{\sqrt{2}}} \left(\frac{\cancel{\sqrt{2}}}{\cancel{m}\cancel{\omega}} \left(n - \frac{1}{4}\right) \hbar \right)$$

$$E_n = \frac{\omega\hbar}{\sqrt{2}} \left(n - \frac{1}{4}\right) \Rightarrow \text{Ans}$$

Baldeo

Lec-3

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WKB Method is used to find :-

① E_n - Energy eigen value

② T - Transmission cofficient.

To Find E_n-

Method-1 → Quantization condition (General method.)
By Integration; we find E_n, for any potential.

Method-2 → Trick
only Applicable for "Even potentials"
like - $V(x) = \alpha x^4$, $V(x) = \alpha x^6$ → For exact form only by Integration
↳ For dependency, we can use trick

Quantization condition - Can be applied to any potential → even, odd, constant, zero.

$$\frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m(E - V(x))}\cdot dx = \begin{cases} n\pi - (\beta_1 - \beta_2) & n = 1,2,3,4 ---- \\ (n+1)\pi - (\beta_1 + \beta_2) & n = 0,1,2,3, ---- \end{cases}$$

Phase fector $\beta \Rightarrow \begin{cases} 0 & \text{Rigid wall } V = \infty \\ \pi/4 & \text{if smooth wall } V = \text{finite} \end{cases}$

$x_1, x_2 \Rightarrow$ turing points. $\Rightarrow T = 0$

$\Rightarrow E = V(x)\big|_{x = x_1}$ & $E = V(x)\big|_{x = x_2}$

$\Downarrow$ $x_1 = ?$ $\quad$ $\Downarrow$ $x_2 = ?$

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Trick : if $V(x) = \alpha |x|^{\nu} \Rightarrow$ even potential

$$E_n = \alpha \left[\left(n - \frac{1}{2}\right) \hbar \sqrt{\frac{\pi}{2m\alpha}} \times \frac{\Gamma\left(\frac{1}{\nu} + \frac{3}{2}\right)}{\Gamma\left(\frac{1}{\nu} + 1\right)} \right]^{\frac{2\nu}{\nu+2}}$$

m = mass of particle
ν = power of x.

where $n = 1, 2, 3, 4$ - - - -.

For $n = 0, 1, 2, 3$ - - -
Replace n by $(n+1)$ in given Formula.

$\Rightarrow$ Dependency of E_n on n for large value of n. $\rightarrow$

$$E_n \propto (n)^{\frac{2\nu}{\nu+2}} \qquad \left(n - \frac{1}{2}\right) \longrightarrow n.$$

$\Rightarrow$ Dependency of E_n on α $\rightarrow$

$$E_n \propto (\alpha)^{1 - \frac{\nu}{\nu+2}} \Rightarrow (\alpha)^{\frac{\cancel{\nu}+2-\cancel{\nu}}{2+\nu}}$$

$$E_n \propto (\alpha)^{\frac{2}{\nu+2}}$$

Note $\rightarrow$
- Trick works even potentials (exception $- \alpha x^4, \alpha x^6$)

- Dependency of E_n on n & α can be obtained for each even potential.

- Dependency of 'E_n on n' can also be obtain by this trick for odd potentials (but not on α)
- Trick does not provide anything for zero or Constant potentials.

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Transmission Cofficient $\Rightarrow$

$$T = e^{-2\gamma}$$

$$\gamma = \frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\cdot dx$$

Problems ; Zetelli

Q→① eg. 9.7 (523) Use the WKB method to estimate the energy levels of One Dimensional Harmonic oscillator.

$V(x) = \frac{1}{2}m\omega^2x^2$ $\qquad E_n = ?$

$$E_n = \alpha\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\alpha}}\,\frac{\sqrt{\frac{1}{\nu}+\frac{3}{2}}}{\sqrt{\frac{1}{\nu}+1}}\right]^{\frac{2\nu}{\nu+2}}$$

$\alpha = \frac{1}{2}m\omega^2$ $\qquad \nu = 2$

$$E_n = \left(\frac{1}{2}m\omega^2\right)\left[\left(n-\frac{1}{2}\right)\hbar\sqrt{\frac{\pi}{2m\times\frac{1}{2}m\omega^2}}\times\frac{\sqrt{\frac{1}{2}+\frac{3}{2}}}{\sqrt{\frac{1}{2}+1}}\right]^{\frac{2\times2}{2+2}}$$

$$E_n = \frac{1\cancel{m}\omega^{\cancel{2}}}{\cancel{2}}\left[\left(n-\frac{1}{2}\right)\hbar\times\frac{\cancel{\sqrt{\pi}}}{\cancel{m}\omega}\times\frac{1}{\frac{1}{\cancel{2}}\times\cancel{\sqrt{\pi}}}\right]$$

$$E_n = \hbar\omega\left(n-\frac{1}{2}\right) \qquad n = 1, 2, 3, 4 - - -$$

Ans

For $n = 0, 1, 2, 3 - - - -$

$$E_n = \hbar\omega\left(n+1-\frac{1}{2}\right) = \hbar\omega\left(n+\frac{1}{2}\right) \qquad n = 0, 1, 2, 3 - - -$$

"Exact energy"

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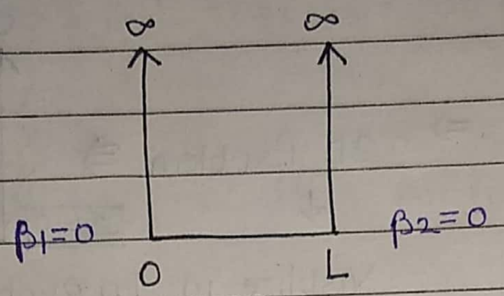
Que-② eg-9.8 Pg-526 Use the WKB approximation to calculate the energy levels of a spinless particle of mass 'm' moving in a 1D box with walls at $x = 0$ & $x = L$, $E_n = ?$



$$V(x) = \begin{cases} 0 & 0 < x < L \\ L & \text{otherwise} \end{cases}$$

0 to L ⇒ No trick ⇒ By Quantization Condition ✓

Soln :- We know, Quantization condition is –

$$\frac{1}{\hbar}\int_{x_1}^{x_2} \sqrt{2m\{E - V(x)\}}\cdot dx = \begin{matrix} n\pi - (\beta_1 + \beta_2) & n = 1, 2, 3 \ldots \\ (n+1)\pi - (\beta_1 + \beta_2) & n = 0, 1, 2, \ldots \end{matrix}$$

also $\beta = \begin{cases} 0 & \text{Rigid wall } (V \to \infty) \\ \frac{\pi}{4} & V \to \text{finite} \end{cases}$

$x_1, x_2 \Rightarrow ?$

$$E = V(x)\Big|_{x=0} \quad ⇒ \quad \begin{matrix} x_1 = 0 \\ x_2 = L \end{matrix}$$

$$\Rightarrow \frac{1}{\hbar}\int_0^L \sqrt{2m(E - 0)}\cdot dx = n\pi \quad \text{for } n = 1\ 2\ 3.$$

$$\sqrt{2mE}\int_0^L dx = n\pi\hbar$$

$$\sqrt{2mE} \times L = n\pi\hbar \quad \text{(squaring both side)}$$

$$2mE \times L^2 = n^2\pi^2\hbar^2$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \Rightarrow \text{exact solution}$$

Ans