Non-Linear Partial differential Equation

#. Compatible System of first-order equations :-

Consider the first order partial differential equations

$$f(x,y,z,p,q)=0 \quad \text{----(i)}$$

and $$g(x,y,z,p,q)=0 \quad \text{----(ii)}$$

Then equation (i) and (ii) are known as compatible system if they have atleast one common solution.

Necessary and sufficient condition :-

Necessary and sufficient condition for compatible system is $[f, g]=0$

or, $$\boxed{\frac{\partial(f,g)}{\partial(x,p)} + p\cdot\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\cdot\frac{\partial(f,g)}{\partial(z,q)} = 0}$$

where $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ are first oder partial differential equation.

Note: Let the partial differential differential equations $p=P(x,y)$ and $q=Q(x,y)$ are compatible iff $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$.

Example :- Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible, and solve them.

Solution :-

Let $\quad f(x, y, z, p, q) = xp - yq = 0 \quad \text{---} (1)$

and $\quad g(x, y, z, p, q) = z(xp + yq) - 2xy = 0. \quad \text{---} (2)$

Therefore,

$f_x = p, \quad f_y = -q, \quad f_z = 0, \quad f_p = x, \quad f_q = -y.$

and $g_x = zp - 2y, \quad g_y = zq - 2x, \quad g_z = xp + yq, \quad g_p = zx;$

$g_q = zy.$

$$\therefore \quad \frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & xz \end{vmatrix}$$

$$= \cancel{2}xzp - xzp + 2xy$$
$$= 2xy.$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + yq & xz \end{vmatrix} = -(x^2 p + xyq)$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq - 2x & zy \end{vmatrix} = -zyq + zyq - 2xy$$
$$= -2xy.$$

and $\dfrac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2 q.$

$$\therefore \frac{\partial(f,g)}{\partial(x,p)} + p\cdot\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\cdot\frac{\partial(f,g)}{\partial(z,q)}$$

$$= 2xy - x^2p^2 - xypq - 2xy + xypq + y^2q^2$$

$$= -x^2p^2 + y^2q^2$$

$$= -(x^2p^2 - y^2q^2)$$

$$= -(xp+yq)\cdot(xp-yq)$$

$$= -(xp+yq)\cdot 0$$

$$= 0$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p\cdot\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\cdot\frac{\partial(f,g)}{\partial(z,q)} = 0.$$

Hence equations 1 and 2 are compatible.

Now, solving 1 and 2, for p and q we get

$$p = \frac{y}{z}, \quad q = \frac{x}{z}.$$

$\because dz = p\,dx + q\,dy$ **

$\Rightarrow dz = \left(\frac{y}{z}\right)dx + \left(\frac{x}{z}\right)dy$

$\Rightarrow z\,dz = y\,dx + x\,dy$

$\Rightarrow \boxed{z^2 = 2xy + c}$ (on Integration)

**

$\because z = f(x,y)$

$\Rightarrow dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\cdot dy$

$\Rightarrow dz = p\cdot dx + q\,dy$

where $p = \frac{\partial f}{\partial x}$

$q = \partial f/\partial y$.

Hence we get

$$z^2 = 2xy + c,$$ where c is arbitrary constant.

(Which represent the common solution for the eqn ① and ②).

Exercise:

1). Show that $\frac{\partial z}{\partial x} = 7x + 8y - 1$ and $\frac{\partial z}{\partial y} = 8x + 9y + 10$ are compatible.

2). Show that the equations $p = ax + hy + g$ and $q = hx + by + f$ are compatible and hence solve them.

where a, h, g, b and f all are real constants.

Charpit's Method :-

(General method of solving PDE of order one but of any degree.)

Consider a first order non-linear PDE

$$f(x, y, z, p, q) = 0 \quad --- (1)$$

A family of PDE

$$g(x, y, z, p, q, a) = 0 \quad --- (2)$$

is said to be one parameter family of PDE, comptible with (1) if (2) is compatible with (1) for each value of of 'a'.

$$\Rightarrow \quad [f, g] = 0$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p \cdot \frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q \frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$\Rightarrow \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} + p \cdot \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} = 0$$

$$\Rightarrow f_x g_p - g_x f_p + p(f_z g_p - g_z f_p) + (f_y g_q - g_y f_q) + q(f_z g_q - g_z f_q) = 0$$

$$\Rightarrow -f_p g_x - f_q g_y - (p f_p + q f_q) g_z + (f_x + p f_z) g_p + (f_y + q f_z) g_q = 0$$

which is a quasi linear of Ist order PDE for g with x, y, z, p, q as independent variable.

its solution is given as

$$\boxed{\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + q\cdot f_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}}$$

this is known as Charpit's auxiliary equation.

Note. Charpit's method consist in choosing a one parameter family of PDE to which is such that each member of the family is compatible with the given eqn 1.

Example :- Solve : $px + qy = pq$ --- 1

Solution:

Let $f(x,y,z,p,q) = px + qy - pq = 0$

then

$f_x = p, \quad f_y = q, \quad f_z = 0, \quad f_p = x - q, \quad f_q = y - p.$

Now Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + q\cdot f_z}.$$

$$\Rightarrow \frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q)+q(y-p)} = \frac{-dp}{p} = \frac{-dq}{q} \quad \text{--- 2}$$

from 2, we have

$$\frac{-dp}{p} = -\frac{dq}{q}$$

$$\Rightarrow \quad \frac{dp}{p} = \frac{dq}{q}$$

on integration, we have

$$\log p = \log q + \log a$$

$$\Rightarrow \quad p = aq \quad \text{----} \ (3)$$

putting value of p in (1), we have

$$axq + qy = a.q.q$$

$$\Rightarrow \quad (ax+y).q = aq^2$$

$$\Rightarrow \quad (ax+y) = aq$$

$$\Rightarrow \quad \boxed{q = \frac{ax+y}{a}} \quad \text{---} \ (4).$$

therefore, we get

$$\boxed{p = ax+y.}$$

Now, $\quad dz = p\,dx + q\,dy$

$$\Rightarrow \quad dz = (ax+y)\,dx + \frac{ax+y}{a}.\,dy$$

$$\Rightarrow \quad a\,dz = (ax+y)\,(a\,dx + dy)$$

on integrating, we get

$$az = \frac{(ax+y)^2}{2} + b'$$

$$\Rightarrow \quad 2az = (ax+y)^2 + 2b'$$

$$\Rightarrow \quad \boxed{2az = (ax+y)^2 + b}$$

which is required solution, known as "<u>Complete Integral</u>"

$$\Rightarrow \boxed{p = \frac{2x(z-ay)}{x^2-a}}$$

$\because$ $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{2x(z-ay)}{x^2-a}\cdot dx + a\cdot dy$$

$$\Rightarrow dz - a\,dy = \frac{2x(z-ay)}{x^2-a}\,dx$$

$$\Rightarrow \frac{dz - a\,dy}{z-ay} = \frac{2x}{x^2-a}\,dx$$

On Integrating, we get

$$\log(z-ay) = \log(x^2-a) + \log b$$

or, $z - ay = b(x^2-a)$

or, $\boxed{z = ay + b(x^2-a)}$ ---- (*)

where a & b are arbitrary constant.

Hence, eqn (*) is the <u>complete integral</u> for PDE (1).

Singular Integral:

Differentiating eqn ⊛ partially w.r.t. a & b, we get

$$0 = y - b \quad ---- (3)$$

& $$0 = x^2 - a \quad --- (4)$$

from (3), we get $b = y$.

from (4), $a = x^2$.

Now substituting the values of a & b, given by above in ⊛, we get

$$z = x^2 y + y(x^2 - x^2)$$

$$\Rightarrow \boxed{z = x^2 y}$$

which is the required singular integral.

OR

Differentiating eqn (1) w.r.t. p and q respectively, we get

$$0 - x^2 - 0 + q = 0 \quad ---- 3(i)$$

& $$0 - 0 - 2xy + p = 0 \quad ---- 4(i)$$

from 3(i) & 4(i), we have.

$$q = x^2 \quad \& \quad p = 2xy.$$

using above values of p & q in (1), we get

$$2xz - 2xy.x^2 - 2x^2.xy + 2xy.x^2 = 0 \Rightarrow \boxed{z = x^2 y.}$$

which is required singular integral.

General Integral:

Replacing b by $\phi(a)$ in eqn ✱, we get

$$z = ay + \phi(a)(x^2-a) \quad ---- \text{(5)}$$

Differentiating eqn (7) w.r.t 'a' we get

$$0 = y + \phi'(a).(x^2-a) - \phi(a) \quad --- \text{(6)}$$

Thus the general integral is obtained by eliminating 'a' from (5) and (6).

Exercise:-

(A). Find the complete integral of the following equations:

i). $zpq = p+q$

ii). $q = (z+px)^2$

iii). $(p^2+q^2)x = pz$

(B). Find the complete, singular and general integrals of the pde $p^2x + q^2y = z$.

#> Special methods of solutions applicable to certain standard forms:

A Standard Form: Only p and q are present:

In this case we consider the pde of the form

$$f(p,q)=0 \quad ---- (1)$$

Therefore Charpit's auxiliarly equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{-dp}{f_x+pf_z} = \frac{-dq}{f_y+qf_z}$$

$\Rightarrow \quad \frac{-dp}{0} = \frac{-dq}{0}$ as $f_x = f_y = f_z = 0$ as f is independent of x, y, z.

$\Rightarrow \quad dp = 0 \quad | \quad dq = 0$

$\Rightarrow \quad p = \text{constt} \quad | \quad q = \text{Constant}$

$$\Rightarrow \boxed{p = a \text{ (say)} \qquad q = b \text{ (say)}}. \quad ---- (2)$$

Now, we get

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = a\,dx + b\,dy$$

On Integrating,

$$\Rightarrow \boxed{z = ax + by + c} \quad ---- (3)$$

where c is arbitrary constant.

putting $p=a$ and $q=b$ in ①, we get

$$f(a,b)=0$$

$$\Rightarrow \quad b=\phi(a) \quad \text{-----} \ ④$$

using eqn ④ in eqn ③, we have

$$\boxed{z = ax + \phi(a)\cdot y + c} \quad \text{----} \ \boxed{5}$$

which is the required complete integral of pde ①.

Singular Solution:

differentiating eqn ⑤ partially w.r.t. 'a' and 'c', we have

$$0 = x + \phi'(a)\cdot y \quad \text{----} \ ⑥$$

$$\& \quad 0 = 1 \quad \text{----} \ ⑦$$

since the last eqn ⑦ is meaning-less, Hence we conclude that the pde's of the form Ⓐ have no singular solution.

General solution:

first we assume that

$$c = \psi(a)$$

so, we have

$$z = ax + \phi(a)\cdot y + \psi(a) \quad \text{---} \ ⑧$$

Diff. eqn 8 partially w.r.t a, we get

$$0 = x + \phi'(a).y + \psi'(a) \quad \text{----} (9)$$

Eleminating a between 8 and 9, we get the general solution of pde 1.

Example:—

1). Find complete and general solutions (integrals) of the pde

$$p^2+q^2 = m^2, \text{ where } m \text{ is constant.}$$

Solution:

Since we have

$$p^2+q^2 = m^2 \quad \text{-----} (1)$$

where m is constant.

∵ eqn 1 is of the form $f(p,q)=0$. Therefore the solution of 1 is -

$$z = ax + by + c \quad \text{----} (2)$$

by putting $p=a$ & $q=b$ in 1, we get

$$a^2+b^2 = m^2$$

$$\Rightarrow \quad b = \pm\sqrt{(m^2-a^2)}$$

Hence the complete integral is

$$z = ax \pm \sqrt{m^2-a^2} + c. \quad \text{---} (3)$$

General solution :

Let $c = \phi(a)$.

we get
$$z = ax \pm \sqrt{m^2-a^2}\,.\,y + \phi(a) \quad \text{----} ④$$

diff. eqn ④ partially w.r.t a we get
$$0 = x \mp \frac{ay}{\sqrt{m^2-a^2}} + \phi'(a). \quad \text{---} ⑤$$

eliminating a from eqn ④ and ⑤, we get the required general solution of the pde ①.

Exercise:

Find the complete and general integrals of the following eqns –

i). $pq = a$, where a is constant

ii). $p+q = pq$

iii). $p^2+q^2 = mpq$, where m is constant

- Standard from (B): Clairaut Equation:

A first order PDE is said to be of clairaut form if it can be written as -

$$z = px + qy + f(p, q) \quad --- (1).$$

Let $F \equiv px + qy + f(p, q) - z \quad --- (2).$

Therefore, Charpit's auxiliary eqns are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z} = \frac{-dq}{F_y + qF_z}$$

$$\Rightarrow \frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{p(x+f_p) + q.(y+f_q)} = \frac{-dp}{0} = \frac{-dq}{0} \quad -(3)$$

From last two fraction, we get

$$\frac{dp}{0} = \frac{dq}{0}$$

$\Rightarrow$	$dp = 0$	$dq = 0$
$\Rightarrow$	$p = \text{constt}$	$q = \text{constt}$
$\Rightarrow$	$\boxed{p = a}$	$\boxed{q = b}$ (say)

Substituting $p=a$ and $q=b$ in eqn (1), the complete integral is

$$\boxed{z = ax + by + f(a,b)} \quad --- (4)$$

Note:

Singular integral:

Diff eqn (4) partially w.r.t. a and b we have

$$0 = x + f'(a,b) \quad ---- (5)$$

$$0 = y + f'(a,b) \quad ---- (6)$$

eliminate (a) and (b) from eqn (5) and (6), we get required singular integral of pde (1).

Example:— Find the complete and singular integrals of the equations

$$z = px + qy + p^2 + q^2 \quad --- (1).$$

Solution:

Here given that :—

$$z = px + qy + p^2 + q^2 \quad --- (1).$$

the pde (1) is in clairaut's form, Hence its complete integral is

$$\boxed{z = ax + by + a^2 + b^2} \quad --- (2)$$

Singular Solution:

diff. eqn (2) partially w.r.t. a & b, we have

$$0 = x + 2a. \Rightarrow a = -\frac{x}{2}$$

$$0 = y + 2b \Rightarrow b = -\frac{y}{2}$$

putting the values of a and b in eqn (2), we get the singular integral of pde (1) as

$$z = \left(-\frac{x}{2}\right).x + \left(\frac{-y}{2}\right).y + \left(\frac{-x}{2}\right)^2 + \left(\frac{-y}{2}\right)^2$$

$$\Rightarrow z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$\Rightarrow \boxed{z = -\left(\frac{x^2+y^2}{4}\right)}$$

Exercise Find the complete and singular integrals of the pdes

i). $z = px + qx - \sqrt{pq}$

ii). $z = px + qx + \sqrt{p^2+q^2}$

iii). $z = px + qx + \log(pq)$.

iv). $pqz = p^2qx + pq^2y + (p^2+q^2)$.

Standard form C: Only p, q and z are present:

Consider the first order PDE of the form

$$f(p, q, z) = 0 \quad \text{-----} (1)$$

Charpit's auxiliary eqns are.

$$\frac{dx}{+f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p.f_p + q.f_q} = \frac{-dp}{p.f_z} = \frac{-dq}{qf_z} \quad -(2)$$

$\{\because f_x = f_y = 0$ as $f(p, q, z) = 0\}$

From last two fraction, we get

$$\frac{dp}{pf_z} = \frac{dq}{q.f_z}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

on integration,

$\boxed{q = ap.}$, where a is an arbitrary constant.

Now,

$$dz = p\,dx + q\,dy$$

$$\Rightarrow \quad dz = p\,dx + ap\,dy.$$

$$\Rightarrow \quad dz = p\,(dx + a\,dy)$$

$$\Rightarrow \quad dz = p.d(x+ay)$$

$$\Rightarrow \quad dz = p\,du \quad \text{-------} \quad (3)$$

where $u = x + ay.$ & $p = \frac{dz}{du}$ --- (4)

so we get

$$q = ap = a\frac{dz}{du}. \quad \text{----} \quad (5).$$

Solving 4 and 5 we get z as function of u. complete integral is obtained when by replacing $u = x + ay$.

$\boxed{\text{Working method}} \rightarrow$

Let pde is

$$f(p, q, z) = 0 \quad \text{---} (1).$$

Step 1>. Let $u = x + ay$, where a is arbitrary constant.

Step 2>. Replace p and q by $\frac{dz}{du}$ and $\frac{a\,dz}{du}$ respectively in 1, and ~~the~~ solve the pde by usual methods.

Step 3>. Replace u by $x+ay$ in the solution obtained by step 2.

Step 4>. Hence we get required complete integral of pde ①.

Example – Find the complete integral of

$$p(1+q) = qz.$$

Solution :–

∵ Given that

$$p(1+q) = qz \quad \text{----} ①$$

putting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in ①

where a is arbitrary constant and $u = x+ay$.

Therefore, we get

$$\frac{dz}{du}\left(1 + a.\frac{dz}{du}\right) = a.z\,\frac{dz}{du}.$$

$$\Rightarrow \quad 1 + a.\frac{dz}{du} = az$$

$$\Rightarrow \quad a\frac{dz}{du} = az - 1$$

$$\Rightarrow \quad a\left(\frac{dz}{az-1}\right) = du$$

On integration we have

$$\log(az-1) = u + c.$$

$$\Rightarrow \quad \boxed{\log(az-1) = x+ay+c.} \quad \Big|\ \because u = x+ay.$$

Example: Find the complete integral of

$$z^2(p^2+q^2+1)=k^2, \qquad k = \text{const.}$$

Solution:

$\because$ Here given that

$$z^2(p^2+q^2+1)=k^2 \quad \text{---}(1)$$

Now putting. $p=\frac{dz}{du}$ and $q = a\frac{dz}{du}$ in eqn (1) where a is an arbitrary constant and $u = x+ay$.

Then, we get

$$z^2.\left(\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2+1\right)=k^2$$

$$\Rightarrow z^2\left\{\left(\frac{dz}{du}\right)^2+a^2.\left(\frac{dz}{du}\right)^2\right\}=k^2-z^2$$

$$\Rightarrow (1+a^2).\left(\frac{du}{dz}\right)^2=\frac{k^2-z^2}{z^2}$$

$$\Rightarrow \pm\sqrt{1+a^2}.\frac{du}{dz}=\frac{\sqrt{k^2-z^2}}{z}$$

$$\Rightarrow \pm\sqrt{1+a^2}.\frac{z}{\sqrt{k^2-z^2}}.dz=du$$

On integrating, we have

$$\pm \sqrt{1+a^2} \cdot \sqrt{k^2 - z^2} = u + c.$$

$$\Rightarrow \quad (1+a)^2 (k^2 - z)^2 = (u+c)^2$$

$$\Rightarrow \quad \boxed{(1+a)^2 (k^2 - z)^2 = (x + ay + c)^2} \quad --- \text{ ②}$$

where c is arbitrary constant.

⇒ Eqn ② is required complete integral of the pde ①.

Exercise:

Find the complete integral of the following equations

i). ①. $p(1+q^2) = q(z-1)$

ii). $p^3 + q^3 - 3pqz = 0$

iii). $z^2 (p^2 z^2 + q^2) = 1.$

Standard form D: Equation of the form $f_1(x,p)=f_2(y,q)$:

consider the non-linear pde of first order of the form

$$f_1(x,p) = f_2(y,q) \quad \text{----} (1)$$

i.e $$F \equiv f_1(x,p) - f_2(y,q) = 0 \quad \text{----} (2)$$

Then charpit's auxiliary eqns are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p + qF_q} = \frac{-dp}{F_x + pF_z} = \frac{-dq}{F_y + qF_z}$$

$$\Rightarrow \frac{dx}{+\frac{\partial f_1}{\partial p}} = \frac{dy}{-\frac{\partial f_2}{\partial q}} = \frac{dz}{p.\frac{\partial f_1}{\partial p} - q\frac{\partial f_2}{\partial q}} = \frac{-dp}{\frac{\partial f_1}{\partial x}}$$

$$= \frac{-dq}{-\frac{\partial f_2}{\partial y}} \quad \text{—} (2)$$

Now taking first and fourth fraction, we get

$$\frac{dx}{\frac{\partial f_1}{\partial p}} = -\frac{dp}{\frac{\partial f_1}{\partial x}}$$

$$\Rightarrow \left(\frac{\partial f_1}{\partial x}\right)dx + \left(\frac{\partial f_1}{\partial p}\right).dp = 0$$

$$\Rightarrow df_1 = 0$$

$$\Rightarrow f_1(x,p) = a \quad \text{where } a \text{ is constt.} \text{—} (3)$$

which is of the form $f_1(x,p) = f_2(y,q)$.

Now, we get

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \quad (\text{say}) \quad --- \; (2)$$

then, we get

$$\frac{p^2(1+x^2)}{x^2} = a$$

$$\Rightarrow p^2 = \frac{ax^2}{1+x^2}$$

$$\Rightarrow p = \pm \frac{\sqrt{a}\,x}{\sqrt{1+x^2}}$$

$$\frac{q}{y} = a.$$

$$\Rightarrow q = ay$$

$$\Rightarrow q = ay \quad --- \; (3)$$

Substituting these values of p & q, in

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \pm \frac{\sqrt{a}\,x}{\sqrt{1+x^2}}\,dx + ay\,dy.$$

on integrating, we get

$$\Rightarrow \boxed{z = \pm\sqrt{a}.\sqrt{1+x^2} + \frac{a}{2}y^2 + c}^{**} \quad — \; (4)$$

where c is an arbitrary constant.

**which is the complete integral of the pde (1).

Exercise:

Find the complete and singular integrals of the following pdes —

1). $(p^2+q^2) = x^2+y^2$.

2). $p^2 y(1+x^2) = qx^2 + x^2$.

3). $p^2+y = q+x$.

4). $z^2(p^2+q^2) = x^2+y^2$.

Jacobi's Method :→

Consider the pde

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 0 \quad --- (1)$$

where $P_1 = \frac{\partial z}{\partial x_1}$, $P_2 = \frac{\partial z}{\partial x_2}$, $P_3 = \frac{\partial z}{\partial x_3}$

and the dependent variable z does not occurs except by its partial derivatives P_1, P_2, P_3.

Then the solution of PDE (1) is given by Jacobi's method.

Note Jacobi's method is used for solving pde having three or more than three indebendent variables.

Working Method:—

Step1 : Consider a pde having three independent variables x_1, x_2, x_3 s.t.

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 0 \quad --- (1)$$

Step2 : Jacobi's auxiliary eqn for pde (1) is

$$\frac{dP_1}{f_{x_1}} = \frac{dP_2}{f_{x_2}} = \frac{dP_3}{f_{x_3}} = \frac{dx_1}{-f_{P_1}} = \frac{dx_2}{-f_{P_2}} = \frac{dx_3}{-f_{P_3}} \quad --- (2)$$

Step 3: Solving these eqns 2, we obtain two eqns s.t.

$$F_1(x_1, x_2, x_3, P_1, P_2, P_3) = a_1 \quad ---- 3(i)$$

$$F_2(x_1, x_2, x_3, P_1, P_2, P_3) = a_2 \quad --- 3(ii)$$

where a_1 and a_2 are arbitrary constants.

Note While obtaining 3(i) & 3(ii) try to select simple eqns so that we get simblest form of solution.

Step 4:

Verify that eqns 3(i) & 3(ii) satisfy the conditions

$$(F_1, F_2) = \sum_{i=1}^{3} \left(\frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial P_i} - \frac{\partial F_1}{\partial P_i} \cdot \frac{\partial F_2}{\partial x_i} \right) = 0 \quad --- 4$$

if eqn 4 is satisfied the solve eqn 1, 3(i) and 3(ii) for P_1, P_2, P_3 in terms of x_1, x_2, x_3.

Step 5: Complete integral of pde 1 is given by-

$$\because \boxed{dz = P_1 dx_1 + P_2 dx_2 + P_3 dx_3}$$

$$\Rightarrow z = \int P_1 dx_1 + \int P_2 dx_2 + \int P_3 dx_3.$$

Example ①: Find a complete integral of $p_1x_1 + p_2x_2 = p_3^2$. ③

Solution:

Let

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 + p_2x_2 - p_3^2 = 0 \quad \text{--- ①.}$$

$\therefore$ Jacobi's auxiliary eqns are

$$\frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}}.$$

$$\Rightarrow \frac{dp_1}{p_1} = \frac{dp_2}{p_2} = \frac{dp_3}{0} = \frac{dx_1}{-x_1} = \frac{dx_2}{-x_2} = \frac{dx_3}{2p_3} \quad \text{--- ②}$$

Now, taking first and fourth fraction, we get

$$\frac{dp_1}{p_1} = -\frac{dx_1}{x_1}$$

on integration,

$$\Rightarrow \log p_1 = -\log x_1 + \log a_1$$

$$\Rightarrow \log p_1x_1 = \log a_1$$

$$\Rightarrow \underline{p_1x_1 = a_1} \quad \text{--- ③} \quad \& \quad \underline{F_1 \equiv p_1x_1 = a_1}$$

Similarly, taking fraction second & fifth, we get

$$\frac{dp_2}{p_2} = -\frac{dx_2}{x_2}$$

on integration, we get

$$\underline{p_2x_2 = a_2} \quad \text{--- ④} \quad \& \quad \underline{F_2 \equiv p_2x_2 = a_2}$$

Now we have to verify the relation $(f_1, f_2) = 0$

$$\because \quad (f_1, f_2) = \sum_{i=1}^{3} \left(\frac{\partial f_1}{\partial x_i} \cdot \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \cdot \frac{\partial f_2}{\partial x_i} \right)$$

$$= \left(\frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial p_1} - \frac{\partial f_1}{\partial p_1} \cdot \frac{\partial f_2}{\partial x_1} \right) + \left(\frac{\partial f_1}{\partial x_2} \cdot \frac{\partial f_2}{\partial p_2} - \frac{\partial f_1}{\partial p_2} \cdot \frac{\partial f_2}{\partial x_2} \right) + \left(\frac{\partial f_1}{\partial x_3} \cdot \frac{\partial f_2}{\partial p_3} - \frac{\partial f_1}{\partial p_3} \cdot \frac{\partial f_2}{\partial x_3} \right)$$

$$= (p_1 \cdot 0 - x_1 \cdot 0) + (0 \cdot x_2 - 0 \cdot p_2) + (0 \cdot 0 - 0 \cdot 0)$$

$$= 0$$

$$\Rightarrow \boxed{(f_1, f_2) = 0} \quad \text{(verified)}$$

Next, solving (1), (3), (4) for p_1, p_2, p_3 in terms of x_1, x_2, x_3 as –

$$p_1 = \frac{a_1}{x_1} \quad \& \quad p_2 = \frac{a_2}{x_2}$$

therefore from (1) –

$$\frac{a_1}{x_1} \cdot x_1 + \frac{a_2}{x_2} \cdot x_2 = p_3^2$$

$$\Rightarrow \quad p_3 = \sqrt{a_1 + a_2}$$

Hence the complete integral is given by -

$$dz = p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3$$

$$\Rightarrow \quad dz = \left(\frac{a_1}{x_1}\right)\cdot dx_1 + \left(\frac{a_2}{x_2}\right)dx_2 + (a_1+a_2)^{1/2}\, dx_3$$

on integration, we have

$$\boxed{z = a_1 \log x_1 + a_2 \log x_2 + \sqrt{a_1+a_2}\, x_3 + a_3.}$$

where a_1, a_2 and a_3 are arbitrary constant.

Example: Find the complete integral of $2P_1x_1x_3 + 3P_2x_3^2 + P_2^2P_3 = 0$.

Solution: Let

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 2P_1x_1x_3 + 3P_2x_3^2 + P_2^2P_3 = 0 \quad \text{—(1)}$$

$\therefore$ Jacobi's auxiliary eqns are

$$\frac{dP_1}{f_{x_1}} = \frac{dP_2}{f_{x_2}} = \frac{dP_3}{f_{x_3}} = \frac{-dx_1}{f_{P_1}} = \frac{-dx_2}{f_{P_2}} = \frac{-dx_3}{f_{P_3}}$$

$$\Rightarrow \frac{dP_1}{2P_1x_3} = \frac{dP_2}{0} = \frac{dP_3}{2P_1x_1 + 6P_2x_3} = \frac{-dx_1}{+2x_1x_3} = \frac{-dx_2}{-3x_3^2 - 2P_2P_3}$$

$$= \frac{dx_3}{-P_2^2} \quad \text{---(2)}$$

taking first and fourth fraction, we get ;

$$\frac{dP_1}{2P_1x_3} = \frac{-dx_1}{2x_1x_3}.$$

on integration, we have

$$\Rightarrow \quad P_1x_1 = a_1 \quad \text{or} \quad F_1 \equiv P_1x_1 = a_1 \quad \text{—(3)}$$

Similarly, taking second and fifth fraction, we get

$$dP_2 = 0$$

$$\Rightarrow P_2 = a_2 \quad \text{or} \quad f_2 \equiv P_2 = a_2 \text{ ---- } (4).$$

Now eqns (3) and (4) verify $(f_1, f_2) = 0$ --- (*)

$$\therefore (f_1, f_2) = \sum_{i=1}^{3}\left(\frac{\partial f_1}{\partial x_i}\cdot\frac{\partial f_2}{\partial P_i} - \frac{\partial f_1}{\partial P_i}\cdot\frac{\partial f_2}{\partial x_i}\right)$$

$$= \left(\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial P_1} - \frac{\partial f_1}{\partial P_1}\cdot\frac{\partial f_2}{\partial x_1}\right) + \left(\frac{\partial f_1}{\partial x_2}\cdot\frac{\partial f_2}{\partial P_2} - \frac{\partial f_1}{\partial P_2}\cdot\frac{\partial f_2}{\partial x_2}\right) +$$

$$\left(\frac{\partial f_1}{\partial x_3}\cdot\frac{\partial f_2}{\partial P_3} - \frac{\partial f_1}{\partial P_3}\cdot\frac{\partial f}{\partial x_3}\right)$$

$$= (0.0 - 1.0) + (0.0 - (-1).0) + (0.x_3 - 0.P_3)$$

$$= 0$$

Hence we get, $(f_1, f_2) = 0$.

Now, from (3) and (4) we get –

$$\underline{P_1 = \frac{a_1}{x_1}} \quad \text{and} \quad \underline{P_2 = a_2}.$$

using the values of P_1 and P_2 in eqn (1) –

$$2.\frac{a_1}{\cancel{x_1}}.\cancel{x_1}.x_3 + 3.a_2.x_3^2 + a_2^2.P_3 = 0.$$

$$\underline{P_3 = -\frac{(2a_1x_3 + 3a_2x_3^2)}{a_2^2}}.$$

$\Rightarrow$ $\because dz = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$.

$$\Rightarrow dz = \frac{a_1}{x_1} dx_1 + a_2 dx_2 + \left\{ -\frac{(2a_1x_3 + 3a_2x_3^2)}{a_2^2} \right\} dx_3.$$

$$\Rightarrow dz = \frac{a_1}{x_1} \cdot dx_1 + a_2 dx_2 - \frac{(2a_1x_3 + 3a_2x_3^2)}{a_2^2} dx_3.$$

on integration, we have

$$\boxed{z = a_1 \log x_1 + a_2 x_2 - \frac{1}{a_2^2}\left\{ a_1 x_3^2 + a_2 x_3^3 \right\} + a_3}$$

where a_1, a_2, a_3 are arbitrary constant.

Exercise. :

- Find a complete integral of $P_1 P_2 P_3 = z^3 x_1 x_2 x_3$.
- Find a complete integral of $x_1 P_1^2 + x_2 P_2^2 - x_3 P_3^2 = 0$.

Linear partial Differential equation with constant coefficient

A pde in which dependent variable and its derivatives appear only in the first order and not multiplied together, their coeff. being constants known as a linear (pde) partial differential equation with constant coefficient.

Consider the general form of eqn

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial^{n-1} x \cdot \partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \cdots \cdots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right) + \cdots + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right) + N_0 z = f(x,y) \quad (*)$$

where the coeff. $A_0, A_1, A_2, \cdots A_n, B_0, B_1 \cdots B_n, \cdots$ M_0, M_1 & N_0 are constant, and $f(x,y)$ is a constant or continuous function of x & y.

or, $$\left[\left(A_0 D^n + A_1 D^{n-1} D' + \cdots + A_n D'^n\right) + \left(B_0 D^{n-1} + B_1 D^{n-2} + \cdots \cdots + B_{n-1} D'^{n-1}\right) + \cdots + \left(M_0 D + M_1 D'\right) + N_0\right] z = f(x,y)$$

or $$\boxed{F(D, D') = f(x,y). \text{ where } D \equiv \frac{d}{dx},\ D' \equiv \frac{d}{dy}.} \quad *$$

Case 1>. If $f(x,y)=0$ then

$$\boxed{F(D, D')z=0} \quad \text{---- } (2)$$

Eqn (2) is known as Homogeneous Linear partial Differential Equation with constant coeff.

Case 2>. if $f(x,y) \neq 0$ then

$$\boxed{F(D, D')z = f(x,y)} \quad \text{---- } (3)$$

Eqn (3), is known as non-homogeneous linear partial differential equation with constant coefficients.

#2. Working Rule for finding complementary function of Linear homogeneous pde with constant coefficient:

Step 1: Put the given eqn in standard form-

$$(A_1 D^n + A_2 D^{n-1} + \cdots + A_n D'^n) z = f(x,y) \quad \text{---} (1)$$

Step 2: Replacing D by m and D' by 1 in the coeff. of z, we obtain the auxiliary eqn of pde (1) as,

$$A_1 \cdot m^n + A_2 m^{n-1} + \cdots + A_n = 0 \quad \text{---} (2)$$

Step 3: Solve eqn (2) for m, following cases arise.

Case (i) Let $m = m_1, m_2, \cdots m_n$ (distinct roots)

then

$$\boxed{C.F. = \phi_1(y+m_1 x) + \phi_2(y+m_2 x) + \cdots + \phi_n(y+m_n x)} \text{ — } (3)$$

OR

$$\boxed{C.F. = \phi_1(x+m_1 y) + \phi_2(x+m_2 y) + \cdots + \phi_n(x+m_n y)}$$

(*)

Case ii): if $m = m_1$ (n-times repeated)

then $C.F. = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + x^2\cdot\phi_3(y+m_1x) + \cdots$

$$\cdots + x^{n-1}\phi_n(y+m_1x)$$

Case III: Corresponding to a non-repeated factor D on LHS. of (2), the part of C.F. is $\phi(y)$.

Case IV: Corresponding to a non-repeated factor D' on LHS. of (2), the part of C.F is $\phi(x)$.

Case V: Corresponding to a repeated factor D^m on LHS of (2) the part of C.F is

$$C.F. = \phi_1(y) + x\cdot\phi_2(y) + x^2\phi_3(y) + \cdots + x^{m-1}\phi_m(y)$$

Case VI: Corresponding to a repeated factor D'^m on LHS of (2) the part of C.F is

$$C.F = \phi_1(x) + y\cdot\phi_2(x) + y^2\phi_3(x) + \cdots + y^{m-1}\phi_m(x)$$

Example: Solve: $2r + 5s + 2t = 0$

Solution:

$$\Rightarrow (2D^2 + 5DD' + 2D'^2)z = 0 \quad \text{---(1)}$$

putting $D \equiv m$ & $D' \equiv 1$ we have the auxiliary eqn as.

$$2m^2 + 5m + 2 = 0$$

$$\Rightarrow \quad 2m^2 + 4m + m + 2 = 0$$

$$\Rightarrow \quad 2m(m+2) + 1(m+2) = 0$$

$$\Rightarrow \quad (m+2)(2m+1) = 0$$

$$\Rightarrow \quad \boxed{m = -2 \quad \text{or} \quad m = -1/2}$$

Therefore general solution of pde ① is

$$z = \phi_1(y - 2x) + \phi_2(y - x/2)$$

or $\quad z = \phi_1(y - 2x) + \phi_2\left\{\frac{1}{2}(2y - x)\right\}$

or, $\quad \boxed{z = \phi_1(y - 2x) + \psi_1(2y - x)}$

where ϕ_1 and ψ_1 are arbitrary function.

Example: Solve $(D^4D' - D'^5)z = 0$

Solution: $\because$ given that

$$(D^4D' - D'^5)z = 0$$

$$\Rightarrow \quad D'(D^4 - D'^4)z = 0 \quad ---- ①$$

Replacing D by m and D' by 1, we have auxiliary eqn as

$$1.(m^4 - 1) = 0$$

$$\Rightarrow \quad m^4 - 1 = 0$$

$\Rightarrow (m^2-1)(m^2+1)=0$

$\Rightarrow (m+1)(m-1)(m^2+1)=0$

$\Rightarrow m=-1,+1,\pm i$

$\Rightarrow \boxed{m=\pm 1, \pm i}$

From eqn ①, it is clear that there is a non-repeated factor of D', therefore corresponding C.F is $\phi(x)$.

Hence general solution of pde ① is given by

$$z=\phi_1(y-x)+\phi_2(y+x)+\phi_3(y+ix)+\phi_4(y-ix)$$
$$+\phi_5(x)$$

where ϕ_i's are arbitrary functions for $i=1$ to 5.

Example 3: Solve: $(D^3D'^2)z=0$ - - - .

Solution:

$\because$ Here given that

$$(D^3 \cdot D'^2)z=0 \quad \text{- - - - ①}$$

In the L.H.S. of eqn ① there is repeated factor ~~of~~ D^3 and D'^2.

Replacing D by m and D' by 1, we get

$$m^3=0$$

$$\Rightarrow \quad m=0 \quad (\text{3-times repeated})$$

Therefore general solution of pde (1) is -

$$\cancel{z = \phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \psi_1(y) + x}$$

$$\boxed{z = \phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \psi_1(x) + y^2\psi_2(y)}$$

where ϕ_1, ϕ_2, ϕ_3, ψ_1 and ψ_2 are arbitrary functions

Exercise:

Solve the following equations:

1). $r + t + 2s = 0$

2). $(D^3 . D'^2 + D^2 . D'^3)z = 0$

3). $(D^3 - 4D^2D' + 4DD'^2)z = 0$

4). $(D^4 D'^2)z = 0.$

#>. Method of finding particular integral (PI) of (LHPDE) Linear homogeneous partial differential equation :

Let $$F(D,D')z = f(x,y).$$

Case I : When $f(x,y) = \phi(ax+by)$:

if $F(D,D')$ be homogeneous function of D and D' of degree n, then

$$P.I. = \frac{1}{F(D,D')} \cdot \phi^{(n)}(ax+by) = \frac{1}{F(a,b)} \phi(ax+by).$$

provided $F(a,b) \neq 0$, $\phi^{(n)}$ being n-th derivative of ϕ w.r.t. $(ax+by)$ as a whole.

Exceptional case :- When $F(a,b)=0$, then the above theorem is not valid. In such case we have

$$\frac{1}{(bD-aD')^n} \cdot \phi(ax+by) = \frac{x^n}{b^n . n!} \phi(ax+by).$$

Note : ①. if $F(a,b)=0$ then $(bD-aD')$ is a factor of $F(D,D')$.

②: if $F(a,b) \neq 0$.

$$P.I. = \frac{1}{F(D,D')} \cdot \phi(ax+by) = \frac{1}{F(a,b)} \cdot \iint \cdots \int f(v)\,dv\,dv \cdots dv$$

where $v = ax+by$.

Example:- Solve $(D^2+3DD'+2D'^2)z = x+y$

Solution:-

$\because$ Here given pde is

$$(D^2+3DD'+2D'^2)z = (x+y) \quad \text{---} (1)$$

Auxiliary eqn of pde (1) is

$$m^2 + 3m.1 + 2 = 0$$

$$\Rightarrow \quad m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0$$

$$\Rightarrow \quad \boxed{m = -1, -2}$$

$$C.F. = \phi_1(y-x) + \phi_2(y-2x) \quad \text{---} (2).$$

Now particular integral of pde (1) -

$$P.I. = \frac{1}{(D^2+3DD'+2D'^2)} \cdot (x+y)$$

$$= \frac{1}{(1+3.1.1+2.1)} \iint (x+y)\, d(x+y) \cdot d(x+y)$$

$$= \frac{1}{(1+3+2)} \iint v.dv.dv$$

where $v = x+y$

$$= \frac{1}{6} \int \frac{v^2}{2}.dv$$

$$= \frac{1}{6} \cdot \frac{v^3}{2.3}$$

$$\boxed{P.I = \frac{1}{36}(x+y)^3.} \quad \text{—} (3)$$

Hence required general solution is given by-

$$z = C.F. + P.I.$$

$$\Rightarrow \quad z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{36}(x+y)^3.$$

Example: Solve $(D^3 - 6D^2D' + 11DD'^2 + 6D'^3)z = e^{5x+6y}$

Solution: $\because$ the given pde is-

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y} \quad \text{---} (1)$$

Auxiliary eqn of pde (1) is

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow \quad (m-1)(m-2)(m-3) = 0$$

$$\Rightarrow \quad \boxed{m = 1, 2, 3}$$

therefore,

$$C.F. = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x) \quad \text{----} (2).$$

Now particular integral of pde (1) is

$$P.I. = \frac{1}{(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)} \cdot e^{5x+6y}$$

$$= \frac{1}{(5^3 - 6\times5^2\times6 + 11\times5\times6^2 - 6.6^3)} \iiint e^v \, dv\, dv\, dv$$

where $v = 5x+6y$.

$$\Rightarrow \quad P.I. = -\frac{1}{91}\iint e^{v}\,dv\,dv$$

$$= -\frac{1}{91}\int e^{v}\,dv$$

$$= -\frac{1}{91}e^{v}$$

$$P.I. = -\frac{1}{91}e^{5x+6y}$$

Hence general solution of pde ① is given by

$$\boxed{z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y+3x) - \frac{1}{91}e^{5x+6y}}$$

Example: Solve: $r - 2s + t = \sin(2x+3y)$.

Solution: Since the given pde is

$$r - 2s + t = \sin(2x+3y)$$

or, $(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$ —— ①

Now, auxiliary eqn is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \boxed{m = 1, 1}$$

therefore,

$$C.F. = \phi_1(y+x) + x\phi_2(y+x) \quad \text{---} ②$$

and particular integral is-

$$P.I. = \frac{1}{(D^2 - 2DD' + D'^2)}\cdot \sin(2x+3y) = \frac{1}{(D-D')^2}\sin(2x+3y)$$

$$= \frac{1}{(2-3)^2}\iint \sin v\,dv\,dv$$

where $v = 2x+3y$

$\Rightarrow$ $P.I. = \frac{1}{1}\int -\cos v \, dv$ $\quad \left| \because \int \sin v \, dv = -\cos v \right.$

$\Rightarrow$ $\boxed{P.I. = -\sin v}$

$\Rightarrow$ $P.I. = -\sin(2x+3y).$

Hence general solution of pde ① is-

$$\boxed{z = \phi_1(y+x) + x\phi_2(y+x) - \sin(2x+3y).}$$

Example: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y).$

Solution: $\because$ the given pde is

$$(D^3 - 4D^2D' + 4D'^2D)z = 4\sin(2x+y) \quad \text{---} ①$$

Here the auxiliary eqn is

$$m^3 - 4m^2 + 4m = 0$$

$\Rightarrow$ $m(m^2 - 4m + 4) = 0$

$\Rightarrow$ $m(m-2)^2 = 0$

$\Rightarrow$ $\boxed{m = 0, 2, 2}$

$$C.F. = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) \quad \text{---} ②$$

Now P.I. corresponding to $\sin(2x+y)$

$$P.I. = \frac{1}{D^3 - 4D^2D' - 4DD'^2}\sin(2x+y)$$

$$= \frac{1}{D.(D-2D)^2} \cdot \sin(2x+y)$$

$$P.I. = \frac{4}{D(D-2D')^2} \sin(2x+y)$$

$$= \frac{4}{(D-2D')^2} \left\{ \frac{1}{D} . \sin(2x+y) \right\}.$$

$$= \frac{4}{(D-2D')^2} . \left(-\frac{1}{2} \cos(2x+y) \right)$$

$$= -2 . \frac{1}{(D-2D')^2} . \cos(2x+y).$$

$$= -2 . \frac{x^2}{1^2 \, 2!} \cos(2x+y)$$ ** (By formula)

<u>Exceptional case</u>

$$P.I. = -x^2 \cos(2x+y)$$

Hence general solution of pde (1) is

$$\boxed{z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - x^2 \cos(2x+y)}$$

Exercise :-

✓ Solve the following pde-

i). $(D^2 - 5DD' + 4D'^2) z = \sin(4x+y)$

ii). $(D^2 - 2a DD' + a^2 D'^2) z = f(y+ax)$

iii). $(D^3 - 4D^2 D' + 4DD'^2) z = \sin(y+2x).$

IV). $(D^2 - 3DD' + 2D'^2) z = e^{2x-y} + e^{x+y} + \cos(x+2y).$

Case 2: When $f(x,y) = x^m y^n$:

→ if $n<m$, $\frac{1}{f(D,D')}$ should be expanded in powers of $\frac{D'}{D}$.

→ if $m<n$, $\frac{1}{f(D,D')}$ should be expanded in power of $\frac{D}{D'}$.

→ if $m=n$. then $\frac{1}{f(D,D')}$ can be expanded in power of $\frac{D}{D'}$ or $\frac{D'}{D}$.

Example: Solve $(D^2+3DD'+2D'^2)z = x+y$ by expanding the particular integral.

Solution: ∵ the given pde is

$$(D^2+3DD'+2D'^2)z = xy \quad --- (1)$$

Here auxiliary eqn is

$$m^2+3m+2=0$$

$$\Rightarrow (m+1)(m+2)=0$$

$$\Rightarrow \boxed{m=-1,-2}$$

$$C.F. = \phi_1(y-x) + \phi_2(y-2x). \quad --- (2)$$

Now particular integral -

$$P.I = \frac{1}{(D^2 + 3DD' + 2D'^2)} x \cdot y.$$

$$= \frac{}{2D'^2\left(1 + \frac{D^2 + 3DD'}{2D'^2}\right)} xy.$$

$$= \frac{1}{2D'^2} \cdot \left(1 + \frac{D^2 + 3DD'}{2D'^2}\right)^{-1} xy.$$

$$= \frac{1}{2D'^2}\left\{1 - \frac{D^2 + 3DD'}{2D'^2} + \cdots\right\} xy$$

$$= \frac{1}{2D'^2}\left\{xy - \frac{1}{2D'^2}(3)\right\}$$

$$= \frac{1}{2D'^2}\left\{xy - \frac{3x^2}{4}\right\}.$$

$$= \frac{1}{2}\left\{\frac{x \cdot y^3}{6} - \frac{3x^2 \cdot y^2}{4 \cdot 2}\right\}$$

$$P.I. = \left(\frac{xy^3}{12} - \frac{3x^2y^2}{16.}\right)$$

Hence general solution is given by-

$$\boxed{z = \phi_1(y - x) + \phi_2(y - 2x) + \frac{xy^3}{12} - \frac{3x^2y}{16.}}$$

General method of finding particular Integral:

Let the given pde is

$$F(D, D')\, z = f(x, y) \quad ---① $$

where $F(D, D')$ is homogeneous function of D & D'.

of degree n, so that

$$F(D, D') = (D - m_1 D')(D - m_2 D') ---- (D - m_n D')$$

$$P.I = \frac{1}{F(D, D')} f(x, y)$$

$$P.I. = \frac{1}{(D - m_1 D')(D - m_2 D') --- (D - m_n D')} f(x, y) \quad ---- ②$$

In order to calculate P.I. given by (2), Let

$$(D - m_1 D')\, z = f(x, y) \quad ----- ③$$

$$\Rightarrow \quad p - m_1 q = f(x, y)$$

$$\Rightarrow \quad \frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)} \quad --- ④$$

taking first two fraction,

$$\frac{dx}{1} = \frac{dy}{-m}$$

$$\Rightarrow \quad dy + m\, dx = 0$$

$$\Rightarrow \quad \boxed{y + mx = c_1} \quad ---- ⑤$$

from eqn 4,

$$\frac{dx}{1} = \frac{dz}{f(x,y)}$$

$$\Rightarrow \quad dz = f(x,y)\,dx$$

$$\Rightarrow \quad dz = f(x, c-mx)\,dx \qquad \text{where } mx+y=c.$$

$$\Rightarrow \quad \boxed{z = \int f(x, c-mx)\,dx.}$$

After integration, the constant of integration c must be replaced by $c = y+mx$.

Hence P.I. 2 can be obtained by applying the operation 6 by the factors in succession.

Note: Formulae:

a). $\frac{1}{D-mD'} f(x,y) = \int f(x, c-mx)\,dx$

where $c = y+mx$

b). $\frac{1}{D-mD'} f(x,y) = \int f(x, c+mx)\,dx$

where $c = y-mx$.

Example: Solve $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x$

Solution: $\because$ Here given pde is

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x \quad \text{---} \; (1)$$

$\therefore$ Here auxiliary eqn is

$$m + 1 = 0$$

$$\Rightarrow \boxed{m = -1}$$

$$C.F = \phi_1(y - x) \quad \text{---} \; (2).$$

$$P.I. = \frac{1}{D + D'} \cdot \sin x$$

$$= 1 \cdot \int \sin x \, dx$$

$$= -\cos x$$

Hence general solution is given by

$$\boxed{z = \phi_1(y - x) - \cos x.}$$

Example: Solve $r+s-6t = y\cos x$

Solution: Since the given pde is

$$r+s-6t = y\cos x.$$

$$\text{or, } (D^2+DD'-6D'^2)z = y\cos x \quad \text{---} (1)$$

Here auxiliary eqn is-

$$m^2+m-6=0$$

$$\Rightarrow (m+3)(m-2)=0$$

$$\Rightarrow \boxed{m=-3,\ 2}$$

$$\text{C.F.} = \phi_1(y+2x) + \phi_2(y-3x) \quad \text{----} (2)$$

Now particular integral is

$$\text{P.I.} = \frac{1}{D^2+DD'-6D'^2} \cdot y\sin x$$

$$= \frac{1}{(D-2D')(D+3D')} y\sin x$$

$$= \frac{1}{D-2D'}\left[\frac{1}{D+3D'} y\sin x\right]$$

$$= \frac{1}{D-2D'} \int (3x+c).\cos x\, dx$$

(By formula $\frac{1}{D+mD'} f(x,y) = \int f(x, c+mx)\,dx$.)

$$\Rightarrow \quad P.I. = \frac{1}{(D-2D')}\left[(3x+c)\sin x - 3\int \sin x\, dx\right]$$

$$= \frac{1}{(D-2D')}\left[(3x+c)\sin x + 3\cos x\right]$$

$$= \frac{1}{(D-2D')}\left[y\sin x + 3\cos x\right] \quad \Big| \because y = 3x+c.$$

$$= \int [(c'-2x)\sin x + 3\cos x]\, dx \;**$$

$$= \int (c'-2x)\cdot \sin x\, dx + 3\int \cos x\, dx$$

$$= (c'-2x)\cdot(-\cos x) + 2\int -\cos x\, dx + 3\sin x$$

$$= -(c'-2x)\cos x - 2\sin x + 3\sin x$$

$$= -y\cos x + \sin x \qquad \Big| \because y = c'-2x.$$

$$\boxed{P.I. = \sin x - y\cos x}$$

Hence ~~complete~~ General solution of pde (1) is -

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x}$$

Note: ** $\frac{1}{D-mD'} f(x,y) = \int f(x, c-mx)\, dx,$

where

$y = c - mx$

Classification of second order Partial differential Equations:

consider pde of the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad --- (1)$$

where $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$.

and R, S and T are functions of x and y.

Eqn (1) is hyperbolic if $S^2 - 4RT > 0$.

Eqn (1) is parabolic if $S^2 - 4RT = 0$.

Eqn (1) is elliptic if $S^2 - 4RT < 0$.

$\boxed{\text{Case.1}}$ When eqn (1) is hyperboic i.e $S^2 - 4RT > 0$

The characteristic eqn of (1) is

$$R\lambda^2 + S\lambda + T = 0 \quad --- (2)$$

Here the roots of eqn (2) are real and distinct, say λ_1 and λ_2 are roots of eqn (2).

Now, the characteristic ~~eq~~ curves are given by

$$\frac{dy}{dx} + \lambda_1 = 0 \qquad \& \qquad \frac{dy}{dx} + \lambda_2 = 0$$

$$\downarrow \text{solution} \qquad\qquad \downarrow$$

$$\xi(x, y) = c_1 \qquad\qquad \eta(x, y) = c_2$$

#

$\Rightarrow$ $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$ are characteristic curves of pde ①.

Case 2: When equation ① is parabolic i.e. $S^2 - 4RT = 0$:

Here roots are real and equal.

i.e. $\lambda_1 = \lambda_2 = \lambda$

Now characteristic curves is given by

$$\frac{dy}{dx} + \lambda = 0$$

$$\Rightarrow \xi(x,y) = c_1$$

there is only one characteristic curve.

Case III: When eqn ① is elliptic $S^2 - 4RT < 0$:

then roots are real and complex ($\alpha + i\beta$ type)

$\Rightarrow$ Here characteristic curve can not be calculated.

Que: The variable ξ and η which reduce to pde ① to canonical form

$$\frac{\partial^2 u}{\partial x^2} - x^2 \cdot \frac{\partial^2 u}{\partial y^2} = 0$$

$\boxed{\text{Solution}}$:

$\because$ Here pde is

$$\frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} (1)$$

$R = 1, \quad S = 0, \quad T = -x^2$

$\because \quad S^2 - 4RT = 0 + 4 \cdot x^2 = 4x^2 > 0 \qquad (\because x \neq 0)$

$\Rightarrow$ Transformation is hyperbolic.

Now, characteristic curves are –

$$\frac{dy}{dx} \pm \lambda_1 = 0$$

$\because R\lambda^2 + S\lambda + T = 0$

$\Rightarrow \lambda^2 - x^2 = 0$

$\Rightarrow \boxed{\lambda = \pm x}$

therefore, we have

$$\frac{dy}{dx} + x = 0 \quad \& \quad \frac{dy}{dx} - x = 0$$

$\Rightarrow \quad dy + x\,dx = 0 \quad \& \quad dy - x\,dx = 0$

on integration, we have

$$y + \frac{x^2}{2} = c_1 \quad \& \quad y - \frac{x^2}{2} = c_2$$

$$\Rightarrow \boxed{\xi \equiv y + \frac{x^2}{2} = c_1 \quad \& \quad \eta \equiv y - \frac{x^2}{2} = c_2}$$

Canonical Form :- The PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad \text{——} ①$$

Now, we change independent variable x and y into ξ, η then ① becomes

$$\bar{A} u_{\xi\xi} + \bar{B} u_{\xi\eta} + \bar{C} u_{\eta\eta} + \bar{D} u_\xi + \bar{E} u_\eta + \bar{F} u = \bar{G} \quad \text{——} ②$$

Where

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y.$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y.$$

$$\bar{F} = F$$

$$\bar{G} = G$$

Problem :- Reduce this transformation

$$\frac{\partial^2 u}{\partial x^2} - \operatorname{sech}^4 x \, \frac{\partial^2 u}{\partial y^2} = 0$$

in canonical form.

Solution :- $\frac{\partial^2 u}{\partial x^2} - \operatorname{sech}^4 x \, \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow u_{xx} - \operatorname{sech}^4 x \, u_{yy} = 0 \quad \text{——} ①$$

Comparing ① with $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we get

$$A = 1, \; B = 0, \; C = -\operatorname{sech}^4 x, \; D = 0, \; E = 0, \; F = 0, \; G = 0$$

$\therefore B^2 - 4AC = 0 + 4\operatorname{sech}^4 x = 4\operatorname{sech}^4 x > 0$ (Hyperbolic)

Roots of the auxiliary equations are

$$A\lambda^2 + B\lambda + C = 0 \Rightarrow \lambda^2 - \operatorname{sech}^4 x = 0 \Rightarrow \lambda = \pm \operatorname{sech}^2 x$$

Characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + \operatorname{sech}^2 x = 0 \quad \& \quad \frac{dy}{dx} - \operatorname{sech}^2 x = 0$$

$$\Rightarrow \xi \equiv y + \tanh x = c_1 \quad \& \quad \eta \equiv y - \tanh x = c_2$$

Now, changing the independent variables x and y into ξ & η, equation ① becomes.

$$\bar{A} u_{\xi\xi} + \bar{B} u_{\xi\eta} + \bar{C} u_{\eta\eta} + \bar{D} u_{\xi} + \bar{E} u_{\eta} + \bar{F} u = \bar{G} \quad \text{—— ②}$$

$$\bar{A} = A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2$$

$$= 1\cdot(\operatorname{sech}^2 x)^2 + 0\cdot \xi_x \xi_y + (-\operatorname{sech}^4 x)\cdot 1^2 = 0 = \bar{C}$$

$$\bar{B} = 2A\xi_x \eta_x + B(\xi_x \eta_y + \eta_x \xi_y) + 2C\xi_y \eta_y$$

$$= 2\times 1 \times \operatorname{sech}^2 x \cdot (-\operatorname{sech}^2 x) + 0\cdot(\xi_x \eta_y + \eta_x \xi_y) - 2\operatorname{sech}^4 x\cdot 1\cdot 1$$

$$= -4\operatorname{sech}^4 x.$$

$$\bar{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$= 1\cdot 2\operatorname{sech} x\cdot \operatorname{sech} x \tanh x + 0 + (-\operatorname{sech}^4 x)\cdot 0 + 0 + 0$$

$$= 2\operatorname{sech}^2 x \tanh x$$

$$\bar{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$= 1\cdot(-2\operatorname{sech} x\cdot \operatorname{sech} x \tanh x) + 0 + 0 + 0 + 0$$

$$= -2\operatorname{sech}^2 x \tanh x$$

$$\bar{F} = F = 0$$

$$\bar{G} = G = 0$$

Substituting these values in ②, we get.

$$-4\,\text{sech}^4 x\, u_{\xi\eta} + 2\,\text{sech}^2 x \tanh x\, u_\xi - 2\,\text{sech}^2 x \tanh x\, u_\eta = 0$$

$$\Rightarrow \boxed{u_{\xi\eta} = \frac{\tanh x}{2\,\text{sech}^2 x}\left(u_\xi - u_\eta\right)} \quad \underline{\text{Ans}}$$

$\xi = y + \tanh x$

$\eta = y - \tanh x$

$\frac{\xi - \eta}{2} = \tanh x$

$\text{sech}\, x = \sqrt{1 - \tanh^2 x}$

$= \sqrt{1 - \left(\frac{\xi - \eta}{2}\right)^2}$

NET (SUB)

Problem :- Solve

one dimensional wave equ^n.

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{——— (1)}$$

Initial value only

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty \ \& \ t > 0$$

Initial displacement

ANS :. $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \rightarrow$ D'Alembert's sol^n of wave equ^n.

Solution :- Here $A = -c^2$, $B = 0$, $C = 1$, $D = 0$, $E = 0$, $F = G = 0$

$\therefore B^2 - 4AC = 4c^2 > 0 \Rightarrow$ Hyperbolic

$\therefore$ Roots of auxiliary equations

$$A\lambda^2 + B\lambda + C = 0$$

$$\Rightarrow -c^2\lambda^2 - 0 + 1 = 0 \Rightarrow \lambda = \pm 1/c$$

$\therefore$ Characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{c} = 0 \quad \& \quad \frac{dy}{dx} - \frac{1}{c} = 0$$

$$\Rightarrow \xi \equiv x + cy = c_1 \quad \& \quad \eta \equiv cy - x = c_2$$

The given PDE converted into

$$u_{\xi\eta} = 0$$

$$C.F = u = \phi_1(x+ct) + \phi_2(x-ct) \quad \text{——— (2)}$$

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GATE-05

$\boxed{\text{Problem}}$:- PDE

$$xu_{xx} + 2xyu_{xy} + yu_{yy} + xu_y + yu_x = 0,$$ then

(A) elliptic in the region $x<0,\ y<0,\ xy>1$.

(B) elliptic in the region $x>0,\ y>0,\ xy>1$.

(C) parabolic in the region $x<0,\ y<0,\ xy>1$

(D) hyperbolic in the region $x<0,\ y<0,\ xy>1$.

$\boxed{\text{Solution}}$:- Here $R = x,\ S = 2xy,\ T = y$

$$\therefore\ S^2 - 4RT = (2xy)^2 - 4xy$$
$$= 4x^2y^2 - 4xy$$
$$= 4xy(xy-1) > 0.$$

$\therefore\ S^2 - 4RT > 0$

$\Rightarrow$ hyperbolic in the region $x<0,\ y<0,\ xy>1$.

NET JUNE 06

$\boxed{\text{Problem}}$:- Reduce this transformation

$$\frac{\partial^2 u}{\partial x^2} - \operatorname{sech}^4 x \frac{\partial^2 u}{\partial y^2} = 0$$

in canonical form.

$\boxed{\text{Solution}}$:- Put $u = z(x,y)$

$$\Rightarrow r - \operatorname{sech}^4 x \cdot t = 0 \quad \text{—— ①}$$

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

Here $R = 1$, $S = 0$, $T = -\operatorname{sech}^4 x$

$$S^2 - 4RT = 4\operatorname{sech}^4 x > 0 \quad \text{(always)}$$

(Hyperbolic)

Characteristic equation is

$$R\lambda^2 + S\lambda + T = 0$$

$$\lambda^2 - \operatorname{sech}^4 x = 0$$

$$\Rightarrow \lambda = \pm \operatorname{sech}^2 x$$

Characteristic curves are given by

$$\frac{\partial^2 u}{\partial x^2} - \operatorname{sech}^4 x \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + \operatorname{sech}^2 x = 0 \quad \& \quad \frac{dy}{dx} - \operatorname{sech}^2 x = 0$$

$$\Rightarrow \xi \equiv y + \tanh x = c_1 \quad \& \quad \eta \equiv y - \tanh x = c_2$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\Rightarrow p = \operatorname{sech}^2 x \left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$= \left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) \Rightarrow \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[\operatorname{sech}^2 x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right)\right]$$

$$= -2\operatorname{sech}^2 x \tanh x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) + \operatorname{sech}^2 x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right)$$

$$= \operatorname{sech}^2 x\left[-2\tanh x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) + \frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]$$

$$= \operatorname{sech}^2 x\left[-2\tanh x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) + \left(\frac{\partial^2 z}{\partial \xi^2} - \frac{\partial^2 z}{\partial \xi \partial \eta}\right)\operatorname{sech}^2 x + \left(\frac{\partial^2 z}{\partial \xi \partial \eta} - \frac{\partial^2 z}{\partial \eta^2}\right)(-\operatorname{sech}^2 x)\right]$$

$$\Rightarrow r = \operatorname{sech}^2 x\left[-2\tanh x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) + \operatorname{sech}^2 x\left(\frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}\right)\right]$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right)$$

$$= \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

Put all the values in ①, we get

$$\operatorname{sech}^2 x\left[-2\tanh x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) + \operatorname{sech}^2 x\left(\cancel{\frac{\partial^2 z}{\partial \xi^2}} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 z}{\partial \eta^2}}\right)\right]$$

$$- \operatorname{sech}^4 x\left(\cancel{\frac{\partial^2 z}{\partial \xi^2}} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \cancel{\frac{\partial^2 z}{\partial \eta^2}}\right) = 0$$

$$\Rightarrow -2\tanh x \operatorname{sech}^2 x\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) - 4\operatorname{sech}^4 x\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \quad 2\operatorname{sech}^2 x \frac{\partial^2 z}{\partial \xi \partial \eta} + \tanh x \left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) = 0$$

$$\xi = y + \tanh x$$

$$\eta = y - \tanh x$$

$$\frac{\xi - \eta}{2} = \tanh x$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1$$

$$\Rightarrow \operatorname{sech}^2 x = 1 - \tanh^2 x \Rightarrow \operatorname{sech} x = \sqrt{1 - \tanh^2 x}$$

$$\Rightarrow \operatorname{sech} x = \sqrt{1 - \left(\frac{\xi - \eta}{2}\right)^2}$$

$$\Rightarrow \boxed{\left[1 - (\xi - \eta)^2\right] \frac{\partial^2 z}{\partial \xi \partial \eta} + (\xi - \eta)\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) = 0}$$

Ans

Parabolic :-

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{——} (1)$$

If $B^2 - 4AC = 0 \Rightarrow$ Parabola.

Characteristic equations are

$$\frac{dy}{dx} + \lambda = 0 \Rightarrow \xi(x,y) = c_1$$

$\hookrightarrow$ Characteristic curve.

We assume another solution $\eta(x,y) = c$ s.t. ξ & η are L.I.

i.e. $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$ in the given region.

NET JUNE 07

Problem :- Reduce $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} = 0$ —— (1) into canonical form.

Solution :- Here $A = e^{2x}$, $B = 2e^{x+y}$, $C = e^{2y}$

$\therefore B^2 - 4AC = 4e^{2x+2y} - 4e^{2x}\cdot e^{2y} = 0$

Hence, the given differential equation to be parabolic.

Roots are given by

$$A\lambda^2 + B\lambda + C = 0$$

$$e^{2x}\lambda^2 + 2e^{x+y}\lambda + e^{2y} = 0$$

$$\Rightarrow (\lambda e^x + e^y)^2 = 0 \Rightarrow \lambda = -e^{y-x}$$

$\therefore$ Characteristic equations are given by

$$\frac{dy}{dx} - e^{y-x} = 0 \Rightarrow e^{-y}dy - e^{-x}dx$$

$$\Rightarrow e^{-y} - e^{-x} = c_1 = \xi \quad \text{——} (2)$$

Suppose $\eta = e^{-y} + e^{-x} = c_2$ —— (3)

Now, $\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} e^{-x} & -e^{-y} \\ -e^{-x} & e^{-y} \end{vmatrix}$

$$= e^{-(x+y)}\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = 2e^{-(x+y)} \neq 0.$$

$$\therefore \xi = e^{-y} - e^{-x} \quad \& \quad \eta = e^{-y} + e^{-x}$$

$$p = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = e^{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$q = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\right] = -e^{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$+ e^{-x}\left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]$$

$$\therefore r = -e^{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) + e^{-2x}\left[\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[-e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\right]$$

$$= -e^{-y}\left[\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x}\right]$$

$$= -e^{-(x+y)}\left[\frac{\partial^2 u}{\partial \xi^2} + \cancel{\frac{\partial^2 u}{\partial \xi \partial \eta}} - \cancel{\frac{\partial^2 u}{\partial \eta \partial \xi}} - \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$= -e^{-(x+y)}\left[\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[-e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\right]$$

$$= e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) - e^{-y}\left[-e^{-y}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\right)\right.$$

$$\left. - e^{-y}\left(\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right)\right]$$

$$= e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + e^{-2y}\left[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right]$$

$$= e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + e^{-2y}\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)$$

Substituting these values in ①, we get

$$e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$$

$$\Rightarrow e^{2x}\left[-e^{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) + e^{-2x}\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)\right]$$

$$+ 2e^{x+y}\left[-e^{-(x+y)}\left(\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2}\right)\right]$$

$$+ e^{2y}\left[e^{-y}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + e^{-2y}\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)\right] = 0$$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{\partial^2 u}{\partial \xi^2} = \phi\left(\xi, \eta, u, \dfrac{\partial u}{\partial \xi}, \dfrac{\partial u}{\partial \eta}\right) \\ \boxed{\text{OR}},\ \dfrac{\partial^2 u}{\partial \eta^2} = \phi\left(\xi, \eta, u, \dfrac{\partial u}{\partial \xi}, \dfrac{\partial u}{\partial \eta}\right) \end{array}}$$

Always converted into that form.

$$\Rightarrow (1-2+1)\frac{\partial^2 u}{\partial \xi^2} + (1+2+1)\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \xi}(e^{y} - e^{-x}) + (e^{x}+e^{y})\frac{\partial u}{\partial \eta} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{4}\left[(e^{x} - e^{y})\frac{\partial u}{\partial \xi} - (e^{x}+e^{y})\frac{\partial u}{\partial \eta}\right]$$

$$\because \left.\begin{array}{l}\xi = e^{-y} - e^{-x} \\ \eta = e^{-y} + e^{-x}\end{array}\right] \Rightarrow \frac{\xi - \eta}{2} = -e^{-x} \Rightarrow e^{x} = \frac{2}{\eta - \xi} \ \& \ e^{y} = \frac{2}{\xi + \eta}$$

$$\therefore \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{4}\left[\left(\frac{2}{\eta - \xi} - \frac{2}{\xi + \eta}\right)\frac{\partial u}{\partial \xi} - \left(\frac{2}{\xi+\eta} - \frac{2}{\xi - \eta}\right)\frac{\partial u}{\partial \eta}\right]$$

$$= \frac{\cancel{2}}{\cancel{4}_2(\xi^2 - \eta^2)}\left[-(\xi + \cancel{\eta}) - (\xi - \cancel{\eta})\right]\frac{\partial u}{\partial \xi} - \left[(\xi - \eta) - (\xi + \eta)\right]\frac{\partial u}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{2(\xi^2 - \eta^2)}\left[2\eta\frac{\partial u}{\partial \eta} - 2\xi\frac{\partial u}{\partial \xi}\right] = \frac{1}{\xi^2 - \eta^2}\left[\eta\frac{\partial u}{\partial \eta} - \xi\frac{\partial u}{\partial \xi}\right]$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{\xi^2 \eta^2}\left(\eta\frac{\partial u}{\partial \eta} - \xi\frac{\partial u}{\partial \xi}\right)}$$ Ans

↳ which is parabolic.

$\boxed{\text{Elliptic}}$:-

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad \text{--- (1)}$$

If $B^2 - 4AC < 0 \Rightarrow$ Elliptic.

Characteristic roots are

$$A\lambda^2 + B\lambda + C = 0 \quad [\text{Roots are complex}]$$

Characteristic equations are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \xi(x,y) = C_1 \quad \& \quad \eta(x,y) = C_2$$

We use another independent variables say α and β s.t.

$$\alpha = \frac{\xi + \eta}{2}, \quad \beta = \frac{\xi - \eta}{2i}$$

$\boxed{\text{Problem}}$:- Reduce $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ into canonical form.

$\boxed{\text{Solution}}$:- Here $A = 1$, $B = 0$, $C = x^2$

$$\therefore \quad B^2 - 4AC = -4x^2 < 0$$

Hence the given PDE are elliptic type in nature.

Roots are given by

$$R\lambda^2 + S\lambda + T = 0$$

$$\lambda^2 + x^2 = 0 \Rightarrow \boxed{\lambda = \pm ix}$$

Characteristic equations are given by

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \frac{dy}{dx} + ix = 0 \quad \& \quad \frac{dy}{dx} - ix = 0$$

$$\Rightarrow y + \frac{ix^2}{2} = C_1 \quad \& \quad y - \frac{ix^2}{2} = C_2$$

$$\Rightarrow \xi \equiv y + \frac{ix^2}{2} = C_1 \quad \& \quad \eta \equiv y - \frac{ix^2}{2} = C_2$$

Let α and β are another independent variables such that

$$\alpha = \frac{\xi+\eta}{2} = y$$

$$\beta = \frac{\xi-\eta}{2i} = \frac{x^2}{2}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \alpha}{\partial x} = 0$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

$$\boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(z, \xi, \eta, \frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}\right)} \to \text{Always converted into that form.}$$

Substituting these values in (1), we have.

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2}\frac{\partial z}{\partial \beta} = -\frac{1}{2\beta}\left(\frac{\partial z}{\partial \beta}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\left(\frac{\partial z}{\partial \beta}\right)} \text{ Ans.}$$

$\boxed{\text{Note}}$:-

i). $\boxed{A=1,\ B=0,\ C=0}$,
Heat equation $\left(\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u}{\partial t} \right) \rightarrow$ Parabolic

ii). $\boxed{A=1,\ B=0,\ C=1}$,
Laplace equation $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right) \rightarrow$ Elliptic

iii). $\boxed{A=c^2,\ B=0,\ C=-1}$,
Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $(c \neq 0) \rightarrow$ Hyperbolic

$\boxed{\text{Problem}}$:- 1). $u_{xx} - u_{yy} - \frac{2}{x} u_x = 0 \rightarrow$ Hyperbolic

~~2). $u_{xx} - y u_{xy} + x u_x + y u_y + x = 0 \rightarrow$ Hyperbolic.~~

3). $y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0 \rightarrow$ Parabolic.

4). $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \rightarrow$ Parabolic. ANS $\frac{\partial^2 z}{\partial \eta^2} = 0$

5). $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + 5 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial y} - 3z \rightarrow$ Elliptic

$\boxed{\text{Ans}}$ 1). $2x u_{\xi\eta} + u_\xi - u_\eta = 0$, $\xi = x+y$, $\eta = y-x$.

$\boxed{\text{Ans}}$ 3). $\frac{\partial^2 z}{\partial \eta^2} = 0$

$\boxed{\text{Ans}}$ 4). $\frac{\partial^2 z}{\partial \eta^2} = 0$

In three Variables :-

$$A = \begin{bmatrix} \text{coeff. of } u_{xx} & \text{coeff. } \frac{u_{xy}}{2} & \text{coeff. of } \frac{u_{xz}}{2} \\ \frac{u_{yx}}{2} & u_{yy} & \frac{u_{yz}}{2} \\ \frac{u_{zx}}{2} & \frac{u_{zy}}{2} & u_{zz} \end{bmatrix}$$

$$A = \begin{bmatrix} R & S/2 \\ S/2 & T \end{bmatrix}$$

$$|A| = \frac{4RT - S^2}{4}$$

Here if $|A| < 0 \Rightarrow$ Hyperbolic

$|A| = 0 \Rightarrow$ Parabolic

$|A| > 0 \Rightarrow$ Elliptic.

Classification of PDE in three independent variables :-

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0$$

$a_{ij} = a_{ji}$ & b & c are constants or some function of independent variables x_1, x_2, x_3.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \text{Real symmetric matrix.}$$

i). If all eigen values of A are non-zero and have same sign except one eigen value then transformation is $\boxed{\text{hyperbolic}}$.

ii). If one eigen value of A is zero then the transformation is $\boxed{\text{parabolic}}$

iii). If all eigen values of A are non-zero and of same sign then the transformation is $\boxed{\text{elliptic}}$.

$\boxed{\text{Problem}}$:- $u_{xx} + u_{yy} + u_{zz} = 0$

$\boxed{\text{Solution}}$:- This is Laplace's equation in three-dimension.

$\therefore$ Elliptic.

& Here $a_{11} = 1$, $a_{22} = 1$, $a_{33} = 1$

$|A| = 1 > 0$
i.e. $|A| > 0 \Rightarrow$ Elliptic

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here eigen values are 1, 1, 1.

$\therefore$ All eigen values are non-zero and of same sign.

$\Rightarrow$ Elliptic.

$\boxed{\text{Problem}}$:- $u_{xx} + u_{yy} = u_{zz}$ $\Rightarrow$ $u_{xx} + u_{yy} - u_{zz} = 0$

Here $a_{11} = 1$, $a_{22} = 1$, $a_{33} = -1$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Here eigen values are 1, 1, −1.

Here two eigen values are same sign and one eigen value is of different sign.

$\Rightarrow$ Hyperbolic.

$|A| = -1 < 0$
i.e. $|A| < 0 \Rightarrow$ Hyperbolic

$\boxed{\text{Problem}}$:- $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$

$\boxed{\text{Solution}}$:- Here $a_{11} = 1$, $a_{22} = 2$, $a_{33} = 1$, $a_{12} = -2$

$a_{23} = -2$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$|A| = 1(-2) + 2(-2) + 0\cdot(-4)$

$= -2 - 4 = -6$

Here $|A| = -6 < 0 \Rightarrow$ Hyperbolic.

$\boxed{\text{2nd method}}$:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)\{(\lambda-2)(\lambda-1) - 4\} + 2(-2+2\lambda)$

$= (1-\lambda)[\{(\lambda-2)(\lambda-1) - 4\} - 4]$

$= (1-\lambda)\,[\lambda^2 - 3\lambda + 2 - 8]$

$= (1-\lambda)\,(\lambda^2 - 3\lambda - 6)$

$\dfrac{3 \pm \sqrt{9+24}}{2}$

$\dfrac{3 \pm \sqrt{33}}{2}$

$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda - 6) = 0$

$\Rightarrow \lambda = 1, \dfrac{3+\sqrt{33}}{2}, \dfrac{3-\sqrt{33}}{2}$

$(+) \quad (+) \quad (-)$

Here two eigenvalues are +ve & one eigen value is −ve.

$\Rightarrow$ Hyperbolic.

$\boxed{\text{Problem}}$:- $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$

$\boxed{\text{Solution}}$:- Here $a_{11} = 1$, $a_{22} = 2$, $a_{33} = 1$, $a_{12} = -2$

$a_{23} = -2$

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

$|A| = 1(-2) + 2(-2) + 0\cdot(-4)$

$= -2 - 4 = -6$

Here ~~$|A| = -6 < 0$~~ $\Rightarrow$ Hyperbolic.

$\boxed{\text{2nd method}}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)\{(\lambda-2)(\lambda-1) - 4\} + 2(-2+2\lambda)$

$= (1-\lambda)[\{(\lambda-2)(\lambda-1) - 4\} - 4]$

$= (1-\lambda)\,[\lambda^2 - 3\lambda + 2 - 8]$

$= (1-\lambda)\,(\lambda^2 - 3\lambda - 6)$

$\frac{3 \pm \sqrt{9+24}}{2}$

$\frac{3 \pm \sqrt{15}}{2}$

$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda - 6) = 0$

$\Rightarrow \lambda = 1, \ \frac{3+\sqrt{33}}{2}, \ \frac{3-\sqrt{33}}{2}$

$(+) \quad (+) \quad (-)$

Here two eigenvalues are +ve & one eigen value is −ve.

$\Rightarrow$ Hyperbolic.